# ON THE WELL POSEDNESS FOR HYPERBOLIC SYSTEMS OF CONSERVATION LAWS WITH LARGE BV DATA

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Contents



Abstract. We study the Cauchy problem for a strictly hyperbolic $n \times n$ system of conservation laws in one space dimension

$$u_t + f(u)_x = 0,$$

assuming that the initial data $u(0,x) = \bar{u}(x)$ has bounded but possibly large total variation. Under a linearized stability condition on the Riemann problems generated by the jumps in $\bar{u}$, we prove existence and uniqueness of a (local in time) $BV$ solution, depending continuously on the initial data in $L^1_{loc}$. The last section contains an application to the $3 \times 3$ system of gas dynamics.

## 1. Introduction and statement of the main results

The system of conservation laws in one space dimension takes the form:

$$u_t + f(u)_x = 0. \tag{1.1}$$

In this paper we are mainly concerned with the existence and stability of solutions to (1.1) in the vicinity of a self-similar entropy solution $u_0 = u_0(x/t)$ to a given Riemann problem:

$$u(0,x) = \begin{cases} u_l & x < 0, \\ u_r & x > 0 \end{cases} \tag{1.2}$$

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with possibly large data. Indeed, because of the finite propagation speed, the stability of Riemann problems will imply the local in time well posedness of the Cauchy problem (1.1) with initial data:

$$u(0,x) = \bar{u}(x), \tag{1.3}$$

having bounded but possibly large total variation. We shall thus first construct a continuous semigroup of solutions on a domain $\mathcal{D}$ of functions $\bar{u}$ containing all suitably small $BV$ perturbations of the Riemann solution. In the simplest case where $u_r = u_l$, the well posedness of the Cauchy problem for small $BV$ perturbation of the constant solution $u_0(t,x) \equiv u_l$ was proven in [BCP] and in [BLY] (see also [B]). On the other hand, in the general case where the reference solution $u_0$ contains large waves, one needs to impose additional stability assumptions to control the size of a first order perturbation, measured in $TV$ or in the $L^1$ norm. As proven in [BM], such perturbation $v = v(t,x)$ satisfies a linear hyperbolic system of the form

$$v_t + \big[\mathrm{D}f(u_0)\cdot v\big]_x = 0, \tag{1.4}$$

supplemented by appropriate boundary conditions across the jumps in $u_0$. For each $i = 1, \ldots, n$, let $v_i$ be the $i$-th component of $v$ in a basis $\{r_1, \ldots, r_n\}$ of eigenvectors of $\mathrm{D}f(u)$ so that $v = \sum_i v_i r_i$. Our basic stability assumption, roughly speaking, will be the existence of positive weights $W_i = W_i(x/t)$ such that the integral

$$\int \sum_{i=1}^{n} W_i(t,x)\,\big|v_i(t,x)\big|\,dx$$

is non-increasing in time, for every solution $v$ of the linearized system (1.4).

Following this approach, in earlier papers we have analyzed the cases where the Riemann solution $u_0$ consisted of a number of large shocks [Le1, Le2] or when it contained a single rarefaction [Le3, Le4]. The present work deals with the general case of noninteracting shocks, rarefactions and contact discontinuities.

The outlay of the paper is the following. In section 2 we briefly comment on the stability of shocks and rarefactions and then turn to studying contact discontinuities. Section 3 introduces the $BV$ and $L^1$ stability conditions (BV) and (L1) on the wave pattern in the reference solution $u_0$, which generalise the conditions in [BC, BM, Scho, Le1, Le3]. In section 4 we prove the existence of solutions to (1.1) (1.3) under condition (BV) (Theorem 1.1), by means of the wave front tracking algorithm. In section 5, relying on (L1), we construct the Lyapunov functional, measuring the distance between the solutions to Cauchy problems. This yields the existence of a Lipschitz continuous flow of solutions (Theorem 1.2). Section 6 contains the stability estimates needed in section 5; we focus on the modifications of the estimates from [BLY, Le1, Le3] due to the presence of different kinds of large elementary waves. In section 7 we deduce a local well posedness result for the class of functions having bounded total variation. Finally, section 8 concerns the validation of the stability conditions in the setting of gas dynamics.

We start by stating our basic hypotheses and setting the notation. Assume that the Riemann problem (1.1) (1.2) has a self-similar solution $u_0(t,x) = u_0(1, x/t)$. For a small parameter $c > 0$ define the domain:

$$\Omega = \Omega_c = \{\omega \in \mathbf{R}^n : \quad ||\omega - u_0(1,x)|| < c \quad \text{for some } x \in \mathbf{R}\}\,; \tag{1.5}$$

all the subsequent reasoning will be restricted to this domain, with the parameter $c$ appropriately small.

We work with the following hypotheses:

(H1) The smooth flux $f : \Omega \longrightarrow \mathbf{R}^n$ is strictly hyperbolic. More precisely, for each $u \in \Omega$ the Jacobian matrix $\mathrm{D}f(u)$ has $n$ distinct and real eigenvalues: $\lambda_1(u) < \ldots < \lambda_n(u)$.

(H2) Each characteristic field of (1.1) in $\Omega$ is either genuinely nonlinear or linearly degenerate. That is, with a basis $\{r_i(u)\}_{i=1}^n$ of the right eigenvectors of $\mathrm{D}f(u)$; $\mathrm{D}f(u)r_i(u) = \lambda_i(u)r_i(u)$, each of the $n$ directional derivatives $\langle \mathrm{D}\lambda_i, r_i\rangle$ is either $> 0$ in $\Omega$ or it vanishes identically.

In the case of linearly degenerate fields we set $||r_i(u)|| = 1$, while when the $i$-th field is genuinely nonlinear we choose the normalization of $r_i(u)$ so that $\langle \mathrm{D}\lambda_i(u), r_i(u)\rangle = 1$ for all $u \in \Omega$. The dual basis of left eigenvectors is denoted $\{l_j(u)\}_{j=1}^n$. We have $\langle l_j(u), r_i(u)\rangle = \delta_{ij}$ for all $i, j : 1 \ldots n$ and $u \in \Omega$.

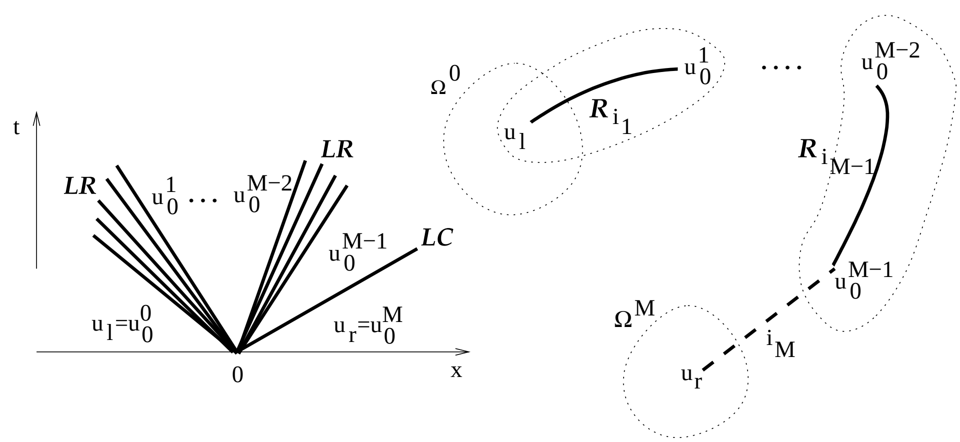


FIGURE 1.1

We assume that the reference solution $u_0(t,x)$ is composed of $M \in \{2, \ldots, n\}$ elementary waves, consecutively connecting $M+1$ distinct states $\{u_0^q\}_{q=0}^M$. For each $q : 1 \ldots M$, the $q$-th wave is a self-similar solution of the Riemann problem $(u_0^{q-1}, u_0^q)$ and is associated with the $i_q$-th characteristic family, $1 \le i_1 < i_2 < \ldots < i_M \le n$. We have $u_0^0 = u_l$ and $u_0^M = u_r$, see Figure 1.1.

The elementary waves can be of three types:

($i$). $q \in \mathcal{LS}$. Stable, compressive shock of a genuinely nonlinear family $i_q$. That is, calling $\Lambda^q$ the speed of the shock we have:

$$\Lambda^q \cdot (u_0^q - u_0^{q-1}) = f(u_0^q) - f(u_0^{q-1}), \tag{1.6}$$

$$\lambda_{i_q-1}(u_0^{q-1}) < \Lambda^q < \lambda_{i_q}(u_0^{q-1}) \quad \text{and} \quad \lambda_{i_q}(u_0^q) < \Lambda^q < \lambda_{i_q+1}(u_0^q), \tag{1.7}$$

$$\det\Big[r_1(u_0^{q-1}) \ldots r_{i_q-1}(u_0^{q-1}),\ u_0^q - u_0^{q-1},\ r_{i_q+1}(u_0^q) \ldots r_n(u_0^q)\Big] \neq 0. \tag{1.8}$$

$(ii)$. $q \in \mathcal{LC}$. Stable contact discontinuity of a linearly degenerate family $i_q$. That is, (1.6) holds together with:

$$\lambda_{i_q}(u_0^{q-1}) = \Lambda^q = \lambda_{i_q}(u_0^q) \tag{1.9}$$

and the stability conditions (1.8) and

$$\langle l_{i_q}(u_0^{q-1}), u_0^q - u_0^{q-1}\rangle \cdot \langle l_{i_q}(u_0^q), u_0^q - u_0^{q-1}\rangle \neq 0 \tag{1.10}$$

are satisfied.

$(iii)$. $q \in \mathcal{LR}$. Stable rarefaction wave of a genuinely nonlinear family $i_q$. That is:

$$u_0(t,x) = \mathcal{R}_{i_q}(\theta) \quad \text{if} \quad x/t = \lambda_{i_q}(\mathcal{R}_{i_q}(\theta)), \quad \theta \in [0, \Theta_q], \tag{1.11}$$

where $\mathcal{R}_{i_q}$ is the rarefaction curve joining states $u_0^{q-1}$ and $u_0^q$:

$$\begin{gathered} \frac{\mathrm{d}}{\mathrm{d}\theta}\mathcal{R}_{i_q}(\theta) = r_{i_q}\left(\mathcal{R}_{i_q}(\theta)\right), \\ u_0^{q-1} = \mathcal{R}_{i_q}(0), \quad u_0^q = \mathcal{R}_{i_q}(\Theta_q), \quad \Theta_q = \lambda_{i_q}(u_0^q) - \lambda_{i_q}(u_0^{q-1}) > 0. \end{gathered} \tag{1.12}$$

By strict hyperbolicity, the elementary waves in (i) - (iii) have speeds ordered increasingly and thus they can be put together to obtain the weak entropy admissible solution $u_0$ to (1.1). Conditions (1.8) and (1.10) are the stability conditions for shocks and contacts. We do not formulate here the related condition for rarefactions as it is implied by more restrictive conditions (BV) or (L1) on the stability of the whole wave pattern in $u_0$ (introduced in section 3). The discussion of stability conditions on single waves will be the objective of section 2.

We now turn to formulating the main results of the paper. The precise form of the stability conditions (BV) and (L1) will be given in sections 3.2 and 3.3.

**Theorem 1.1.** *Assume that (H1), (H2) and the BV stability condition (BV) hold. There exists $\delta > 0$ such that for every $\bar{u}$ in the set:*

$$\begin{aligned} \mathrm{cl}\,\big\{w : \mathbf{R} \longrightarrow \mathbf{R}^n; \quad & ||w\circ\phi - u_0(1,\cdot)||_{L^\infty} + TV(w\circ\phi - u_0(1,\cdot)) < \delta \\ & \textit{for some increasing diffeomorphism } \phi : \mathbf{R} \longrightarrow \mathbf{R}\big\}, \end{aligned} \tag{1.13}$$

*where* cl *denotes the closure in $L^1_{loc}$, the Cauchy problem (1.1) (1.3) has a global entropy weak solution $u(t,x)$.*

**Theorem 1.2.** *Assume that (H1), (H2) and the $L^1$ stability condition (L1) are satisfied. Then there exists a closed domain $\mathcal{D} \subset L^1_{loc}(\mathbf{R}, \Omega)$, containing the set in (1.13) for some $\delta > 0$, and there exists a semigroup $S : \mathcal{D} \times [0,\infty) \longrightarrow \mathcal{D}$ such that:*

(i) $||S(\bar{u},t) - S(\bar{v},s)||_{L^1} \leq L \cdot (|t-s| + ||\bar{u}-\bar{v}||_{L^1})$ *for all $\bar{u}, \bar{v} \in \mathcal{D}$, all $t, s \geq 0$ and a uniform constant $L$, depending only on the system (1.1),*

(ii) *for all $\bar{u} \in \mathcal{D}$, the trajectory $t \mapsto S(\bar{u},t)$ is the solution to (1.1) (1.3) given in Theorem 1.1.*

Throughout the paper, by $\mathcal{O}(1)$ we mean any uniformly bounded function, depending only on the system (1.1). Any sufficiently small but positive constant is denoted by $c$. The Riemann data is for simplicity denoted by $(u_l, u_r)$.

