ON THE L^1 WELL POSEDNESS OF SYSTEMS OF CONSERVATION LAWS NEAR SOLUTIONS CONTAINING TWO LARGE SHOCKS

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Contents

1. Introduction

We consider the Cauchy problem for $n \times n$ system of conservation laws in one space dimension:

$$u_t + f(u)_x = 0, \tag{1.1}$$

$$u(0,x) = \bar{u}(x). \tag{1.2}$$

As in the classical paper of Lax [L], we assume here that the system is strictly hyperbolic with each characteristic field either linearly degenerate or genuinely nonlinear. In this setting, the recent progress in the field has shown that within the class of initial data $\bar{u} \in L^1 \cap BV(\mathbf{R}, \mathbf{R}^n)$ having the total variation suitably small, the problem (1.1) (1.2) is well posed in $L^1(\mathbf{R}, \mathbf{R}^n)$. Namely, as proved in [BC1] [BCP] [BLY], the entropy solutions of (1.1) (1.2) constitute a semigroup which is Lipschitz continuous with respect to time and initial data. A major question which remains open is whether the uniqueness of solutions also holds for arbitrarily large initial data. We observe that, because of the finite propagation speed, this is essentially a local problem. Moreover, given any BV function $\bar{u} : \mathbf{R} \longrightarrow \mathbf{R}^n$, for each point $x_0 \in \mathbf{R}$ one can find left and right neighbourhoods $[x_0 - \delta, x_0)$ and $(x_0, x_0 + \delta]$ on which $\bar{u}$ has arbitrarily small variation.

By the previous remarks the problem is thus reduced to proving the well posedness of the Cauchy problem (1.1) (1.2) with the initial data $\bar{u}$ being a small perturbation of a fixed Riemann problem (u_0^-, u_0^+). The solution of the latter consists of m (large) waves of different characteristic families.

In this paper we study the case where the Riemann problem is solved by two large shocks, travelling with the speeds Λ^i and Λ^j. The more general case of m shocks, $2 < m \leq n$ will be addressed in the forthcoming work [Le1].

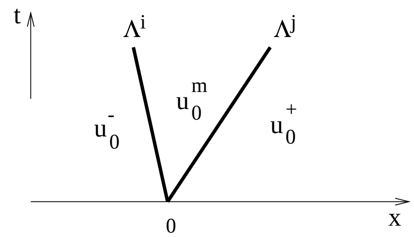

FIGURE 1.1

The following questions arise naturally:

A. Do we have the (global) existence of an 'admissible' solution u to (1.1) (1.2) when $\bar{u}$ stays 'close' to the Riemann data (u_0^-, u_0^+)?
B. In case the answer to A is positive, is the solution u stable under small perturbations of its initial data?

Several authors have given contribution to a better understanding of the above issues in various contexts. In particular, we mention here two papers which are closely related to our work. Schochet was the first to introduce in [Scho] the so-called finiteness condition, giving a positive answer to question A for general $n \times n$ systems. In [BC2] Bressan and Colombo consider the general Riemann problem for systems of two equations and assuming a stronger stability condition, answer question B positively. In particular, they establish the existence of a Lipschitz semigroup of solutions, defined on a domain containing all suitably small BV perturbations of Riemann data (u_0^-, u_0^+). They also construct an example whose aim is to show that with the stability condition being violated, the system (1.1) in general does not generate a Lipschitz continuous flow of entropy solutions.

The goal of this article is to discuss both questions A and B, for a general $n \times n$ system of conservation laws. We formulate a Finiteness Condition and a new Stability Condition. We show that the Finiteness Condition guarantees the positive answer to question A (Theorem A); while the Stability Condition is essential in giving positive answer to question B (Theorem B), yielding the existence of a Lipschitz semigroup of entropy solutions. Different finiteness and stability conditions appearing in the literature are discussed in the paper [Le2].

The paper is organized as follows. In Section 2 we discuss the setting of the problem, introduce our Finiteness and Stability Conditions and state the main theorems, that will be proved in Section 6. In Section 3 we describe the wave front tracking algorithm, working in the presence of large shocks, and list the main features of the piecewise constant approximate solutions, produced by the algorithm (Theorem 3.5). In particular, we explain the role of the Finiteness Condition for the stability of the algorithm. The limiting process, applied to the wave front tracking approximations, yields a weak 'admissible solution' to the Cauchy problem (1.1) (1.2), as shown in Theorem A. Section 4 contains the definition of the entropy functional and the basic L^1 stability estimates (4.10) – (4.13) for the wave front tracking approximations, that we prove in Section 5.