## 2. Stability of shocks, contacts and rarefactions

In this section we study stability of elementary waves in $u_0$. For each $q : 0 \dots M$ let $\Omega^q$ be an open neighbourhood of the state $u_0^q$. Given an elementary wave connecting states $u_0^{q-1}$ and $u_0^q$ we want that every Riemann problem $(w^-, w^+) \in \Omega^{q-1} \times \Omega^q$ has a unique self-similar solution containing $n-1$ weak waves and a single large wave $(u^-, u^+)$ of the type and with the stability properties of $(u_0^{q-1}, u_0^q)$. A convenient tool is the constitutive function $\Psi^q : \Omega^{q-1} \times \Omega^q \longrightarrow \mathbf{R}^{n-1}$ whose zero locus consists of such pairs of states $(u^-, u^+)$. We will treat separately each kind of elementary waves.

**2.1. Shocks.** Fix $q \in \mathcal{LS}$ and assume (1.6) – (1.8) that is: the Rankine-Hugoniot conditions, the Lax compressibility condition [L] and the Majda stability condition [M]. Define $\Psi^q$ as follows:

$$\Psi^q(u^-, u^+) = \left\{ \langle f(u^+) - f(u^-), V_k(u^+ - u^-) \right\}_{k=1}^{n-1}. \tag{2.1}$$

Here $V_k \in \mathbf{R}^n$ are smooth vector functions defined on a neighbourhood of the vector $u_0^q - u_0^{q-1}$, and such that for each $v$ the space

$$\text{span } \{V_1(v), \dots, V_{n-1}(v)\}$$

is the orthogonal complement of $v$. Obviously $(\Psi^q)^{-1}(0)$ consists of pairs of states that satisfy the Rankine-Hugoniot condition. By continuity, (1.7) and (1.8) must also hold for these pairs. Further, via implicit function theorem, the Majda stability condition (1.8) implies the solvability of each Riemann problem $(w^-, w^+) \in \Omega^{q-1} \times \Omega^q$ within the class of self-similar functions containing a single shock $(u^-, u^+) \in (\Psi^q)^{-1}(0)$. In particular, we have:

The $n-1$ vectors:

$$\left\{ \frac{\partial \Psi^q}{\partial u^-}(u_0^{q-1}, u_0^q) \cdot r_k(u_0^{q-1}) \right\}_{k=1}^{i_q - 1} \cup \left\{ \frac{\partial \Psi^q}{\partial u^+}(u_0^{q-1}, u_0^q) \cdot r_k(u_0^q) \right\}_{k=i_q+1}^{n} \tag{2.2}$$

are linearly independent.

and

$$\text{rank } \frac{\partial \Psi^q}{\partial u^-}(u_0^{q-1}, u_0^q) = \text{rank } \frac{\partial \Psi^q}{\partial u^+}(u_0^{q-1}, u_0^q) = n - 1. \tag{2.3}$$

These are all folklore results. The details can be found, for example in [Le2]. When $u_0^q - u_0^{q-1}$ is small then (1.7) and (1.8) are authomatic on the negative part of the shock curve $\mathcal{S}_{i_q}$.

**2.2. Contact discontinuities.** Let the $i_q$ characteristic field be linearly degenerate in $\Omega^{q-1} \cup \Omega^q$ and assume (1.6), (1.8) and (1.9). As in the case of shocks, the candidate for the constitutive function $\Psi^q$ is given by (2.1), and by the Majda condition (1.8) it satisfies (2.2). Again $(\Psi^q)^{-1}(0)$ is an $(n+1)$-dimensional manifold near $(u_0^{q-1}, u_0^q)$, containing all pairs of states $(u^-, u^+)$ which satisfy the Rankine-Hugoniot equations:

$$f(u^+) - f(u^-) = \Lambda(u^-, u^+) \cdot (u^+ - u^-). \tag{2.4}$$

Unfortunatelly $\Psi^q(u^-, u^+) = 0$ does not imply that $(u^-, u^+)$ is a contact discontinuity:

$$\lambda_{i_q}(u^-) = \Lambda(u^-, u^+) = \lambda_{i_q}(u^+). \tag{2.5}$$

As an example, take $n = 2, q = 1$ and $u_0^0 = (0,0), u_0^1 = (1,0)$. Define the flux $f$ in (1.1) to be $f(u_1, u_2) = (u_1, u_1 \cdot u_2)$ in a neighbourhood of $u_0^0$ and $f(u_1, u_2) = (0, u_2)$ in a neighbourhood of $u_0^1$. The first characteristic family is linearly degenerate and the jump $(u_0^0, u_0^1)$ satisfies (1.6), (1.8) and (1.9). However, $\lambda_1$ is identically equal to 0 in the neighbourhood of $u_0^1$ so any 1-contact discontinuity must have speed $\Lambda(u^-, u^+) = 0$. Thus the set of all 1-contacts $(u^-, u^+)$ with (2.4) and (2.5) is only a 2-dimensional submanifold of the 3-dimensional $(\Psi^q)^{-1}(0)$.

There are two cases when all elements $(u^-, u^+)$ of $(\Psi^q)^{-1}(0)$ satisfy (2.5). The first case is when $u_0^q$ lies on the $i_q$-th rarefaction curve through $u_0^{q-1}$, as in (1.12). The flow of the ODE (1.12) yields then the rarefaction curve $\mathcal{R}_{i_q}$ through each point $u^- \in \Omega^{q-1}$. The segments of these curves corresponding to parameter values $\theta \in [\Theta_q - c, \Theta_q + c]$ foliate the manifold $(\Psi^q)^{-1}(0)$. Similarily, for each $u^+ \in \Omega^q$ we have a curve of admissible left states $u^- \in \Omega^{q-1}$ and (2.5) is clear since $\lambda_{i_q}$ is constant along each $\mathcal{R}_{i_q}$. We are more interested in the situation when the two sets $\Omega^{q-1}$ and $\Omega^q$ are not a priori connected by a curve of the admissible right or left states. The second case is thus when the same foliation is forced by our additional stability condition (1.10). More precisely, we have:

**Theorem 2.1.** *Let $q \in \mathcal{LC}$ (so that (1.6), (1.8) – (1.10) hold). Then every pair of states $(u^-, u^+) \in \Omega^{q-1} \times \Omega^q$ satisfying the Rankine-Hugoniot equations (2.4) is again a stable contact discontinuity, that is we have (2.5) and the relevant Majda stability (1.8) and condition (1.10) still hold. In particular (2.2) and (2.3) are satisfied.*

Note that (1.10) does not cover the case when $u_0^{q-1}$ and $u_0^q$ are connected by the $i_q$-th rarefaction curve. Condition (1.10) is however quite general and in particular it is always satisfied for sufficiently weak contact discontinuities.

The proof of Theorem 2.1 will be given through two lemmas.

**Lemma 2.2.** *The vectors $\{V_k\}_{k=1}^{n-1}$ in (2.1) can be chosen so that*

$$V_k(v) = -[\mathrm{D}V_k(v)]^t \cdot v$$

*for every $k : 1 \dots n-1$ and every $v$ in a neighbourhood of $u_0^q - u_0^{q-1}$.*

**Lemma 2.3.** *In the setting of Theorem 2.1, (2.2) and (2.3) hold. Consequently, (2.5) holds for every $(u^-, u^+) \in (\Psi^q)^{-1}(0)$.*

**Proof of Lemma 2.2.** Call $e_1, \dots, e_n$ the standard Euclidean basis of $\mathbf{R}^n$. For $v$ close to $e_n$ define vectors $\{\widetilde{V}_k(v)\}_{k=1}^{n-1}$ applying the Gramm-Schmidt ortogonalization process to $n$ linearly independent vectors: $v, e_1, \dots, e_{n-1}$:

$$\begin{aligned} &\widetilde{V}_1(v) = e_1 - \langle e_1, v\rangle \cdot \frac{v}{\| v \|^2} \\ &\widetilde{V}_k(v) = e_k - \Big[\langle e_k, v\rangle \cdot \frac{v}{\| v \|^2} + \sum_{s=1}^{k-1} \langle e_k, \widetilde{V}_s(v)\rangle \cdot \widetilde{V}_s(v)\Big] \quad \forall k : 2 \dots n-1. \end{aligned} \tag{2.6}$$

We will first prove that

$$\Big[\mathrm{D}\widetilde{V}_k(v)\Big]^t \cdot v = -\widetilde{V}_k(v). \tag{2.7}$$

Using the formula $[\mathrm{D}(v/\| v \|^2)]^t \cdot v = -v/ \| v \|^2$ we have:

$$\left[\mathrm{D}\widetilde{V}_1(v)\right]^t \cdot v = \frac{\langle e_1, v\rangle}{\| v \|^2} v - \left(\frac{v}{\| v \|^2} \cdot e_1^t\right)^t \cdot v = -\widetilde{V}_1(v).$$

Similarily, for $k \geq 2$:

$$\begin{aligned}\mathrm{D}\widetilde{V}_k(v) = &- \langle e_k, v\rangle \cdot \mathrm{D}\left(\frac{v}{\| v \|^2}\right) - \frac{v}{\| v \|^2} \cdot e_k^t \\ &+ \sum_{s=1}^{k-1} \langle e_k, \widetilde{V}_s(v)\rangle \cdot \mathrm{D}\widetilde{V}_s(v) + \sum_{s=1}^{k-1} \widetilde{V}_s(v) \cdot e_k^t \cdot \mathrm{D}\widetilde{V}_s(v).\end{aligned}$$

Assuming (2.7) for each $s < k$ and recalling that $\widetilde{V}_s(v)^t \cdot v = 0$ for every $s$, we conclude:

$$\begin{aligned}\left[\mathrm{D}\widetilde{V}_k(v)\right]^t \cdot v = &\langle e_k, v\rangle \cdot \frac{v}{\| v \|^2} - e_k - \sum_{s=1}^{k-1} \langle e_k, \widetilde{V}_s(v)\rangle \cdot \widetilde{V}_s(v) \\ &+ \sum_{s=1}^{k-1} \left[\mathrm{D}\widetilde{V}_s(v)\right]^t \cdot e_k \cdot \widetilde{V}_s(v)^t \cdot v = -\widetilde{V}_k(v).\end{aligned}$$

Now for $v$ close to $u_0^q - u_0^{q-1}$ define:

$$V_k(v) = A^{-1} \cdot \widetilde{V}_k(Av), \tag{2.8}$$

where $A$ is an orthogonal transformation composed with a dilatation such that $A(u_0^q - u_0^{q-1}) = e_n$. Obviously $\{V_k\}_{k=1}^n$ are smooth functions and they span the orthogonal complement of their argument vector. By (2.7), (2.8) and noticing that $A^{-1} = \| u_0^q - u_0^{q-1} \|^2 \cdot A^t$ we finally obtain:

$$\begin{aligned}[\mathrm{D}V_k(v)]^t \cdot v &= A^t \cdot \left[\mathrm{D}\widetilde{V}_k(v)\right]^t \cdot (A^t)^{-1} \cdot v = A^{-1} \cdot \left[\mathrm{D}\widetilde{V}_k(v)\right]^t \cdot Av \\ &= -A^{-1} \cdot \widetilde{V}_k(Av) = -V_k(v).\end{aligned}$$

■

**Proof of Lemma 2.3.** By Lemma 2.2, (2.1) and (2.4), we can calculate the derivative of $\Psi^q$ at each $(u^-, u^+) \in (\Psi^q)^{-1}(0)$:

$$\frac{\partial \Psi^q}{\partial u^-}(u^-, u^+) = -V(u^+ - u^-) \cdot \left[\mathrm{D}f(u^-) - \Lambda(u^-, u^+) \cdot \mathrm{Id}\right] \tag{2.9}$$

$$\frac{\partial \Psi^q}{\partial u^+}(u^-, u^+) = V(u^+ - u^-) \cdot \left[\mathrm{D}f(u^+) - \Lambda(u^-, u^+) \cdot \mathrm{Id}\right], \tag{2.10}$$

where $V$ is the $(n-1) \times n$ matrix, whose rows are the vectors $V_1, \ldots, V_{n-1}$. Note that (1.10) is equivalent to:

$$u_0^q - u_0^{q-1} \notin \mathrm{span}\left\{r_k(u_0^{q-1})\right\}_{k \neq i_q} \cup \ \mathrm{span}\left\{r_k(u_0^q)\right\}_{k \neq i_q}. \tag{2.11}$$

By (2.9) and (1.9) we have:

$$
\begin{aligned}
&\mathrm{Im}\ \frac{\partial \Psi^q}{\partial u^-}(u_0^{q-1}, u_0^q)\\
&\qquad = \mathrm{span}\left\{V(u_0^q - u_0^{q-1})\cdot\Big(\lambda_k(u_0^{q-1}) - \lambda_{i_q}(u_0^{q-1})\Big)\cdot r_k(u_0^{q-1})\right\}_{k=1}^n\\
&\qquad = \mathrm{span}\left\{V(u_0^q - u_0^{q-1})\cdot r_k(u_0^{q-1})\right\}_{k\neq i_q}.
\end{aligned}
\tag{2.12}
$$

Similarily, Im $\partial\Psi^q/\partial u^+(u_0^{q-1}, u_0^q)$ is spanned by the vectors $V(u_0^q - u_0^{q-1})\cdot r_k(u_0^q)$, for $k \neq i_q$. In view of (2.11) this implies that the rank of both derivatives is maximal, which yields (2.3). To show (2.2) note that it is equivalent to the linear independence of the vectors

$$\left\{V(u_0^q - u_0^{q-1})\cdot r_k(u_0^{q-1})\right\}_{k=1}^{i_q-1} \cup \left\{V(u_0^q - u_0^{q-1})\cdot r_k(u_0^q)\right\}_{k=i_q}^n, .$$

which is in turn equivalent to the Majda condition (1.8), as

$$\ker V(u^+ - u^-) = \mathrm{span}\ (u^+ - u^-).$$

Now we turn to proving (2.5) for a pair of states $(u^-, u^+) \in (\Psi^q)^{-1}(0)$. Since the $n-1$ vectors in (2.12) are linearly independent, by a continuity argument also the vectors:

$$\left\{V(u^+ - u^-)\cdot\left[\mathrm{D}f(u^+) - \Lambda(u^-, u^+)\cdot\mathrm{Id}\right]\cdot r_k(u^+)\right\}_{k\neq i_q}$$

are linearly independent. Thus the 1-dimensional space

$$\mathrm{Ker}\ \frac{\partial\Psi^q}{\partial u^+}(u^-, u^+) = \mathrm{Ker}\ \left\{V(u^+ - u^-)\cdot\left[\mathrm{D}f(u^+) - \Lambda(u^-, u^+)\cdot\mathrm{Id}\right]\right\} \tag{2.13}$$

is spanned by the vector

$$v = r_{i_q}(u^+) + \sum_{k\neq i_q}\alpha_k\cdot r_k(u^+). \tag{2.14}$$

On the other hand,

$$u^+ - u^- \notin \mathrm{Im}\ \left[\mathrm{D}f(u^+) - \Lambda(u^-, u^+)\cdot\mathrm{Id}\right]$$

as it is true for $(u^-, u^+) = (u_0^{q-1}, u_0^q)$ by (2.11). Hence $[\mathrm{D}f(u^+) - \Lambda(u^-, u^+)\cdot\mathrm{Id}]\cdot v = 0$ and so by (2.14) there must be $v = r_{i_q}(u^+)$ and $\Lambda(u^-, u^+) = \lambda_{i_q}(u^+)$.

Indeed, the space in (2.13) is tangent to the curve of states $w^+$ such that $\Psi^q(u^-, w^+) = 0$. This curve must be the $i_q$-th local rarefaction curve through $u^+$.

The other equality $\Lambda(u^-, u^+) = \lambda_{i_q}(u^-)$ is proven similarly, using (2.10). This establishes Lemma 2.3. ∎

**2.3. Rarefactions.** Let $q \in \mathcal{LR}$ so that the $i_q$-th field is genuinely nonlinear and (1.12) holds. The solvability of Riemann problems in $\Omega^{q-1}\times\Omega^q$ requires that the matrix $F_q(0, \Theta_q)$ is invertible [Le3]. This matrix is defined in the following way. Let the $n\times n$ transport matrix $\mathbf{T}_q(\theta)$ be the solution of the ODE:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}\theta}\mathbf{T}_q(\theta) = \mathrm{D}r_{i_q}(\mathcal{R}_{i_q}(\theta))\cdot\mathbf{T}_q(\theta), \quad \theta\in[0,\Theta_q],\\ \mathbf{T}_q(0) = \mathrm{Id}_n. \end{cases} \tag{2.15}$$

For $\theta, \tilde\theta \in [0, \Theta_q]$ with $\theta \le \tilde\theta$, let $F_q(\theta, \tilde\theta)$ be the $n \times n$ matrix whose columns $c_k^q(\theta, \tilde\theta) \in \mathbf{R}^n,\ k : 1 \dots n$ are given by:

$$\begin{aligned} c_k^q(\theta, \tilde\theta) &= \mathbf{T}_q(\tilde\theta) \cdot \mathbf{T}_q(\theta)^{-1} \cdot r_k(\mathcal{R}_{i_q}(\theta)) \quad \text{for } k : 1 \dots i_q - 1, \\ c_k^q(\theta, \tilde\theta) &= r_k(\mathcal{R}_{i_q}(\tilde\theta)) \quad \text{for } k : i_q \dots n. \end{aligned} \tag{2.16}$$

The constitutive function $\Psi^q$ can be defined as:

$$\Psi^q(u^-, u^+) = \left(\sigma_1 \dots \sigma_{i_q - 1}, \sigma_{i_q+1} \dots \sigma_n\right), \tag{2.17}$$

where $\{\sigma_k\}_{k=1}^n$ stand for the strengths of the waves in the solution of the Riemann problem $(u^-, u^+)$. It can be seen that (2.2) and (2.3) continue to hold [Le4].

In case of shocks and contacts the stability conditions (1.6) – (1.10) guarantee not only the solvability of the nearby Riemann problems but also the $BV$ and $L^1$ stability of these solutions. In case of a single rarefaction the situation is much different, as it takes time for the perturbation to pass through the rarefaction fan, which in turn yields continuous creation and anihilation of waves. The extra conditions guaranteeing the control of the amount of perturbation inside the rarefaction fan, measured in various norms, were discussed in [Le4]. They are strictly stronger than the invertibility of the matrix $F_q(0, \Theta_q)$ and they will be a part of our stability assumptions on the wave pattern in $u_0(t, x)$. The introduction of these assumptions will be carried out in section 3.

## 3. Stability conditions on wave patterns

**3.1. Riemann problems.** In the previous section we discussed conditions guaranteeing the solvability of Riemann problems whose data are close to the end states of a single large wave (shock, contact or a rarefaction). Assuming (1.6) – (1.12) is however not enough to ensure the existence of a self-similar solution to $(u^-, u^+) \in \Omega^0 \times \Omega^M$. Our first condition deals with this obstacle.

For each $q : 1 \dots M$ define a $(n-1) \times (n-1)$ matrix $C_q$ whose negative first $i_q - 1$ columns, and last $n - i_q$ columns are the vectors in (2.2). Notice that for $q \in \mathcal{LR}$ we have $C_q = \mathrm{Id}_{n-1}$ and thus $C_q$ is invertible for each $q$. Call

$$\begin{aligned} F_q^{left} &= -C_q^{-1} \cdot \frac{\partial \Psi^q}{\partial u^-}(u_0^{q-1}, u_0^q) \cdot \left[r_{i_q}(u_0^{q-1}) \dots r_n(u_0^{q-1})\right], \\ F_q^{right} &= C_q^{-1} \cdot \frac{\partial \Psi^q}{\partial u^+}(u_0^{q-1}, u_0^q) \cdot \left[r_1(u_0^q) \dots r_{i_q}(u_0^q)\right]. \end{aligned} \tag{3.1}$$

The $(n-1) \times i_q$ matrix $F_q^{right}$ expresses strengths of the weak outgoing waves in terms of strengths of waves perturbing the right state of the Riemann problem $(u_0^{q-1}, u_0^q)$. Similarily, the $(n-1) \times (n - i_q + 1)$ matrix $F_q^{left}$ corresponds to perturbations of $u_0^{q-1}$ (see Figure 3.1).

Define now the square $M \cdot (n-1)$ dimensional finiteness matrix $\mathbf{F}$:

$$\mathbf{F} = \begin{bmatrix} [0] & F_1^{right} & & & \\ F_2^{left} & [0] & F_2^{right} & & \\ & F_3^{left} & [0] & F_3^{right} & \\ & & \ddots & \ddots & \\ & & & F_M^{left} & [0] \end{bmatrix}, \tag{3.2}$$

where [0] stands for the $(n-1) \times (n-1)$ zero matrix. We have:

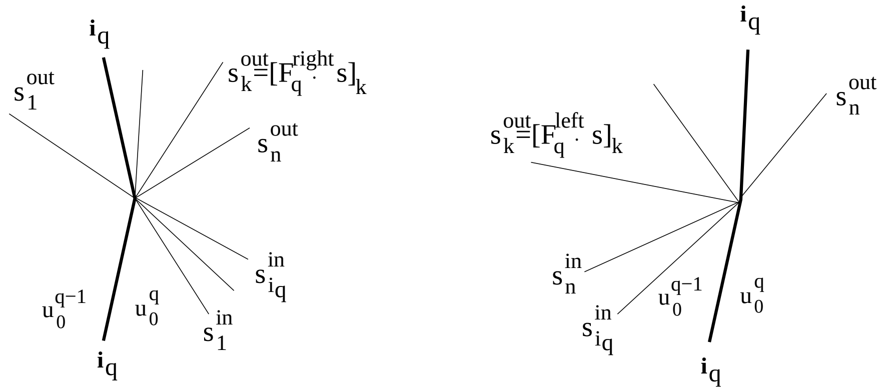


FIGURE 3.1

**Theorem 3.1.** [Le4] *In the above setting, let the following condition be satisfied:*

(F) FINITENESS CONDITION FOR THE WAVE PATTERN $u_0$:
$1$ *is not an eigenvalue of the matrix* $\mathbf{F}$.

*Then any Riemann problem* $(u^-, u^+) \in \Omega^0 \times \Omega^M$ *for (1.1) has a unique self-similar solution in* $\Omega$, *attaining* $n+1$ *states, consequtively connected by* $(n-M)$ *weak waves and* $M$ *strong waves (shocks, contacts or rarefactions) joining states in different sets* $\Omega^q$.

**3.2. BV stability.** Based on the analysis in [Le1, Le3] the following condition was formulated in [Le4]:

(BV) $BV$ STABILITY CONDITION FOR THE WAVE PATTERN $u_0$

There exist positive continuous functions $\{w_i\}_{i=1}^n$ defined on the set of states attained by the reference solution $u_0$ (that is, at the isolated endpoints of shocks and contacts, and along the rarefaction curves), such that for every $q : 1 \dots M$ the following holds.

$(i)$. If $q \in \mathcal{LS} \cup \mathcal{LC}$ then

$$\left|F_q^{left}\right|^t \cdot \begin{bmatrix} w_1(u_0^{q-1}) \\ \vdots \\ w_{i_q-1}(u_0^{q-1}) \\ w_{i_q+1}(u_0^q) \\ \vdots \\ w_n(u_0^q) \end{bmatrix} < \begin{bmatrix} w_{i_q}(u_0^{q-1}) \\ \vdots \\ w_n(u_0^{q-1}) \end{bmatrix} \tag{3.3}$$

$$\text{and} \quad \left|F_q^{right}\right|^t \cdot \begin{bmatrix} w_1(u_0^{q-1}) \\ \vdots \\ w_{i_q-1}(u_0^{q-1}) \\ w_{i_q+1}(u_0^q) \\ \vdots \\ w_n(u_0^q) \end{bmatrix} < \begin{bmatrix} w_1(u_0^q) \\ \vdots \\ w_{i_q}(u_0^q) \end{bmatrix}.$$

We use the following notation. The components of a matrix $|A|$ are absolute values of the components of $A$, the transposition is denoted by $^t$, and the vector inequalities are understood componentwise.

$(ii)$. If $q \in \mathcal{LR}$ then

$$\mathbf{P}_q(\theta) \cdot \begin{bmatrix} w_1(\theta) \\ \vdots \\ w_{i_q-1}(\theta) \\ w_{i_q+1}(\theta) \\ \vdots \\ w_n(\theta) \end{bmatrix} < \begin{bmatrix} w_1'(\theta) \\ \vdots \\ w_{i_q-1}'(\theta) \\ -w_{i_q+1}'(\theta) \\ \vdots \\ -w_n'(\theta) \end{bmatrix} \quad \text{for every } \theta \in [0, \Theta_q]. \tag{3.4}$$

The $(n-1) \times (n-1)$ production matrix $\mathbf{P}_q(\theta)$ is defined to be:

$$\begin{aligned} \mathbf{P}_q(\theta) &= [p_{ij}^q(\theta)]_{\substack{i,j:1\dots n,\\ i,j \neq i_q}}, \qquad \text{for } \ \theta \in [0, \Theta_q], \\ p_{ij}^q(\theta) &= \begin{cases} \big|\langle l_j, [r_i, r_{i_q}]\rangle(\mathcal{R}_{i_q}(\theta))\big| & \text{if } \ i \neq j, \\ \operatorname{sgn}(i_q - i) \cdot \langle l_i, [r_i, r_{i_q}]\rangle(\mathcal{R}_{i_q}(\theta)) & \text{if } \ i = j, \end{cases} \end{aligned} \tag{3.5}$$

where $[r_i, r_j] = \mathrm{D}r_i \cdot r_j - \mathrm{D}r_j \cdot r_i$ stands for the Lie bracket of the vector fields $r_i$ and $r_j$.

In short, condition (BV) claims the existence of a family of nonlinear weights in $\Omega_c$ (along which the strength of waves present in the solution of (1.1) (1.3) is measured) such that assigning to each perturbation wave the weight $w_i$ corresponding to its characteristic family and computed at the wave's left state, the weighted amount of perturbation decreases at each interaction with a large shock or contact as well as with a part of a large rarefaction. Recall that the strengths of waves are computed in terms of change in the eigenvalue for genuinely nonlinear fields, or as the arc-length of the rarefaction curve connecting the two states, for linearly degenerate fields.