2. Preliminaries and main results

Let Ω be an open subset of $\mathbf{R}^n$ and $f : \Omega \to \mathbf{R}^n$ a smooth flux function in (1.1). We assume that the system (1.1) is strictly hyperbolic and that every characteristic field is either linearly degenerate or genuinely nonlinear. For $u \in \Omega$, the eigenvalues of the matrix $Df(u)$ are denoted: $\lambda_1(u) < \ldots < \lambda_n(u)$ while the dual bases of the corresponding right and left eigenvectors $\{r_k(u)\}_{k=1}^n$ and $\{l_k(u)\}_{k=1}^n$ of $Df(u)$ are normalized as follows:

$$\langle r_k(u), l_s(u)\rangle = \delta_{k,s}, \quad |r_k(u)| = 1 \qquad \text{for } k, s = 1 \ldots n.$$

Besides the strict hyperbolicity of (1.1), we assume the stronger condition:

$$\lambda_k(u) < \lambda_s(v), \quad \forall k < s, \ \forall u, v \in \Omega. \tag{2.1}$$

Note that if Ω is a small neighbourhood of a point, then (2.1) is a consequence of the strict hyperbolicity. By continuity, on every compact set $K \subset \Omega$, the differences of the characteristic speeds in different families are bounded away from 0:

$$|\lambda_k(u) - \lambda_s(v)| \geq c, \quad \forall k \neq s, \ \forall u, v \in K, \tag{2.2}$$

for some positive number c depending on K.

Let u_0^l, u_0^r be two different states in Ω. We consider the Cauchy problem for (1.1) with initial data of the Riemann type:

$$u(t,x) = \begin{cases} u_0^l & x > \Lambda^i t, \\ u_0^r & x < \Lambda^i t. \end{cases} \tag{2.3}$$

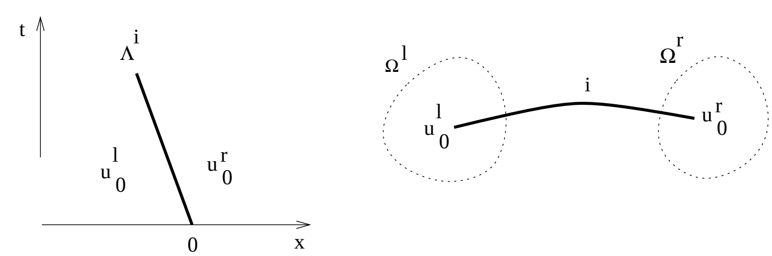

Figure 2.1

The admissibility of (2.3) is defined by the following two conditions:

$$\begin{aligned} &\text{(i)} \ \ f(u_0^l) - f(u_0^r) = \Lambda^i (u_0^l - u_0^r), \\ &\text{(ii)} \ \ \lambda_i(u_0^l) > \Lambda^i > \lambda_i(u_0^r). \end{aligned} \tag{2.4}$$

The first condition is the well-known Rankine-Hugoniot condition, stating that (2.3) is a distributional solution of (1.1), while the second condition says that the shock (u_0^l, u_0^r), traveling with speed Λ^i is a compressible Lax shock of the family i.

If Ω is convex, then for given $u, u' \in \Omega$ one defines the averaged matrix

$$A(u,u') = \int_0^1 A(\theta u + (1-\theta)u')d\theta.$$

Assuming that for every $u, u' \in \Omega$, $A(u,u')$ is strictly hyperbolic (which certainly is the case, if $|u - u'|$ is small enough) and denoting its corresponding bases of right and left eigenvectors: $\{r_k(u,u')\}_{k=1}^n$ and$\{l_k(u,u')\}_{k=1}^n$, we see by the Rankine-Hugoniot equations that u, u' are joined by a shock of the i-characteristic family if and only if

$$\langle l_k(u,u'), u-u'\rangle = 0 \quad \forall k \neq i.$$

In particular, for $n=2$ and $i=1$ the above $n-1$ equations reduce to the scalar condition:

$$\Phi(u,u') = \langle l_2(u,u'), u-u'\rangle = 0, \tag{2.5}$$

The following definition was used in [BC2]. The 1-shock joining the states u_0^l, u_0^r is said to be stable if

$$\left\langle \frac{\partial}{\partial u}\Phi(u_0^l,u_0^r), r_2(u_0^r)\right\rangle \neq 0. \tag{2.6}$$