**3.3. $\mathbf{L^1}$ stability.** For each $q \in \mathcal{LS} \cup \mathcal{LC}$ define the matrix $G_q^{right}$ with the elements $[G_q^{right}]_{ij}$, $i : 1 \dots i_q - 1, i_q + 1 \dots n,\ j : 1 \dots i_q$:

$$\begin{aligned} [G_q^{right}]_{ij} = [F_q^{right}]_{ij} &\cdot \begin{cases} |\lambda_i(u_0^{q-1}) - \Lambda^q| & \text{for } i < i_q \\ |\lambda_i(u_0^q) - \Lambda^q| & \text{for } i > i_q \end{cases} \\ &\cdot \begin{cases} 0 & \text{if } j = i_q \text{ and } q \in \mathcal{LC} \\ 1/|\lambda_j(u_0^q) - \Lambda^q| & \text{otherwise} \end{cases} \end{aligned}$$

and the matrix $G_q^{left}$ with the elements $[G_q^{left}]_{ij}$, $i : 1 \dots i_q{-}1, i_q{+}1 \dots n,\ j : i_q \dots n$:

$$\begin{aligned} [G_q^{left}]_{ij} = [F_q^{left}]_{ij} &\cdot \begin{cases} |\lambda_i(u_0^{q-1}) - \Lambda^q| & \text{for } i < i_q \\ |\lambda_i(u_0^q) - \Lambda^q| & \text{for } i > i_q \end{cases} \\ &\cdot \begin{cases} 0 & \text{if } j = i_q \text{ and } q \in \mathcal{LC} \\ 1/|\lambda_j(u_0^{q-1}) - \Lambda^q| & \text{otherwise} \end{cases} \end{aligned}$$

The matrices $G_q$ express the instanteneous change of the $L^1$ norm (strength $\times$ shift) of the perturbation while it crosses the large wave $(u_0^{q-1}, u_0^q)$; $G_q^{right}$ accounts for the interaction with slower families $i \leq i_q$ and $G_q^{left}$ with faster families.

Further, for each $q \in \mathcal{LR}$ define the corresponding mass production matrix function:

$$(3.6)\qquad \begin{aligned} \mathbf{M}_q(\theta) &= [m_{ij}^q(\theta)]_{\substack{i,j:1\dots n,\\ i,j\neq i_q}} \quad \text{for} \ \ \theta\in[0,\Theta_q],\\ m_{ij}^q(\theta) &= \begin{cases} |\langle l_j, \mathrm{D}r_i\cdot r_{i_q}\rangle(\mathcal{R}_{i_q}(\theta))| & \text{if} \ \ i\neq j,\\ \mathrm{sgn}\ (i_q - i)\cdot\langle l_i, \mathrm{D}r_i\cdot r_{i_q}\rangle(\mathcal{R}_{i_q}(\theta)) & \text{if} \ \ i=j. \end{cases} \end{aligned}$$

The following condition is a generalisation of the $L^1$ stability conditions from [BM, Le1, Le3].

(L1) $L^1$ Stability Condition for the wave pattern $u_0$

There exist positive continuous weights $\{w_i(u)\}_{i=1}^n$ defined on the set of states $u$ attained by the reference solution $u_0$ such that the following holds.

($i$). For every $q\in\mathcal{LS}\cup\mathcal{LC}$ (3.3) holds with $G_q^{left}$, $G_q^{right}$ replacing $F_q^{left}$, $F_q^{right}$ respectively.

($ii$). For every $q\in\mathcal{LR}$ (3.4) is satisfied with $\mathbf{M}_q(\theta)$ replacing the matrix $\mathbf{P}_q(\theta)$.

**3.4. Some remarks.** First, we note the following implications:

**Theorem 3.2.** *In the above setting, the condition (F) is weaker than (BV), which is in turn implied by (L1). In particular, both (BV) and (L1) imply the existence of a self-similar solution to any Riemann problem $(u^-,u^+)\in\Omega_c\times\Omega_c$ such that $||u^- - u_0(1,x_1)|| + ||u^+ - u_0(1,x_2)|| < \delta$ with $x_2 - x_1 < \delta$, for some small $\delta > 0$.*

*Proof.* The implication (BV) $\Longrightarrow$ (L1) was proved in [Le4] for rarefactions and in [Le1] for patterns with large shock waves. In view of these results their generalisation in Theorem 3.2 is straightforward, as well as the implication (L1) $\Longrightarrow$ (BV). ■

Both implications in Theorem 3.2 are strict (see examples in [Le3]). Also, note that once the weights $\{w_i\}$ in (BV) or (L1) are specified, then by restricting their domain to the set of states attained by $u_0(1,\cdot)$ on a bounded space interval, we obtain the stability of any subpattern of $u_0$ composed of a number of consecutive large waves. In particular, for a single rarefaction wave we receive the BV stability condition from [Le3] which implies the invertibility of every $F_q(\theta,\tilde{\theta})$ defined in (2.16).

For some special patterns the stability conditions can be rephrased in more convenient terms:

**Theorem 3.3.** (i) [Le2] *For a pattern containing only large shocks and contacts, the condition (BV) is equivalent to:*

$$\textit{spectral radius of } |\mathbf{F}| < 1,$$

*where the components of the matrix $|\mathbf{F}|$ are absolute values of the components of the finiteness matrix $\mathbf{F}$ in (3.2). The condition (L1) is equivalent to:*

$$\textit{spectral radius of } |\mathbf{G}| < 1,$$

*where $\mathbf{G}$ is defined as $\mathbf{F}$ in (3.2) but with $G_q^{right}$, $G_q^{left}$ replacing $F_q^{right}$, $F_q^{left}$ respectively.*

(ii) [Le4] *For a pattern containing a single rarefaction, conditions (BV) (ii) and (L1)(ii) imply, in particular, the following. Every extreme field ($i_q = 1$ or $n$) rarefaction, every weak rarefaction ($\Theta_q << 1$), or any rarefaction when (1.1) has a system of Riemann invariants, is both BV and $L^1$ stable. When $n = 3, M = 1$ and $i_1 = 2$ then (BV) is equivalent to the existence of a positive solution $v : [0, \Theta_q] \longrightarrow \mathbf{R}_+$ of the equation:*

$$v' = p^1_{31} \cdot v^2 + (p^1_{11} + p^1_{33}) \cdot v + p^1_{13}.$$

*A sufficient condition that the above holds is:*

$$\int_0^{\Theta_q} \int_0^{\theta} p^1_{31}(\theta) \cdot e^{\int_s^\theta p^1_{11} + p^1_{33}} \cdot p^1_{13}(s) \; \mathrm{d}s\mathrm{d}\theta < 1.$$

*The same results are valid for the $L^1$ stability, with $m^1_{ij}$ replacing $p^1_{ij}$.*

We will make use of Theorem 3.3 in section 7, where we validate our conditions in the setting of gas dynamics.

## 4. Proof of Theorem 1.1 - construction of the Glimm functional

Recall that given a Cauchy problem (1.1) (1.3) with $\bar{u}$ having small total variation, its solution can be obtained as the limit of piecewise constant $\epsilon$-approximations $u^\epsilon(t, x)$ constructed via the wave front tracking algorithm [BaJ, HR]. For the detailed description of the algorithm we refer to [B]. The crucial ingredient in proving the global existence of the approximate solutions and the compactness of its sequence is the construction of the suitable Glimm functional to control the total variation of perturbation and the amount of the future interactions. Below we briefly discuss a modification of this standard construction, applicable when the reference pattern $u_0$ is a collection of large noninteracting waves rather than a constant state. We then show that our Glimm-type functional $\Gamma$ is indeed nonincreasing along any wave front tracking approximate solution, thanks to the stability condition (BV).

Define:

$$\begin{aligned} \mathcal{I} = \{q \in \mathcal{LR}\} \cup \{q;\; q \text{ and } q+1 \in \mathcal{LS} \cup \mathcal{LC}\} \\ \cup \{0;\;\; \text{if } 1 \in \mathcal{LS} \cup \mathcal{LC}\} \cup \{M\}. \end{aligned}$$

**Definition 4.1.** *Let $\epsilon_0 > 0$. By $\mathcal{D}_{\epsilon_0}$ we denote the set of piecewise constant functions $v : \mathbf{R} \longrightarrow \mathbf{R}^n$ such that:*

(i) $v(-\infty) = u^0_0,\ v(+\infty) = u^M_0$.

(ii) $v(x) \in \Omega$ *for all* $x \in \mathbf{R}$.

(iii) *All jumps in $v$ either have amplitudes smaller than $\epsilon_0$ or their left and right states belong to some $\Omega^{q-1}$ and $\Omega^q$ respectively, with $q \in \mathcal{LS} \cup \mathcal{LC}$. Thus the corresponding Riemann problems admit the standard self-similar solution. We order the waves in these solutions according to their location and speed; for a wave $\alpha$ by $i_\alpha : 1 \dots n$ we denote its characteristic family, by $\epsilon_\alpha$ its strength and by $x_\alpha$ its location. The strength of any large shock or contact is set to 1.*

(iv) *For each $q \in \mathcal{LR}$ with $q + 1 \in \mathcal{LR}$ there exists a point $x_q \in \mathbf{R}$ such that the following holds. For $q \in \mathcal{LS} \cup \mathcal{LC}$ call $x_q$ the location of the large $i_q$-shock* *or contact. Call $x_0 = -\infty$, $x_{M+1} = +\infty$. For each $q \in \mathcal{I}$ define an open interval $I_q$ as follows. The left endpoint of $I_q$ equals to:*

$$\begin{cases} x_{q-1} & \text{if } q \in \mathcal{LR}, \\ x_q & \text{otherwise.} \end{cases}$$

*The right point of $I_q$ is:*

$$\begin{cases} x_q & \text{if } q, q+1 \in \mathcal{LR}, \\ x_{q+1} & \text{otherwise.} \end{cases}$$

*We see that the intervals $\{I_q\}_{q\in\mathcal{I}}$ partition $\mathbf{R}$ (as in Figure 4.1). Then calling $\epsilon_\alpha^+ = \max(0, \epsilon_\alpha)$, $\epsilon_\alpha^- = \max(0, -\epsilon_\alpha)$, we have:*

$$\begin{aligned} &\sum_{q\in\mathcal{LR}} \sum_{\alpha; x_\alpha \in I_q} \left\{ \left| \left( \sum_{i_\alpha = i_q} \epsilon_\alpha^+ \right) - \Theta_q \right| + \left( \sum_{i_\alpha = i_q} \epsilon_\alpha^- \right) + \left( \sum_{i_\alpha \neq i_q} |\epsilon_\alpha| \right) \right\} \\ &+ \sum_{q \in \mathcal{I}\setminus\mathcal{LR}} \sum_{\alpha; x_\alpha \in I_q} |\epsilon_\alpha| + \sum_{q\in\mathcal{LS}\cup\mathcal{LC}} |u_0^q - v(x_q+)| \leq \epsilon_0. \end{aligned} \tag{4.1}$$

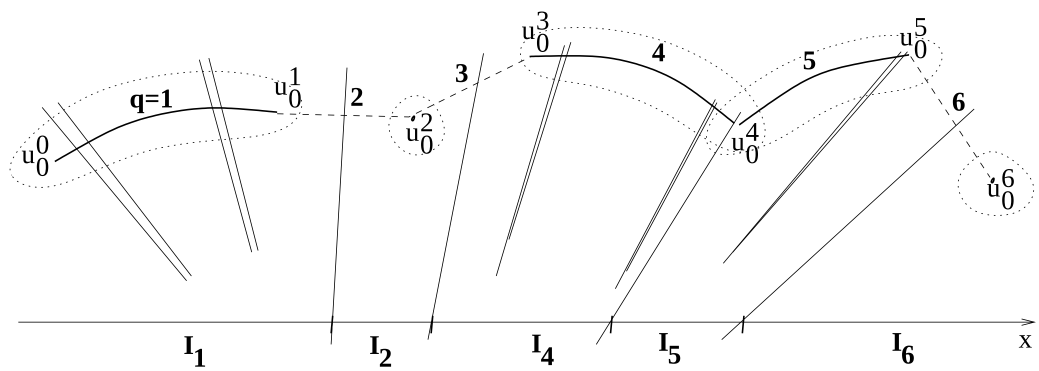


FIGURE 4.1

The intervals $\{I_q\}_{q\in\mathcal{I}}$ are essentially of two types. Namely, either $q \in \mathcal{LR}$ and then for each $x \in I_q$, $v(x)$ belongs to a small neighbourhood of the rarefaction curve $\mathcal{R}_{i_q}$ or $q \notin \mathcal{LR}$ and then $v(x)$ stays close to $u_0^q$ for each $x \in I_q$.

Take a function $u(0,\cdot) \in \mathcal{D}_{\epsilon_0}$ for some small $\epsilon_0 > 0$ and let $\epsilon << \epsilon_0$. Recall that the fundamental block for constructing the approximate solution $u^\epsilon(t,x)$ is provided by piecewise constant approximations of self-similar solutions to Riemann problems. As customary, the non-physical waves generated by the Simplified Riemann Solver are said to belong to $(n+1)$-th characteristic family. The Simplified Riemann Solver is used whenever one of the interacting waves is non-physical or when the product of strenghts of incoming waves is bigger than a treshold parameter $\rho(\epsilon)$. The details can be found in [B], chapter 7. The associated non-physical weight $w_{n+1}$ is defined as a continuous function on the set of states attained by $u_0$ and such that for each $q \in \mathcal{LR}$ one has:

$$w_{n+1}(\theta) = c_q \cdot \exp(-C_q \cdot \lambda_{i_q}(u)) \quad \text{when} \quad u = \mathcal{R}_{i_q}(\theta),\ \theta \in [0, \Theta_q], \tag{4.2}$$

for sufficiently large $C_q > 0$ and a small $c_q > 0$. We also require that $w_{n+1}$ decreases across each shock or contact by the factor $C_q$.

For each $t$ we define the partition $\{I_q(t)\}_{q\in\mathcal{I}}$ as in Definition 4.1, setting:

$$\Lambda_q = (\lambda_{i_q}(u_0^q) + \lambda_{i_q+1}(u_0^q))/2 \quad \text{and} \quad x_q(t) = x_q + t\cdot\Lambda_q \tag{4.3}$$

whenever $q, q+1 \in \mathcal{LR}$ and $x_q(t)$ to be the location of the large $i_q$-shock or contact at time $t$ whenever $q \in \mathcal{LS}\cup\mathcal{LC}$. Notice that the speeds of $\{x_q(t)\}$ are strictly increasing. Using the notation of Definition 4.1 we set:

$$\begin{aligned} V(t) = \sum_{q\in\mathcal{LR}}\ \sum_{\alpha;x_\alpha\in I_q(t)} &\left\{ \left| \left(\sum_{i_\alpha=i_q}\epsilon_\alpha^+\right) - \Theta_q \right| + \left(\sum_{i_\alpha=i_q}\epsilon_\alpha^-\right) + \left(\sum_{i_\alpha\neq i_q}|\epsilon_\alpha|\right)\right\} \\ &+ \sum_{q\in\mathcal{I}\setminus\mathcal{LR}}\ \sum_{\alpha;x_\alpha\in I_q(t)}|\epsilon_\alpha| + \sum_{q\in\mathcal{LS}\cup\mathcal{LC}}|u_0^q - u^\epsilon(t,x_q(t)+)|. \end{aligned}$$

Let $\mathcal{A}_q$, $q\in\mathcal{I}$ be the sets containing all couples of perturbation waves $(\alpha,\beta)$ in $u^\epsilon(t,\cdot)$ approaching each other. More precisely, assuming $x_\alpha < x_\beta,\ x_\alpha, x_\beta \in I_q(t)$, we have $(\alpha,\beta)\in\mathcal{A}_q(t)$ iff $i_\alpha > i_\beta$ or else $i_\alpha = i_\beta$ and at least one of the waves is a genuinely nonlinear shock. In both cases we require that none of the waves $\alpha,\beta$ is a positive $i_q$-wave when $q\in\mathcal{LR}$. Define:

$$Q_0(t) = \sum_{q\in\mathcal{I}}\ \sum_{(\alpha,\beta)\in\mathcal{A}_q(t)} |\epsilon_\alpha\cdot\epsilon_\beta|.$$

Further, let

$$\begin{aligned} Q_{large}(t) = \sum_{q\in\mathcal{LR}}\ \sum_{\alpha;x_\alpha\in I_q(t)} &\left\{ \left(\sum_{i_\alpha\neq i_q}\tilde{w}^q_{i_\alpha}(u^\epsilon(t,x_\alpha-))\cdot|\epsilon_\alpha|\right) + \left(\sum_{i_\alpha=i_q}\tilde{w}^q_{i_q}\epsilon_\alpha^-\right)\right\} \\ &+ \sum_{q\in\mathcal{I}\setminus\mathcal{LR}}\ \sum_{\alpha;x_\alpha\in I_q(t)} \tilde{w}^q_{i_\alpha}|\epsilon_\alpha| \end{aligned}$$

For each $q\in\mathcal{I}$ the functions $\{\tilde{w}_i^q\}_{i=1}^{n+1}$ are smoothly defined in an open neighbourhood of the rarefaction curve $\mathcal{R}_{i_q}(\theta),\ \theta\in[0,\Theta_q]$ if $q\in\mathcal{LR}$, or in $\Omega^q$ if $q\in\mathcal{I}\setminus\mathcal{LR}$. Namely, with the weights $\{w_i\}_{i=1}^n$ satisfying the BV stability condition (BV) and $w_{n+1}$ defined as in (4.2), we first modify:

$$w_{i_q} = \tilde{c} \quad \text{along } \mathcal{R}_{i_q}, \tag{4.4}$$

for some sufficiently small constant $\tilde{c}$. Then let

$$w_i^q(u) = \begin{cases} w_i\left(\mathcal{R}_{i_q}(\lambda_{i_q}(u) - \lambda_{i_q}(u_0^{q-1}))\right) & \text{if } q\in\mathcal{LR}, \\ w_i(u_0^q) & \text{if } q\in\mathcal{I}\setminus\mathcal{LR}. \end{cases} \tag{4.5}$$

Finally we define:

$$\tilde{w}_i^q = w_i^q\cdot\Big(1 + c\cdot(\sharp\{i_m\geq i\} - \sharp\{i_m < i\})\Big) \tag{4.6}$$

where we consider all indices $m\in\mathcal{I}$ such that $x_m$ is a separator between some two adjacent rarefactions, located to the left of the open interval $I_q$. Note that if $c$ in (4.6) is small enough (also with respect to $\tilde{c}$ in (4.4)) then the strict inequalities in (3.3) and (3.4) are preserved, as the $\mathcal{C}^1$ norm of the difference $w_i - \tilde{w}_i^q$ is of the order $\mathcal{O}(1)c$ along $\mathcal{R}_{i_q}$. Thus (BV) continues to hold with the modified weights $\{\tilde{w}_i^q\}$. On the other hand thanks to (4.6) the weight assigned to a wave will decrease each time the wave crosses the separator $x_q$ between two large rarefactions.

Define now the Glimm potentials:

$$Q(t) = Q_0(t) + Q_{large}(t), \qquad \Gamma(t) = V(t) + \kappa \cdot Q(t), \tag{4.7}$$

where $\kappa$ is a sufficiently large constant, to be determined later.

**Lemma 4.2.** *Assume that the condition (BV) holds. Then for some constants $c, \epsilon_0, \kappa > 0$ we have the following. Let $u(0,\cdot) \in \mathcal{D}_{\epsilon_0}$ and let $u^\epsilon$ be the corresponding piecewise constant approximate solution obtained through the wave front tracking algorithm. Then for any $t > 0$ when two wave fronts $\alpha$ and $\beta$ interact, if $\Gamma(u^\epsilon(t-,\cdot)) \leq \epsilon_0$ we have:*

$$\begin{aligned} \Delta Q &= Q(u^\epsilon(t+,\cdot)) - Q(u^\epsilon(t-,\cdot)) \leq -c \cdot |\epsilon_\alpha \epsilon_\beta|, \\ \Delta\Gamma &= \Gamma(u^\epsilon(t+,\cdot)) - \Gamma(u^\epsilon(t-,\cdot)) \leq -c \cdot |\epsilon_\alpha \epsilon_\beta|. \end{aligned} \tag{4.8}$$

*Further, $\Gamma$ is a nonincreasing function of $t$.*

*Proof.* It is clear that $\Delta V \leq \mathcal{O}(1)|\epsilon_\alpha \epsilon_\beta|$. By a standard argument, whenever a couple of fronts belonging to some $\mathcal{A}_q$ interact, we have $\Delta Q_0 \leq -c \cdot |\epsilon_\alpha \epsilon_\beta|$. On the other hand, (BV) guarantees that $\Delta Q_{large} \leq -c \cdot |\epsilon_\alpha \epsilon_\beta|$ when the interaction involves a part of a large rarefaction (see [Le3]) or a large shock/contact discontinuity. These prove (4.8). Further, when a wave crosses $x_q(t)$ (that is the boundary between two rarefactions $i_q$ and $i_{q+1}$) then thanks to (4.6), $Q_{large}$ decreases by the order of the wave strength. This yields the decrease of $\Gamma$, if only we choose $\epsilon_0$ and $1/\kappa$ to be small enough. ∎

We consequently obtain:

**Lemma 4.3.** *Let $u^\epsilon(0,\cdot) \in \mathcal{D}_{\epsilon_0}$. In the setting of Lemma 4.2, an $\epsilon$-approximate solution $u^\epsilon(t,x)$ generated by the algorithm exists for all times $t > 0$ and enjoys the following properties:*

- (i) *$u^\epsilon$ is piecewise constant, with jumps occuring along finitely many lines; jumps are of four types: shocks (and contact discontinuities), rarefaction fronts, non-physical waves and large shocks/contact discontinuities; all jumps of the first three types have strength $< \epsilon_0$, all rarefactions have strength $< \epsilon$,*
- (ii) *the speeds of all $i_q$-waves contained in the interval $I_q(t)$ for $q \in \mathcal{LR}$ are exact; the same is true for the large shocks and contacts; all non-physical waves travel with speed $\hat{\lambda}$; the speed of all other waves differ from the exact speed (Rankine - Hugoniot speed for shocks and the eigenvalue at the left state for rarefaction fronts) at most by $\epsilon$;*
- (iii) *at each time $t \geq 0$ the sum of strengths of non-physical waves in $u^\epsilon$ is bounded by $\epsilon$,*
- (iv) *for all $t \geq 0$ we have: $\Gamma(u^\epsilon(t,\cdot)) \leq \epsilon_0$ and consequently $u^\epsilon(t,\cdot) \in \mathcal{D}_{\epsilon_0}$.*

For the proof of Theorem 1.1 we may without loss of generality assume that our initial data $\bar{u}$ contained in the set (1.13) satisfy additionaly $\bar{u}(-\infty) = u_0^0$, $\bar{u}(\infty) = u_0^M$. Indeed by Theorem 3.1 the self-similar solution to any nearby Riemann problem exists and satisfies (BV). Noticing that $\bar{u}$ belongs then to the $L^1_{loc}$ closure of $\mathcal{D}_{\epsilon_0}$ if only the parameter $\delta$ is small enough with respect to $\epsilon_0$ (the proof is analogous to that of Lemma 4.6 in [Le3]), Theorem 1.1 follows along a standard line as in [B].

**4.1. First order rarefactions.** In the remaining part of this section we remark some properties of the first order rarefaction waves that will be of use in the subsequent analysis.

A positive $i_q$-wave located at $y_0 \in I_q(T),\ q \in \mathcal{LR},\ T > 0$ is called a first order $i_q$-rarefaction wave if there exists a continuous curve $y(t)$ with $y(T) = y_0$, such that for almost all $t \in [0,T]$ we have $y(t) \in I_q(t)$ is the location of a positive $i_q$-wave. For each $t \in [0, +\infty)$ call $L_q(t) \subset I_q(t)$ the set of locations of first order $i_q$-rarefaction waves.

**Lemma 4.4.** *Let $u^\epsilon(t,x)$ be as in Lemma 4.3 (in particular $u^\epsilon(t,\cdot) \in \mathcal{D}_{\epsilon_0}$ for all $t \geq 0$). Then:*

$$\begin{aligned} \tilde{V}(t) := \sum_{q\in\mathcal{LR}} \left| \left( \sum_{x_\alpha \in L_q(t)} \epsilon_\alpha \right) - \Theta_q \right| + \left( \sum_{x_\alpha \notin \bigcup_{q\in\mathcal{LR}} L_q(t)} |\epsilon_\alpha| \right) \\ + \sum_{q\in\mathcal{LS}\cup\mathcal{LC}} |u_0^q - u^\epsilon(t, x_q(t)+)| \\ \leq \mathcal{O}(1)\cdot\epsilon_0. \end{aligned} \tag{4.9}$$

*Moreover if $y(t)$ is continuous and $y(t) \in L_q(t)$ for almost all $t \in [0,T]$ and some $q \in \mathcal{LR}$ then:*

$$\forall t, s \in [0,T] \quad |\lambda_{i_q}(u^\epsilon(t, y(t)-)) - \lambda_{i_q}(u^\epsilon(s, y(s)-))| = \mathcal{O}(1)\cdot\epsilon_0. \tag{4.10}$$

*Proof.* To prove (4.9) one modifies the interaction potentials, defining them as $Q_0$ and $Q_{large}$ but treating $i_q$-positive waves in $I_q(t) \setminus L_q(t)$ as perturbations. Then Lemma 4.2 and its proof are still valid, with $V$ exchanged there to $\tilde{V}$, and thus the estimate in (4.9) follows.

In order to deduce (4.10) we may restrict our attention to the case $t = T$ and $s = 0$. It is convenient to consider the evolution of the related functional:

$$\tilde{\Gamma}(t) = |y'(t) - y'(0)| + \kappa \cdot \tilde{Q}_{large}(t) + \kappa^2 \cdot Q(t),$$

where $\tilde{Q}_{large}(t)$ is defined as $Q_{large}(t)$ but takes into account only perturbation waves $\alpha$ in:

$$\{x_\alpha < y(t) \text{ and } i_\alpha \geq i_q\} \cup \{x_\alpha > y(t) \text{ and } i_\alpha \leq i_q\}$$

and $\kappa > 1$ is a large constant. We see that when $y(t)$ interacts with another wave $\alpha$ then $\Delta Q \leq 0$, $\Delta y' = \mathcal{O}(1)|\epsilon_\alpha|$ and $\Delta\tilde{Q}_{large} \leq -c\cdot|\epsilon_\alpha|$. On the other hand at any other time $\Delta y' = 0$ and $\Delta(\tilde{Q}_{large} + \kappa Q) \leq 0$. Thus $\tilde{\Gamma}$ is a nonincreasing function of $t$ if only $\kappa$ is large. hence $|y'(T) - y'(0)| \leq \tilde{\Gamma}(0) = \mathcal{O}(1)\epsilon_0$ and (4.10) follows since

$$y'(t) = \lambda_{i_q}(u^\epsilon(t, y(t)-))$$

for almost all $t \in [0,T]$. ∎

## 5. Proof of Theorem 1.2 - construction of the Lyapunov functional

Let $u$ and $v$ be two piecewise constant $\epsilon$-approximate solutions of (1.1), as in Lemma 4.3. Recall that by our construction in section 4, for every $t \geq 0$, $u$ and $v$ yield two different partitions of $\mathbf{R}$ into intervals $\{I_k^u(t)\}_{k\in\mathcal{I}}$ and $\{I_k^v(t)\}_{k\in\mathcal{I}}$. The endpoints of these intervals are given by positions of the large discontinuities in $u$ (or in $v$) and by separators $x_k^u(t)$ (respectively $x_k^v(t)$) between $i_k$ and $i_{k+1}$-th field large rarefactions. The speed of the separator $x_k$ is strictly larger than the $i_k$-th and strictly smaller than the $(i_k+1)$-th eigenvalue at $u_0^k$, by (4.3).