Extending this idea, we introduce the following hypothesis:

(2.7) There exist $\Omega^l, \Omega^r \subset \mathbf{R}^n$ neighbourhoods of 'basic' states u_0^l and u_0^r respectively, and a smooth function $\Psi^i : \Omega^l \times \Omega^r \longrightarrow \mathbf{R}^{n-1}$ such that:

- (i) $\Psi^i(u^l,u^r) = 0$ iff the states u^l and u^r can be connected by a (large) shock of the i-th characteristic family, with the speed $\Lambda^i(u^l,u^r)$. The Rankine-Hugoniot condition holds: $f(u^l) - f(u^r) = \Lambda^i(u^l,u^r)(u^l-u^r)$. In particular $\Psi^i(u_0^l,u_0^r) = 0$ and $\Lambda^i(u_0^l,u_0^r) = \Lambda^i$.
- (ii) $$\text{rank } \frac{\partial\Psi^i}{\partial u^l}(u_0^l,u_0^r) = \text{rank } \frac{\partial\Psi^i}{\partial u^r}(u_0^l,u_0^r) = n-1.$$
- (iii) The $n-1$ vectors: $$\left\{\frac{\partial\Psi^i}{\partial u^l}(u_0^l,u_0^r)\cdot r_k(u_0^l)\right\}_{k=1}^{i-1} \cup \left\{\frac{\partial\Psi^i}{\partial u^r}(u_0^l,u_0^r)\cdot r_k(u_0^r)\right\}_{k=i+1}^{n}$$ are linearly independent.

The above conditions require only that the function f is defined in a small neighbourhood of the basic states u_0^l, u_0^r (in particular the set Ω does not need to be connected). A more detailed discussion of (2.7) can be found in [Le2].

Another remark is that if the basic shock (u_0^l, u_0^r) is weak enough then the existence of Ψ^i (for any $i = 1\dots n$) is ensured by the fundamental theorem of Lax [L]. Moreover, the well known proof of this result via the implicit function theorem allows us to introduce the C^2 functions $\Psi_k : \Omega\times I \to \Omega$, $k = 1\dots n$, (here I is a small open interval containing zero) which, for fixed $u \in \Omega$, coincide with the rarefaction curves for the positive part of I and for $\epsilon \in I$ negative follow the shock curves $\mathcal{S}$. These shock curves constitute the Hugoniot locus of the neighbourhood states, which can be connected with u to the left by a k-admissible shock [L]. We parameterize the curves $\Psi_k(u,\cdot)$ by arc-length, equal to the strength of the discontinuity (left state, right state). In this case,

$$\frac{\partial}{\partial\epsilon}\Psi_k(u,0) = r_k(u).$$

We denote by $\tilde\Psi_k : \Omega\times I \to \Omega$ the C^2 functions for which

$$\Psi_k(u^l,\epsilon) = u^+ \ \text{ iff } \ \tilde\Psi_k(u^+,-\epsilon) = u^l.$$

As previously we have that

$$\frac{\partial}{\partial \epsilon}\tilde{\Psi}_k(u,0) = r_k(u).$$

Proposition 2.1. *Let the conditions (2.7) and the regularity assumptions (2.1) (2.4) hold. Then (possibly replacing the sets Ω^l and Ω^r with suitably small neighbourhoods of u_0^l and u_0^r respectively), the following statements are true:*

(i) *Every Riemann problem $(u^l, u^r) \in \Omega^l \times \Omega^r$ has a unique 'admissible' self-similar solution, composed of n shock or rarefaction waves, connecting the states $u^l = u_0, u_1, \dots, u_{i-1} \in \Omega^l$ and $u_i, u_{i+1}, \dots, u_n = u^r \in \Omega^r$, as in Figure 2.2.*

(ii) *The 'admissibility' of this solution is understood in the following sense:*

$$u_k = \Psi_k(u_{k-1}, \epsilon_k),$$

for every $k \neq i$ and some small parameter ϵ_k, which will be called the strength of the k-wave (u_{k-1}, u_k), and

$$\Psi^i(u_{i-1}, u_i) = 0.$$

(iii) *The wave (u_{i-1}, u_i) is a compressive Lax shock, that is:*

$$\lambda_i(u_{i-1}) > \Lambda^i(u_{i-1}, u_i) > \lambda_i(u_i).$$

The speed $\Lambda^i(u_{i-1}, u_i)$ depends in a C^2 way on (u^l, u^r).

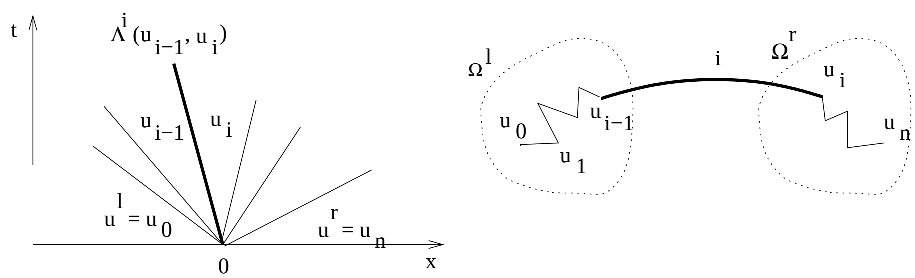

FIGURE 2.2

Before we give the proof of Proposition 2.1, we proceed towards the main point of our work. Fix three distinct states $u_0^-, u_0^m, u_0^+ \in \Omega$. Assume that the states u_0^-, u_0^m are connected by a large Lax compressive stable shock of the i-th characteristic family, that is (2.4) (2.7) hold with the superindices l, r replaced by $-, m$ respectively. The states u_0^m, u_0^+ are assumed to be connected by a large shock of the j-th family ($j > i$), travelling with the speed $\Lambda^j > \Lambda^i$ and also having the properties (2.4) (2.7), with l, r, i replaced by $m, +, j$ respectively.

Consider a small wave of a family $k \leq i$, hitting the large initial i-shock (u_0^-, u_0^m) from the right (Figure 2.3). Let ϵ_k^{in} be the strength of the k-small wave and λ_k^{in} its speed. By Proposition 2.1 the Riemann problem (u_0^-, u^m) is solved uniquely by an $(n-1)$-dimensional wave vector $(\epsilon_1^{out}, \dots, \epsilon_{i-1}^{out}, \epsilon_{i+1}^{out}, \dots, \epsilon_n^{out})$. The corresponding speeds of the small solution waves are denoted by λ_s^{out}, $s \in \{1 \dots n\} \setminus \{i\}$. For these indices s, define numbers

$$m_{sk}^i = \frac{\partial \epsilon_s^{out}}{\partial \epsilon_k^{in}}_{|\epsilon_k^{in}=0}.$$

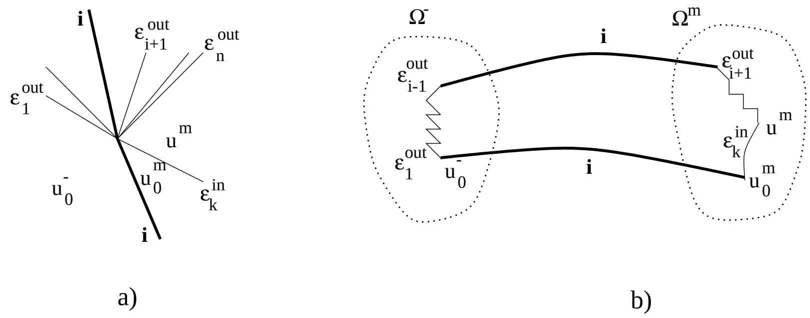

FIGURE 2.3

Similarily, consider a wave pattern, where a small k-wave with $k \geq j$ approaches from the left the large initial j-shock (u_0^m, u_0^+) (Figure 2.4). Solving

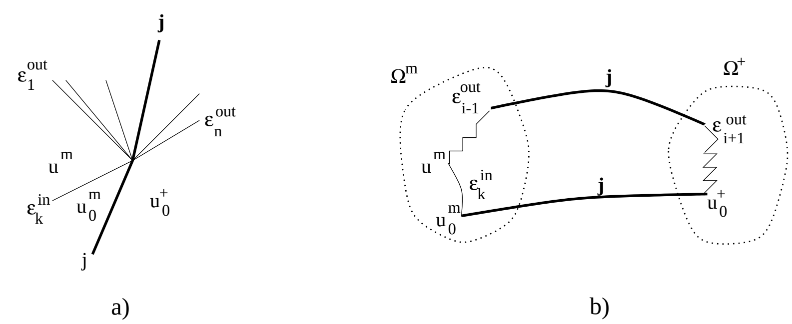

FIGURE 2.4

the Riemann problem (u^m, u_0^+) yields, as before, the unique $(n-1)$-wave vector $(\epsilon_1^{out}, \dots, \epsilon_{j-1}^{out}, \epsilon_{j+1}^{out}, \dots, \epsilon_n^{out})$. For $s \in \{1 \dots n\} \setminus \{j\}$, let