**5.1. Case 1: the profiles u and v are apart from each other.** Let

$$T_1 = \sup\{t;\ \exists k \in \mathcal{I} \quad I_k^u(t) \cap I_k^v(t) = \emptyset\}, \tag{5.1}$$

$$T_2 = \sup\Big\{t > T_1;\ \exists k \in \mathcal{LR} \quad \exists x \in I_k^u(t) \cap I_k^v(t) \quad |\lambda_{i_k}(u(t,x)) - \lambda_{i_k}(v(t,x))| > \nu/2\Big\}, \tag{5.2}$$

where $\nu$ is a positive and small constant to be determined later.

Notice that if $I_k^u(t) \cap I_k^v(t) \neq \emptyset$ for some $t \geq 0$ then the same is true for every $t' \geq t$. Thus, if $T_1 > 0$ then in particular there must be:

$$\exists k \in \mathcal{I} \quad \forall t \in [0, T_1) \quad I_k^u(t) \cap I_k^v(t) = \emptyset. \tag{5.3}$$

In case $T_2 = T_1$, call $I(t)$ the unique bounded connected component of $\mathbf{R} \setminus (I_k^u(t) \cup I_k^v(t))$, for every $t < T_1$.

Now assume that $T_2 > T_1$. Then for some $k \in \mathcal{LR}$ and $x \in I_k^u(t) \cap I_k^v(t)$ the inequality in (5.2) holds; to fix the ideas assume that $\lambda_{i_k}(u(t,x)) > \lambda_{i_k}(v(t,x))$. By (4.9) there exists then a nonempty interval $I(T_2) = [z_0^-, z_0^+] \subset I_k^u(T_2) \cap I_k^v(T_2)$ such that $z_0^- \in L_k^u(T_2),\ z_0^+ \in L_k^v(T_2)$ and:

$$\forall x, y \in I(T_2) \quad \lambda_{i_k}(u(T_2,x)) - \lambda_{i_k}(v(T_2,y)) > \nu/3. \tag{5.4}$$

For $t < T_2$ call $I(t)$ the space interval whose boundary is continuous polygonals $z^-(t) \in L_k^u(t),\ z^+(t) \in L_k^v(t)$ with $z^-(T_2) = z_0^-$ and $z^+(T_2) = z_0^+$. Notice that taking $\epsilon_0$ small enough by Lemma 4.4 we have:

$$\forall t \in [0, T_2) \quad \forall x, y \in I(t) \quad |\lambda_{i_k}(u(t,x)) - \lambda_{i_k}(v(t,y))| > \nu/4. \tag{5.5}$$

For all $t \in [0, T_2)$ the Lyapunov functional $\Phi$ is defined by the formula:

$$\Phi(u,v)(t) = ||u(t) - v(t)||_{L^1} + \kappa_1 \cdot |I(t)|, \tag{5.6}$$

where $|I(t)|$ stands for the length of the interval $I(t)$ and $\kappa_1$ is a sufficiently large integer constant.

**Lemma 5.1.** *If only $\kappa_1$ is large enough then the functional $\Phi$ satisfies:*

$$\Phi(u(t',\cdot), v(t',\cdot)) \leq \Phi(u(t,\cdot), v(t,\cdot)) \tag{5.7}$$

$$||u(t,\cdot) - v(t,\cdot)||_{L^1} \leq \Phi(u(t,\cdot), v(t,\cdot)) \leq C \cdot ||u(t,\cdot) - v(t,\cdot)||_{L^1}, \tag{5.8}$$

*for all $0 \leq t \leq t' < T_2$ and a uniform constant $C > 0$.*

*Proof.* The equivalence (5.8) of $\Phi$ with the $L^1$ distance follows in view of (5.5) if $T_2 > T_1$ and noticing $|u(t,x) - v(t,x)| \geq c > 0$ for $x \in I(t),\ t \in [0, T_1)$ if $T_2 = T_1$.

For each time denote $\mathcal{J}(u)$ and $\mathcal{J}(v)$ the sets of all waves and large rarefactions' separators in $u$ and in $v$, respectively. To prove (5.7) fix $t \in [0, T_2)$ which is not a time of interaction or intersection of any couple of fronts (and separators) in $\mathcal{J}(u) \cup \mathcal{J}(v)$. We calculate:

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\Phi(u,v)(t) = \\ &\quad \sum_{\alpha \in \mathcal{J}(u) \cup \mathcal{J}(v)} \Big| |u(x_\alpha+, t) - v(x_\alpha+, t)| - |u(x_\alpha-, t) - v(x_\alpha-, t)| \Big| \cdot \dot{x}_\alpha \\ &\quad + \kappa_1 \cdot \frac{\mathrm{d}}{\mathrm{d}t}|I(t)|. \end{aligned} \tag{5.9}$$

The first term in (5.9) is of the order of $\mathcal{O}(1)$ because of the finite speed of propagation, boundedness of $TV(u(t))$ and $TV(v(t))$, and:

$$|u(x_\alpha+,t) - v(x_\alpha+,t)| - |u(x_\alpha-,t) - v(x_\alpha-,t)| = \mathcal{O}(1)|\epsilon_\alpha|.$$

On the other hand we have $\mathrm{d}/\mathrm{d}t\ |I(t)| \leq -c$ for $t \in [0, T_1)$ and $\mathrm{d}/\mathrm{d}t\ |I(t)| \leq -\nu/5$ for $t \in [0, T_2)$ (in view of (5.5)) if $T_2 > T_1$. Thus if $\kappa_1$ is large with respect to the system constants and the prechosen $\nu$, we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi(u,v)(t) \leq 0.$$

Integrating in time we conclude (5.7). ■

**5.2. Case 2: u and v close.** For $t \geq T_2$ the Lyapunov functional $\Phi$ is defined in a more complicated way:

$$\Phi(u,v) = \sum_{i=1}^{n} \int_{-\infty}^{+\infty} W_i(x) \cdot w_i(x) \cdot |q_i(x)|\ \mathrm{d}x. \tag{5.10}$$

The scalar quantities $q_i(x)$ are roughly speaking the curvlinear coordinates of the vector $v(t,x) - u(t,x)$ computed along appropriate combinations of wave curves in $\Omega$. The weights $w_i(x)$ will be defined using the $L^1$ stability condition (L1). The functional weights $W_i(x)$ are defined essentially as in [BLY]; they contain a term accounting for the amount of waves in $u(t)$ and $v(t)$ approaching the fictitious wave $q_i(x)$. The presence of the extra terms is connected with the large waves; for rarefactions we employ the ideas from [Le3], for shocks and contacts we refer to [Le1].

Fix $t \geq T_2$ that is not an interaction time of fronts (or separator between two large rarefactions) in $u$ or in $v$. Note that since $t \geq T_1$, by (5.1) there will be either $k = s$ or $s$ is the immediate successor of $k$ in $\mathcal{I}$. Also, since $t \geq T_2$ we have:

$$|\lambda_{i_m}(u(x)) - \lambda_{i_m}(v(x))| \leq \nu/2 \quad \text{ for } m \in \mathcal{LR} \cap \{k,s\}.$$

Let $x \in I_k^u(t) \cap I_s^v(t)$ with $k \leq s$. The decomposition $\{q_i(x)\}_{i=1}^n$ is implicitly given by:

$$v(x) = Z_n(q_n(x)) \circ Z_{n-1}(q_{n-1}(x)) \circ \ldots \circ Z_1(u(x), q_1(x)). \tag{5.11}$$

Set

$$u_i = Z_i(q_i) \circ \ldots \circ Z_1(u(x), q_1). \tag{5.12}$$

If $k < s$ and $s \in \mathcal{LS} \cup \mathcal{LC}$ we have $\Psi^s(u_{i_s-1}, u_{i_s}) = 0$ where $\Psi^s$ is the constitutive function from section 2. The strength $q_{i_s}(x)$ of the large shock/contact is then set to 1. Otherwise $Z_i = \mathcal{S}_i(\cdot, q)$ is the local $i$-shock curve, parametrised by $\lambda_i$ if the $i$-th field is genuinely nonlinear or by the arc-length if it is linearly degenerate.

If $x \in I_k^u \cap I_s^v$ with $s < k$, the quantities $\{q_i(x)\}$ are defined by means of (5.11) with $u(x)$ exchanged with $v(x)$.

**5.3. Definition of weights $\mathbf{w_i(x)}$.** First, let $w_i^q$ be the modification of the weights $w_i$ satisfying the $L^1$ stability condition (L1), given through the formulas (4.4) and (4.5). Given the decomposition $\{q_i(x)\}_{i=1}^n$ at $x \in I_k^u \cap I_s^v$ (to fix the ideas we assume that $k \leq s$ so that (5.11) holds), we now assign the weights $\{w_i(x)\}_{i=1}^n$.

If $k = s$ then we set $\hat{w}_i(x) = w_i^k(u_{i-1})$ for all $i : 1 \ldots n$, where $u_i$ are given by (5.12). Otherwise $s$ is the immediate successor of $k$ in $\mathcal{I}$. Set $\hat{w}_i(x) = w_i^k(u_{i-1})$ for

$i : 1 \dots i_{k+1} - 1$. The weight associated to any large shock/contact is set to $\tilde{c}$. The remaining weights are equal to $w_i^s(u_{i-1})$. Finally, for a small $c > 0$, we set:

$$w_i(x) = \hat{w}_i(x)\Big(1 + c \cdot (\sharp\{i_m \geq i\} - \sharp\{i_m < i\})\Big) \quad \text{for } i : 1 \dots n, \tag{5.13}$$

where we consider all indices $m$ are such that $x_m$ is a separator between some two adjacent large rarefactions ($m, m+1 \in \mathcal{LR}$) located to the left of the point $x$.

For $\epsilon_0$ small (5.13) guarantees the increase of weights $w_i(x)$ for $i \leq i_m$ and decrease of $w_i(x)$ for $i > i_m$ across the separator $x_m$ (in the similar spirit to (4.6)). Note that if $c$ is small then (L1) is still satisfied with the definition (5.13).

**5.4. Definition of functional weights $\mathbf{W_i(x)}$.** Recall that $i_\alpha \in \{1 \dots n+1\}$ is the family of the jump located at $x_\alpha$ with strength $\epsilon_\alpha$. For $k \in \mathcal{I}$ denote $\mathcal{P}^k(u)$ the set of all waves $\alpha$ in $u$ with $\alpha \in I_k^u$, $i_\alpha \neq n+1$ and such that if $k \in \mathcal{LR}$ and $i_\alpha = i_k$ then $\epsilon_\alpha < 0$. Set $\mathcal{P}(u) = \bigcup_{k \in \mathcal{I}} \mathcal{P}^k(u)$. Similarly we define the sets $\mathcal{P}^k(v)$, $\mathcal{P}(v)$.

Let now $x \in I_k^u \cap I_s^v$ with $k \leq s$. The quantities $A_i(x)$ [BLY] measure the total amount of physical perturbation waves in $u$ and $v$ which approach the $i$-th wave $q_i(x)$ located at $x$. More precisely, when the $i$-th field is linearly degenerate we set:

$$A_i(x) = \left[ \sum_{\substack{\alpha \in \mathcal{P}(u) \cup \mathcal{P}(v) \\ x_\alpha < x,\ i_\alpha > i}} + \sum_{\substack{\alpha \in \mathcal{P}(u) \cup \mathcal{P}(v) \\ x_\alpha > x,\ i_\alpha < i}} \right] |\epsilon_\alpha|.$$

For a genuinely nonlinear $i$-th field:

$$\begin{aligned} A_i(x) = &\left[ \sum_{\substack{\alpha \in \mathcal{P}(u) \cup \mathcal{P}(v) \\ x_\alpha < x,\ i_\alpha > i}} + \sum_{\substack{\alpha \in \mathcal{P}(u) \cup \mathcal{P}(v) \\ x_\alpha > x,\ i_\alpha < i}} \right] |\epsilon_\alpha| \\ &+ \begin{cases} \left[ \sum_{\substack{\alpha \in \mathcal{P}(u) \\ x_\alpha < x,\ i_\alpha = i}} + \sum_{\substack{\alpha \in \mathcal{P}(v) \\ x_\alpha > x,\ i_\alpha = i}} \right] |\epsilon_\alpha| & \text{if } q_i(x) < 0, \\ \left[ \sum_{\substack{\alpha \in \mathcal{P}(v) \\ x_\alpha < x,\ i_\alpha = i}} + \sum_{\substack{\alpha \in \mathcal{P}(u) \\ x_\alpha > x,\ i_\alpha = i}} \right] |\epsilon_\alpha| & \text{if } q_i(x) \geq 0. \end{cases} \end{aligned}$$

Define:

$$\begin{aligned} W_i(x) = &\ 1 + \kappa_3(Q(u) + Q(v)) + \kappa_2 A_i(x) \\ &+ \begin{cases} \kappa_4 |q_i(x)| & \text{if } i = i_m,\ m \in \mathcal{LR} \cap \{k, s\}, \\ 0 & \text{otherwise .} \end{cases} \end{aligned} \tag{5.14}$$

Here $Q$ stands for the Glimm's interaction potential defined in (4.7). The (large) constants $\kappa_2, \kappa_3, \kappa_4$ are to be determined later. We see that as soon as they have been assigned, we can impose a suitably small bound on the amount of perturbation in $u$ and $v$ (by taking $\epsilon_0$ small in Definition 4.1, or in particular $\delta$ small in Theorem 1.2), and take $\nu$ in 5.2 so that

$$1 \leq W_i(x) \leq 4 \qquad \text{for all } i, x. \tag{5.15}$$

This ends the definition of the functional $\Phi$.