$$m_{sk}^j = \frac{\partial \epsilon_s^{out}}{\partial \epsilon_k^{in}}_{|\epsilon_k^{in}=0}.$$

Now, we are ready to state the finiteness and stability conditions:

FINITENESS CONDITION

There exist positive weights $w_1, \dots, w_n$ and a number $\theta \in (0,1)$ such that

$$\forall k \leq i \quad \sum_{s=j}^{n} \frac{w_s}{w_k} \cdot |m_{sk}^i| < \theta, \tag{2.8}$$

$$\forall k \geq j \quad \sum_{s=1}^{i} \frac{w_s}{w_k} \cdot |m_{sk}^j| < \theta. \tag{2.9}$$

Stability Condition

There exist positive weights $\tilde{w}_1, \ldots, \tilde{w}_n$ and a number $\Theta \in (0,1)$ such that

$$\forall k \leq i \quad \sum_{s=j}^{n} \frac{\tilde{w}_s}{\tilde{w}_k} \cdot |m^i_{sk}| \cdot \left| \frac{\lambda_s(u_0^m) - \Lambda^i}{\lambda_k(u_0^m) - \Lambda^i} \right| < \Theta. \tag{2.10}$$

$$\forall k \geq j \quad \sum_{s=1}^{i} \frac{\tilde{w}_s}{\tilde{w}_k} \cdot |m^j_{sk}| \cdot \left| \frac{\lambda_s(u_0^m) - \Lambda^j}{\lambda_k(u_0^m) - \Lambda^j} \right| < \Theta. \tag{2.11}$$

It can be shown (see [Le2]) that the above Stability Condition implies the Finiteness Condition. We also remark, that since the weights $w_{i+1}, \ldots, w_{j-1}$ (as $\tilde{w}_{i+1}, \ldots, \tilde{w}_{j-1}$) do not appear in the inequalities (2.8) – (2.11) they may be fixed to 1.

Following [BC2], we define for a given $\delta_0 > 0$ the domain:

$$\widetilde{\mathcal{D}}_{\delta_0} = \text{cl}\Bigg\{ u : \mathbf{R} \longrightarrow \mathbf{R}^n;\ \text{there exist two points } x^i < x^j \text{ in } \mathbf{R}$$

$$\text{such that calling } \widetilde{u}(x) = \begin{cases} u_0^- & x < x^i \\ u_0^m & x^i < x < x^j \\ u_0^+ & x > x^j \end{cases} \text{ we have:} \tag{2.12}$$

$$u - \widetilde{u} \in \mathrm{L}^1(\mathbf{R}, \mathbf{R}^n) \text{ and T.V.}(u - \widetilde{u}) \leq \delta_0 \Bigg\},$$

where the closure is taken in $\mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}, \mathbf{R}^n)$.

Our first result concerns question A, posed in the Introduction.

Theorem A *Assume (2.1) together with (2.4) and (2.7) for both shocks (u_0^-, u_0^m) and (u_0^m, u_0^+). If the Finiteness Condition is satisfied then there exists $\delta_0 > 0$ such that for every $\bar{u} \in \widetilde{\mathcal{D}}_{\delta_0}$ there exists a weak solution to (1.1) (1.2) (defined for all times $t \geq 0$).*

Since, as shown in [Le2], our Finiteness Condition is equivalent to the corresponding condition in [Scho], Theorem A can be seen as a special case of the general result of Schochet [Scho]. Its proof, using the wave front tracking algorithm and the BV stability estimates derived in its course are, nevertheless, important for later purposes of the proof of the L^1 stability result.

The main theorem of our paper is the following.