Taking $\mathcal{D} = \mathrm{cl}\ \mathcal{D}_{\epsilon_0}$ for sufficiently small $\epsilon_0 > 0$ and cl denoting the $L^1_{loc}$ closure, the proof of Theorem 1.2 follows in a standard way ([BLY, B]) from Lemma 5.1 and:

**Lemma 5.2.** *The functional* $\Phi$ *constructed above satisfies:*

$$\Phi(u(t',\cdot), v(t',\cdot)) \leq \Phi(u(t,\cdot), v(t,\cdot)) + C \cdot \epsilon \cdot (t'-t), \tag{5.16}$$

$$\frac{1}{C}||u(t,\cdot) - v(t,\cdot)||_{L^1} \leq \Phi(u(t,\cdot), v(t,\cdot)) \leq ||u(t,\cdot) - v(t,\cdot)||_{L^1}, \tag{5.17}$$

*for all* $t' > t \geq T_2$ *and a uniform constant* $C > 0$ *depending only on the system (1.1).*

*Proof.* The equivalence of $\Phi$ with the $L^1$ distance as in (5.17) follows by (5.15) if we take the weights $\{w_i\}_{i=1}^n$ small enough.

To prove (5.16), define the speed $\lambda_i(x)$ as the Rankine-Hugoniot speed of the shock/contact $q_i(x)$.

Recall that a direct calculation [BLY] gives:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi(u(t), v(t)) = \sum_{\alpha \in \mathcal{J}(u)\cup\mathcal{J}(v)} \sum_{i=1}^n E_{\alpha,i}, \tag{5.18}$$

where $\mathcal{J}(u)$ and $\mathcal{J}(v)$ denote the sets of all waves and large rarefactions' separators in $u$ and in $v$, resepctively. We have:

$$\begin{aligned} E_{\alpha,i} = &\left(W_i \cdot w_i \cdot |q_i|\right)(x_\alpha+) \cdot (\lambda_i(x_\alpha+) - \dot{x}_\alpha) \\ &- \left(W_i \cdot w_i \cdot |q_i|\right)(x_\alpha-) \cdot (\lambda_i(x_\alpha-) - \dot{x}_\alpha). \end{aligned} \tag{5.19}$$

Above $\dot{x}_\alpha$ denotes the speed of propagation of the wave (or a separator) $\alpha$ located at $x_\alpha$. We will prove that:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi(u(t), v(t)) \leq \mathcal{O}(1)\epsilon \tag{5.20}$$

for every time $t \geq T_2$ where the fronts in $u$ or $v$ do not interact. Indeed, this will be the goal of the next section.

Next, let $t$ be such that say fronts $\epsilon_\alpha$ and $\epsilon_\beta$ in $u$ interact. By Lemma 4.2, the quantity $Q(u)$ decreases by the same order of magnitude as $A_i(t,x)$ might increase. Thus if $\kappa_3$ in (5.14) is large enough, all functional weights $W_i(x)$ must decrease across the time $t$. Consequently, the whole functional $\Phi$ decreases as well. Based on these two observations and integrating (5.20) in time, one concludes (5.16). ■

## 6. Stability estimates

In this section we outline the proof of (5.20) by estimating the terms $E_{\alpha,i}$ in (5.19). We will distinguish several cases, depending on the characteristic family $i_\alpha$ of the wave $\alpha \in \mathcal{J}(u)\cup\mathcal{J}(v)$ (it can be a large rarefactions' separator as well) and its location $x_\alpha$. For large shocks/contacts $\alpha$ we will prove that:

$$\sum_{i=1}^n E_{\alpha,i} \leq 0. \tag{6.1}$$

To simplify the matter, we replace each wave $\alpha$ of a genuinely nonlinear family $i_\alpha : 1 \ldots n$ and a positive strength $\epsilon_\alpha > 0$ by a non-entropic shock having its speed

$\dot{x}_\alpha = \lambda_{i_\alpha}$ computed at the left state of $\alpha$ and having the original strength $\epsilon_\alpha$ With this modification we will obtain:

$$\sum_{i=1}^{n} E_{\alpha,i} \leq \mathcal{O}(1)\epsilon_\alpha^2. \tag{6.2}$$

The bounds (6.1) and (6.2) plus the bound on the amount of non-physical waves ($i_\alpha = n+1$) of Lemma 4.3 (iii) yield (5.20) in view of (5.18) and (5.19).

The stability estimates of this kind were first employed in the case of small waves (initial data with small total variation) in [BLY]. They were then adapted for patterns with large shocks in [Le1] and for large rarefactions in [Le3]. In the general setting of the present article further modifications are due to the presence of different kinds of large waves. We focus on these modifications, for the details of various estimates we refer to either [Le3] or [BLY].

First of all note that since $t \geq T_2$ then for every $x \notin \mathcal{J}(u) \cup \mathcal{J}(v)$ with $x \in I_k^u \cap I_s^v$ and say $k \leq s$, the solution of the Riemann problem $(u(x), v(x))$ along shocks (as explained in subsection 5.2) contains at most one large shock/contact. Moreover:

$$|q_{i_m}(x)| < \nu \quad \text{for } m \in \mathcal{LR} \cap \{k, s\}. \tag{6.3}$$

**6.1.** When $\alpha$ is neither a large shock/contact nor a large rarefactions' separator, we have both $x_\alpha-, x_\alpha+ \in I_k^u \cap I_s^v$. If $k = s$ then (6.2) follows exactly as in [Le3] in view of (6.3) for $k \in \mathcal{LR}$ and as in [BLY] for $k \in \mathcal{LS} \cup \mathcal{LC}$.

Assume now $k < s$. If $s \in \mathcal{LR}$ and $\alpha$ is an $i_s$-wave in $u$ then the analysis in [Le3] section 6 again yields (6.2) (in view of the $L^1$ stability condition (L1)).

For $\alpha \in \mathcal{P}(v)$ such that $i_\alpha \neq i_s$ in case when $s \in \mathcal{LR}$, we may again use the reasoning from [BLY] if both $s = k+1,\ k \in \mathcal{LR}$. Otherwise, that is when $k+1 \in \mathcal{LS} \cup \mathcal{LC}$, a large shock/contact is present in the solution of both Riemann problems $(u(x_\alpha-), v(x_\alpha-))$ and $(u(x_\alpha+), v(x_\alpha+))$, so we may employ the estimates in [Le1] to obtain (6.2). In the same manner we treat the case when $\alpha$ is a wave in $u$.

**6.2.** We will prove (6.1) when $\alpha$ is a large rarefactions' separator. First, we focus on the case when $\alpha \in \mathcal{P}(v),\ x_\alpha- \in I_k^u \cap I_k^v$ and $x_\alpha+ \in I_k^u \cap I_s^v$ with both $s = k+1,\ k \in \mathcal{LR}$. The case with $\alpha \in \mathcal{P}(u),\ x_\alpha- \in I_k^u \cap I_s^v,\ x_\alpha+ \in I_s^u \cap I_s^v$ and both $s = k+1,\ k \in \mathcal{LR}$ can be treated analogously. For every $i : 1 \ldots n$ we have:

$$E_{\alpha,i} = \Delta W_i \cdot [w_i^- |q_i^\pm| (\lambda_i^\pm - \dot{x}_\alpha)] + W_i^+ |q_i^\pm| (\lambda_i^\pm - \dot{x}_\alpha) \Delta w_i.$$

Noting $\Delta W_i = \kappa_4 |q_{i_s}^\pm| \cdot \delta_{i i_s}$ and (5.13) we obtain:

$$\sum_{i=1}^{n} E_{\alpha,i} \leq \kappa_4 w_{i_s}^- |q_{i_s}^\pm| \cdot |\lambda_{i_s}^\pm - \dot{x}_\alpha| - c \cdot \sum_{i=1}^{n} W_i^+ w_i^- |q_i^\pm| \cdot |\lambda_i^\pm - \dot{x}_\alpha|,$$

which is nonpositive for small $\nu$, in view of (6.3).

Second, we treat the case when $\alpha \in \mathcal{P}(u),\ x_\alpha- \in I_k^u \cap I_k^v,\ x_\alpha+ \in I_s^u \cap I_k^v$ and both $s = k+1,\ k \in \mathcal{LR}$ (when the "change of direction" occures across a separator $\alpha \in \mathcal{P}(v)$, the estimates are readily the same). Note that $\Delta |q_i| = \mathcal{O}(1) \Big( \sum_{i=1}^{n} |q_i^-| \Big)^2$

and $\Delta\lambda_i = \mathcal{O}(1)\sum_{i=1}^n |q_i^-|$. Hence:

$$
\begin{aligned}
\sum_{i=1}^n E_{\alpha,i} =&\mathcal{O}(1)\cdot\kappa_2\epsilon_0\cdot\left(\sum_{i=1}^n w_i^-|q_i^-|\cdot|\lambda_i^- - \dot{x}_\alpha|\right)\\
&+\kappa_4\Big(\Delta|q_{i_k}|\cdot w_{i_k}^-|q_{i_k}^-|\cdot|\lambda_{i_k}^- - \dot{x}_\alpha| + |q_{i_s}^+|\cdot w_{i_s}^-|q_{i_s}^-|\cdot|\lambda_{i_s}^- - \dot{x}_\alpha|\Big)\\
&+\mathcal{O}(1)\left(\sum_{i=1}^n W_i^+w_i^+|q_i^+|\right)\cdot\left(\sum_{i=1}^n|q_i^-|\right)\\
&+\mathcal{O}(1)\left(\sum_{i=1}^n W_i^+w_i^+|\lambda_i^- - \dot{x}_\alpha|\right)\cdot\left(\sum_{i=1}^n|q_i^-|\right)^2\\
&+\sum_{i=1}^n W_i^+|q_i^-|\cdot(\lambda_i^- - \dot{x}_\alpha)\cdot\Delta w_i.
\end{aligned}
\tag{6.4}
$$

Since the last term in (6.4) may be in view of (5.13) estimated by:

$$-c\sum_{i=1}^n w_i^-|q_i^-|\cdot|\lambda_i^- - \dot{x}_\alpha|,$$

the bound (6.1) follows from (6.4) if only $\nu+\epsilon_0$ is small.

**6.3.** It remains to prove (6.1) when $\alpha$ is a large shock or a contact discontinuity. To fix the ideas, assume that $x_\alpha- \in I_k^u\cap I_k^v,\ x_\alpha+\in I_k^u\cap I_s^v$, both $k,\ s=k+2\in\mathcal{LR}$ and $x_\alpha$ is the location of a large $i_{k+1}$-jump in $v$ (the other configurations are actually easier to deal with). We have:

$$
\begin{aligned}
\sum_{i\neq i_{k+1}} E_{\alpha,i} =& \sum_{i\neq i_{k+1}} W_i^-\cdot\Delta[w_i|q_i|\cdot(\lambda_i-\dot{x}_\alpha)]\\
&+\sum_{i\neq i_{k+1}}(\Delta W_i)\cdot w_i^-|q_i^-|\cdot(\lambda_i^- - \dot{x}_\alpha).
\end{aligned}
\tag{6.5}
$$

Noticing

$$E_{\alpha,i_{k+1}} = W_{i_{k+1}}^+w_{i_{k+1}}^+\cdot(\lambda_{i_{k+1}}^+ - \dot{x}_\alpha) - W_{i_{k+1}}^-w_{i_{k+1}}^-|q_{i_{k+1}}^-|\cdot(\lambda_{i_{k+1}}^- - \dot{x}_\alpha)$$

together with $\sum_{i\neq i_{k+1}}\Delta[w_i|q_i|\cdot(\lambda_i-\dot{x}_\alpha)]\leq -c\sum_{i>k}|q_i^-|$ implied by the condition (L1), the usual manipulations with various terms of (6.5) yield (6.1) for $\tilde{c}$ and $\epsilon_0+\nu$ small enough.

## 7. Local well posedness of the Cauchy problem for general $BV$ data

As a corollary to Theorems 1.1 and 1.2 we obtain the local existence and stability result for arbitrarily large $BV$ initial data that satisfy our stability conditions at all large jumps.