Theorem B *Assume (2.1) together with (2.4) and (2.7) for both shocks (u_0^-, u_0^m) and (u_0^m, u_0^+). If the Stability Condition is satisfied then there exists $\delta_0 > 0$, $L > 0$, a closed domain $\mathcal{D}_{\delta_0} \subset \mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}, \mathbf{R}^n)$ containing $\widetilde{\mathcal{D}}_{\delta_0}$, and a continuous semigroup $S : [0, \infty) \times \mathcal{D}_{\delta_0} \longrightarrow \mathcal{D}_{\delta_0}$ such that:*

(i) $S(0, \bar{u}) = \bar{u}$,
$S(t+s, \bar{u}) = S(t, S(s, \bar{u})) \quad \forall t, s \geq 0\ \forall \bar{u} \in \mathcal{D}_{\delta_0}$.
(ii) $\| S(t, \bar{u}) - S(s, \bar{w}) \|_{\mathrm{L}^1} \leq L(|t-s| + \| \bar{u} - \bar{w} \|_{\mathrm{L}^1}) \quad \forall t, s \geq 0\ \forall \bar{u}, \bar{w} \in \mathcal{D}_{\delta_0}$.
(iii) *Each trajectory $t \mapsto S(t, \bar{u})$ is a weak solution of (1.1) (1.2).*

Note that, by its closedness, the domain $\mathcal{D}_{\delta_0}$ must contain all the initial data that are small BV perturbations of the 'basic' Riemann problem (u_0^-, u_0^+).

The trajectories of the semigroup S will be obtained as the limits of the wave front tracking approximations, described in Section 3. In particular, if $\bar{u} \in \mathcal{D}_{\delta_0}$ is piecewise constant then for $t > 0$ sufficiently small, the function $u(t, \cdot) = S(t, \bar{u})$ coincides with the solution of (1.1) (1.2) obtained by piecing together the standard self-similar solutions of the corresponding Riemann problems.

We now give the omitted:

Proof. of Proposition 2.1. Define the C^2 function $F : \Omega^l \times \Omega^r \times I^{n-1} \to \mathbf{R}^{n-1}$

$$\begin{aligned} &F(u^l, u^r, \epsilon_1, \dots, \epsilon_{i-1}, \epsilon_i, \dots, \epsilon_n) = \\ &\quad \Psi^i \Bigg(\Psi_{i-1}(\dots \Psi_2(\Psi_1(u^l, \epsilon_1), \epsilon_2) \dots \epsilon_{i-1}), \\ &\qquad \tilde{\Psi}_{i+1}(\dots \tilde{\Psi}_{n-1}(\tilde{\Psi}_n(u^r, -\epsilon_n), -\epsilon_{n-1}) \dots - \epsilon_{i+1} \Bigg). \end{aligned}$$

To prove (i), (ii) we proceed by the implicit function theorem. We have that

$$F(u_0^l, u_0^r, 0) = \Psi^i(u_0^l, u_0^r) = 0$$

and

$$\frac{\partial}{\partial(\epsilon_1, \dots, \epsilon_{i-1}, \epsilon_i, \dots, \epsilon_n)} F(u_0^l, u_0^r, 0) = \frac{\partial \Psi^i}{\partial(u^l, u^r)} (u_0^l, u_0^r) \cdot A \cdot B,$$

where:

- A is a $2n \times (n-1)$ matrix, whose first $i-1$ columns are the vectors:

  $$[r_k(u_0^l)^T, 0 \dots 0]^T \in \mathbf{R}^{2n} \quad k : 1 \dots i-1,$$

  and the last $n-i$ columns are the vectors:

  $$[0 \dots 0, r_k(u_0^r)^T]^T \in \mathbf{R}^{2n} \quad k : i+1 \dots n.$$

- The first $i-1$ columns of the $(n-1) \times (n-1)$ matrix B constitute an $(n-1) \times (i-1)$ matrix:

  $$\left[\frac{\partial \Psi^i}{\partial u^r}(u_0^l, u_0^r) \cdot \Big[r_1(u_0^l), \dots, r_{i-1}(u_0^l) \Big] \right],$$

  The last $n-i$ columns of B compose an $(n-1) \times (n-i)$ matrix:

  $$\left[\frac{\partial \Psi^i}{\partial u^l}(u_0^l, u_0^r) \cdot \Big[r_{i+1}(u_0^r), \dots, r_n(u_0^r) \Big] \right].$$