**Corollary 7.1.** *Assume that $f$ is defined on an open set $\Omega$ where (H1) and (H2) hold. Let $K\subset\Omega$ be a compact subset with the following property. For every $u_l, u_r\in K$, the Riemann problem (1.1) (1.2) has a self-similar entropy solution (called subsequently the Riemann solver) satisfying condition ($BV$). Then for every $BV$ initial data $\bar{u}:\mathbf{R}\longrightarrow K$ the Cauchy problem (1.1) (1.3) has an entropy solution $u$ on $[0,T]\times\mathbf{R}$, for some time $T>0$ depending on $\bar{u}$. Moreover $u$ is a Lipschitz continuous function from $[0,T]$ to $L^1_{loc}$.*

*If the stronger condition (L1) holds for all elementary waves inside $K$ then $u(t)$ depends Lipschitz continuously (in $L^1_{loc}$) on the initial data $\bar{u}$.*

This solution is unique within the class of entropy solutions admitting a Lipschitz continuous extension to the (local in time) flow compatible with the prescribed Riemann solver (that satisfies condition (L1)). The proof follows the standard line as in [B] chapter 9.1. Another characterization may be obtained as in [B] chapter 9.3:

**Corollary 7.2.** *The solution $u$ in Corollary 7.1, is unique within the class of functions having the following properties.*

(i) *(conservation equations) The function $u = u(t,x)$ takes values in $K$, $u : [0,T] \longrightarrow L^1_{loc}$ is continuous and $TV(u(t))$ is uniformly bounded. The initial condition (1.3) holds, together with*
$$\int\int \big(u\varphi_t + f(u)\varphi_x\big)\ \mathrm{d}x\mathrm{d}t = 0,$$
*for every $C^1$ function $\varphi$ with compactly supported in $(0,T) \times \mathbf{R}$.*

(ii) *(compatibility with the Riemann solvers) For every $(t,x)$ such that $u(t,x-) \neq u(t,x+)$ there holds*
$$\lim_{\rho\to 0+} \frac{1}{\rho^2}\int_t^{t+\rho}\int_{x-\rho}^{x+\rho}\Big|u(s,y) - U(s-t,y-x)\Big|\ \mathrm{d}x\mathrm{d}t = 0,$$
*where $U$ is the Riemann solver of $(u(t,x-),u(t,x+))$.*

(iii) *(locally bounded variation) For some $\gamma > 0$, whenever $t = t(x)$ is a space-like curve with Lipschitz constant $\gamma$, then the total variation of the composed map $x \mapsto u\big(t(x),x\big)$ is bounded on bounded intervals.*

## 8. Stability of the Riemann problem in gas dynamics

In this section we are concerned with validating the stability conditions of section 3 in the context of gas dynamics. Recall first that for the $p$-system

$$u_t - v_x = 0, \quad v_t + p(u)_x = 0, \tag{8.1}$$

with $p > 0$, $p' < 0$, $p'' > 0$ and $u > 0$, it follows from [BC] that all three conditions (F), (BV) and (L1) are satisfied for any reference pattern containing a 1-wave followed by a 2-wave.

In the setting of the $\gamma$-gas-law Euler equations ($\gamma \geq 1$ is the adiabatic constant):

$$\begin{aligned} &\rho_t + (\rho v)_x = 0, \quad (\rho v)_t + (\rho v^2 + P)_x = 0, \\ &\left(\frac{\gamma-1}{2}\rho v^2 + P\right)_t + \left(\frac{\gamma-1}{2}\rho v^3 + \gamma P v\right)_x = 0. \end{aligned} \tag{8.2}$$

it has been proven in [Scho] that for $\gamma \geq \gamma_1$ the condition (BV) is always satisfied, while for $\gamma \in (1,\gamma_1)$ there indeed exist Riemann problems for which (BV) fails. The treshold exponent $\gamma_1$ is calculated there as:

$$\gamma_1 = \frac{2}{21}(4\sqrt{2}+5) \approx 1.0149.$$

In what follows we study the conditions (L1) and (F) in the same context. Recall that the two extreme fields of (8.2) are genuinely nonlinear, while the intermediate field is linearly degenerate.

**7.1. Patterns with 1-shock, 2-contact and 3-shock.** Notice that all the matrices $\mathbf{F}$, $|\mathbf{F}|$ and $|\mathbf{G}|$ (for definitions see (3.2) and Theorem 3.3 (i)) have the form:

$$M = \begin{bmatrix} 0 & 0 & a & & & \\ 0 & 0 & b & & & \\ & c & 0 & 0 & e & \\ & d & 0 & 0 & f & \\ & & & g & 0 & 0 \\ & & & h & 0 & 0 \end{bmatrix}, \tag{8.3}$$

where $a \dots h$ are the only nonzero entries. One sees that:

$$\det(M - \lambda \mathrm{Id}) = \lambda^2 \cdot \Big( (\lambda^2 - bc)(\lambda^2 - fg) - bedg \Big). \tag{8.4}$$

If $b \dots g$ are nonnegative then the discriminant of the equation $y^2 - (bc + fg)y + bg(cf - ed) = 0$ is nonnegative and the absolute value of its smaller root $y_1$ is not larger than its second, positive root $y_2$. Hence the spectral radius of $M$ is smaller than 1 iff $y_2 < 1$. We leave it to the reader to check that this condition is in turn equivalent to the following couple of inequalities:

$$bc < 1 \quad \text{and} \quad g \cdot \left( f + \frac{bed}{1 - bc} \right) < 1 \tag{8.5}$$

(which indeed constitute the $BV$ stability condition in [Scho], with $b \dots g$ being the appropriate reflexion and transmission coefficients).

We first verify our condition (L1) for a pattern containing a 1-shock, a 2-contact and a 3-shock connecting consecutively the states $u^0$, $u^1$, $u^2$, $u^3$ (we just drop the subscript 0 used in previous sections) and travelling with speeds $\Lambda^1 < \Lambda^2 < \Lambda^3$. Each $u^q = (\rho^q, v^q, P^q)$ with $\rho^q > 0$. For the contact discontinuity we obviously have $\Lambda^2 = v^1 = v^2$. Set $c^q = \sqrt{\gamma \cdot P^q / \rho^q}$. The coefficients of $|\mathbf{G}|$ are then calculated [Scho]:

$$\begin{aligned} b &= |R^{(-)}| \cdot \left| \frac{v^1 + c^1 - \Lambda^1}{v^1 - c^1 - \Lambda^1} \right|, \qquad c = \left| \frac{c^1 - c^2}{c^1 + c^2} \right|, \qquad d = \left| \frac{2c^1}{c^1 + c^2} \right| \cdot \left| \frac{c^2}{c^1} \right|, \\ e &= \left| \frac{2c^2}{c^1 + c^2} \right| \cdot \left| \frac{c^1}{c^2} \right|, \qquad f = \left| \frac{c^2 - c^1}{c^1 + c^2} \right|, \qquad g = |R^{(+)}| \cdot \left| \frac{v^2 - c^2 - \Lambda^3}{v^2 + c^2 - \Lambda^3} \right|. \end{aligned} \tag{8.6}$$

Above $R^{(-)}$ and $R^{(+)}$ are Schochet's notation for, respectively: $[F_1^{right}]_{31}$ and $[F_3^{left}]_{13}$, which are the reflexion coefficients related to the 1-shock and the 3-shock (see subsection 3.1). Introducing the quantity $Q = |c| = |f|$, condition (8.5) becomes

$$Q \cdot b < 1 \quad \text{and} \quad Q(b + g) - 2Q^2 bg + bg - 1 < 0. \tag{8.7}$$

Now as in [Scho] we notice that the range of $(Q, b, g)$ is $[0, 1) \times [0, \tilde{R}(\gamma))^2$, for a positive number $\tilde{R}(\gamma)$ that we discuss later. Therefore (8.7) is equivalent to:

$$\tilde{R}(\gamma) \leq 1 \quad \text{and} \quad 2Q\tilde{R}(\gamma) - 2Q^2 \tilde{R}(\gamma)^2 + \tilde{R}(\gamma)^2 - 1 \leq 0 \quad \forall Q \in [0, 1),$$

which is in turn:

$$\tilde{R}(\gamma) = \sup_{x \in (-1, 0]} h(x, \gamma) \leq 1/\sqrt{2}. \tag{8.8}$$

Defining the 3-shock parameter $x \in (-1, 0]$ to be $x = P^3/P^2 - 1$, $h$ may be computed as in [Scho] pg. 343. Namely:

$$h(x,\gamma) = \frac{\left|1 + \frac{\gamma-1}{\gamma}x - \sqrt{1 + \frac{\gamma+1}{2\gamma}x}\right|}{1 + \frac{\gamma-1}{\gamma}x + \sqrt{1 + \frac{\gamma+1}{2\gamma}x}}. \tag{8.9}$$

Now for $\gamma \in [1, 5/3]$, (8.8) and (8.9) imply:

$$\tilde{R}(\gamma) = \frac{1/\gamma - \sqrt{(\gamma-1)/2\gamma}}{1/\gamma + \sqrt{(\gamma-1)/2\gamma}}$$

(compare formula (5.51) in [Scho]). One checks directly that (8.8) holds for $\gamma \in [\gamma_2, 5/3]$ with

$$\gamma_2 = \frac{1 + \sqrt{137 - 96\sqrt{2}}}{2} \approx 1.05576.$$

When $\gamma > 5/3$, the behaviour of $h(x,\gamma)$ is more complicated. Call $z = (\gamma - 1)/\gamma \in (2/5, 1)$ and let $H(x, z) = -1 + 2(1 + zx)/\big(1 + zx + \sqrt{1 + (1 - z/2)x}\big)$ in $h(x,\gamma) = |H(x, z)|$. We have $\partial H/\partial x \geq 0$ for $z \in [(-3 + \sqrt{17})/2, 1)$ and $H(0, z) = 0$. Thus for $z \geq (-3 + \sqrt{17})/2$ there is $\tilde{R}(\gamma) = -H(-1, z) = 1 - 2(1 - z)/(1 - z + \sqrt{z/2})$. Again, some algebraic manipulations show that in the range of $z \in [(-3 + \sqrt{17})/2, 1)$ the inequality in (8.8) is satisfied for $z \leq z_3 = (21 - 12\sqrt{2} - \sqrt{713 - 502\sqrt{2}})/4 \approx 0.8858$. On the other hand for $z \in (2/5, (-3 + \sqrt{17})/2)$ we have $\partial H/\partial z \leq 0$ and hence (8.8) holds. Finally, we recover the critical adiabatic exponent

$$\gamma_3 = 1/(1 - z_3) \approx 8.7577$$

such that (8.8) is satisfied for $\gamma \in (5/3, \gamma_3]$ and is not satisfied for $\gamma > \gamma_3$. Summarizing, we see that (L1) holds always for $\gamma \in [\gamma_2, \gamma_3]$ while for $\gamma < \gamma_2$ or $\gamma > \gamma_3$ there indeed exist Riemann problems for which (8.8) and thus also the condition (L1) fail.

We now verify the finiteness condition (F). By (8.4), (F) it is equivalent to:

$$(1 - bc)(1 - fg) \neq bedg$$

with

$$b = R^{(-)}, \quad c = (c^1 - c^2)/(c^1 + c^2), \quad d = 2c^1/(c^1 + c^2),$$
$$e = 2c^2/(c^1 + c^2), \quad f = (c^2 - c^1)/(c^1 + c^2), \quad g = R^{(+)}.$$

Hence (F) becomes:

$$(R^{(-)} - R^{(+)}) \cdot \frac{c^1 - c^2}{c^1 + c^2} + R^{(-)}R^{(+)} \neq 1. \tag{8.10}$$

Again, the range of $(c, R^{(-)}, R^{(+)})$ is contained in $(-1, 1) \times (-R(\gamma), R(\gamma))^2$ and therefore (8.10) is implied by:

$$R(\gamma) \leq 1. \tag{8.11}$$

The value of $R(\gamma)$ is estimated for different $\gamma$ in [Scho] formula (5.52) and there we see that (8.11) certainly holds for every $\gamma \geq 1$.

**7.2. Other patterns.** Consider a pattern containing only a 1-shock followed by a 2-contact discontinuity. In view of Theorem 3.3 (i), to validate the condition

(L1) one has to check the spectral radius of the $4 \times 4$ principal minor of $|\mathbf{G}|$. It is smaller than 1 iff $bc < 1$ with $b, c$ given in (8.6), which is equivalent to

$$Q \cdot b < 1 \quad \text{ for every } Q \in [0,1)$$

(see discussion in subsection 7.1). We thus need

$$\tilde{R}(\gamma) \leq 1$$

which always holds because of the definition of $\tilde{R}(\gamma)$ in (8.8) and (8.9).

In the same manner we see that $fg < 1$ and $bg < 1$. Therefore every pattern consisting of a 3-shock preceded by a 2-contact or a 1-shock is $L^1$ stable.

Finally, notice that if a pattern containing only shocks and contacts satisfies (BV) or (L1) then the same remains true if we modify this pattern by adding an extreme field rarefaction on one or on its both sides. Hence, any pattern for (8.2) which is not composed of 3 discontinuities (1-shock, 2-contact, 3-shock), treated in subsection 7.1, satisfies all the stability conditions (F), (BV) and (L1).

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