The matrix B is invertible by the assumption (2.7)(iii). Therefore, for the given pair of states $(u^l, u^r) \in \Omega^l \times \Omega^r$, there exists exactly one $(n-1)$-dimensional wave vector $(\epsilon_1, \dots, \epsilon_{i-1}, \epsilon_{i+1}, \dots, \epsilon_n)$ (that depends in a C^2 way on (u^l, u^r)) such that

$$F(u^l, u^r, \epsilon_1, \dots, \epsilon_{i-1}, \epsilon_{i+1}, \dots, \epsilon_n) = 0.$$

The states $\{u_k\}_{k=0}^n$ are defined as follows:

$$\begin{cases} u_0 = u^l \\ u_k = \Psi_k(\dots \Psi_2(\Psi_1(u^l, \epsilon_1), \epsilon_2) \dots, \epsilon_k) \quad \text{for } k = 1 \dots i-1 \\ u_k = \tilde{\Psi}_{k+1}(\dots \tilde{\Psi}_{n-1}(\tilde{\Psi}_n(u^r, -\epsilon_n), -\epsilon_{n-1}) \dots, -\epsilon_{k+1}) \quad k = i \dots n-1 \\ u_n = u^+. \end{cases}$$

To prove (iii), note that

$$\Lambda^i(u_{i-1}, u_i) = \frac{\langle f(u_{i-1}) - f(u_i), u_{i-1} - u_i \rangle}{|u_{i-1} - u_i|^2}. \tag{2.13}$$

Thus, by the continuity of λ_i and the condition (2.4)(ii) we actually get the stronger condition (implying (ii)):

There exists $c > 0$ such that if $(u^l, u^r), (u_1^l, u_i^r) \in \Omega^l \times \Omega^r$ and $\Psi^i(u^l, u^r) = 0$, then:

$$\begin{aligned} \lambda_i(u_1^l) - \Lambda^i(u^l, u^r) \geq c, \\ \Lambda^i(u^l, u^r) - \lambda_i(u_1^r) \geq c. \end{aligned} \tag{2.14}$$

■

Remark 2.2. From now on, we will tacitly assume that the open sets Ω^l, Ω^r where u_0^l and u_0^r belong respectively, are small enough for all the useful properties such as (2.14) (2.2) to hold. In particular, Ω^l, Ω^r are disjoint and $|u^l - u^r| \geq c$ for all $(u^l, u^r) \in \Omega^l \times \Omega^r$, with c denoting, as usual, a small positive constant.

Remark 2.3. Consider the function $\bar{F} : \Omega^l \times \Omega^r \times I^{n-1} \to \mathbf{R}^{n-1}$, defined exactly as F in the proof of Proposition 2.1 with the functions Ψ_k being replaced by $\mathcal{S}_k(u, \cdot)$ the shock curves through the appropriate states u.

Since $\mathcal{S}_k$ and Ψ_k are second order tangent,

$$\frac{\partial}{\partial(\epsilon_1, \dots, \epsilon_{i-1}, \epsilon_{i+1}, \dots, \epsilon_n)} F(u_0^l, u_0^r, 0) = \frac{\partial}{\partial(\epsilon_1, \dots, \epsilon_{i-1}, \epsilon_{i+1}, \dots, \epsilon_n)} \bar{F}(u_0^l, u_0^r, 0),$$

and as before the implicit function theorem gives us the unique solution to any Riemann problem $(u^l, u^r) \in \Omega^l \times \Omega^r$, with its middle states changing along the shock curves. This solution is not in general entropy admissible, however, since it approximates well the 'real' solution (constructed in Proposition 2.1), if Ω^l, Ω^r are small sets , we will often make use of it, each time stating clearly whether our solution follows the shocks $\mathcal{S}_k$ or the mixed curves Ψ_k.

Consider now the Riemann problem given by the states u_0^- and u_0^+. Its solution is provided by gluing together two large shocks, with u_0^m as middle state, see Figure 1.1.

Proposition 2.4. *Let the Finiteness Condition be satisfied. Then, in the above setting, every Riemann problem $(u^-, u^+) \in \Omega^- \times \Omega^+$ has a unique self similar solution composed of n shocks or rarefaction waves, connecting the states $u^- =$*

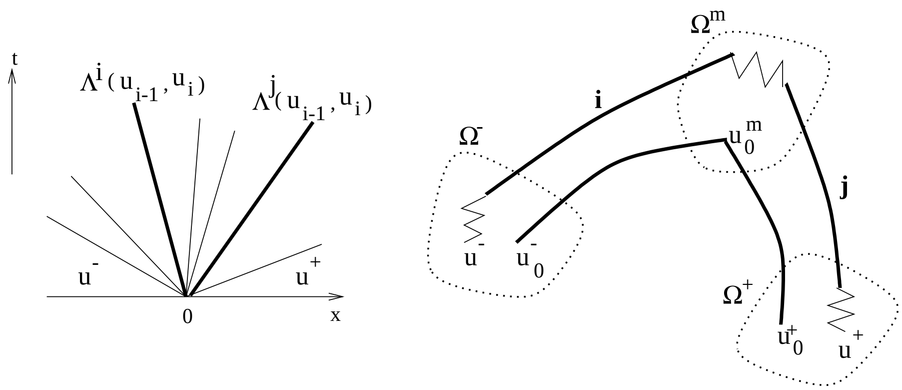

FIGURE 2.5

$u_0, \dots, u_{i-1} \in \Omega^-$, $u_i, \dots, u_{j-1} \in \Omega^m$ and $u_{j+1}, \dots, u_n = u^+ \in \Omega^+$, as in Figure 2.5. For every $k \notin \{i, j\}$, and for some small strength ϵ_k, $u_k = \Psi_k(u_{k-1}, \epsilon_k)$ with

$$\Psi^i(u_{i-1}, u_i) = 0, \quad \Psi^j(u_{j-1}, u_j) = 0$$

and

$$\lambda_i(u_{i-1}) > \Lambda^i(u_{i-1}, u_i) > \lambda_i(u_i)$$
$$\lambda_j(u_{j-1}) > \Lambda^j(u_{j-1}, u_j) > \lambda_j(u_j).$$

The speeds $\Lambda^i(u_{i-1}, u_i)$ and $\Lambda^j(u_{j-1}, u_j)$ depend in a C^2 way on (u^-, u^+).

The above proposition follows from the discussion in [Le2]. Note that, differently from the case of a single large shock, trested in Proposition 2.1, one needs an additional condition (implied by the Finiteness Condition, used to state Proposition 2.4) to guarantee the solvability of Riemann problems (u^-, u^+).

3. WAVE FRONT TRACKING APPROXIMATIONS

Given a Cauchy problem (1.1) (1.2), one of the main strategies [BJ] [BC1] [D] to obtain the existence of its (global in time) solution is the following:

(i) Approximate the initial data $\bar{u}$ by piecewise constant data $\bar{u}_\epsilon$.
(ii) Give a recipe for construction an 'approximate solution' u_ϵ to (1.1) with $u_\epsilon(0, \cdot) = \bar{u}_\epsilon$. The approximating function u_ϵ should have relatively simple structure, e.g. be piecewise constant, with finitely many jumps occuring along straight discontinuity lines.
(iii) Show that for some parameter sequence $\epsilon_n \to 0$, the sequence u_{ϵ_n} has a limit in $\mathrm{L}^1_{\mathrm{loc}}$, and that this limit is a solution to (1.1) (1.2).

This approach will be used to prove Theorem A, in Section 6. In this Section our goal is to realize (ii) by means of so-called wave front tracking algorithm [B1] [BJ] [R], that we carefully adjust to work in the presence of large shocks.

Also, as a preparation for (iii) we state and prove different regularity properties of the approximate solutions. As the basic tool we introduce the Glimm's functional, equivalent to the *T.V.* of the perturbation added to our initial Riemann data (u_0^-, u_0^+), and prove that it decreases at every time when an interaction of two waves takes place. The main features of the approximate solutions are collected in Theorem 3.5.

Riemann solvers

The 'fundamental block' for building the approximate solutions u_ϵ, as announced above is provided by the suitable piecewise constant approximation of the self-similar solution to an arbitrary Riemann problem (u^l, u^r). If both states u^l, u^r are in the same set Ω^-, Ω^m or Ω^+, then the problem (u^l, u^r) is approximately solved by the already standard Accurate or Simplified Riemann Solvers [BJ]. Their constructions depend on two positive parameters: δ which bounds the strength of the wave fronts in every rarefaction fan approximating centered rarefaction wave in the real solution, and $\hat{\lambda}$ (strictly larger than all characteristic speeds of (1.1)) that is the speed of non-physical waves, generated whenever the simplified method is used. Below we present the corresponding solvers for the 'large' Riemann problems $(u^-, u^m) \in \Omega^- \times \Omega^m$. The case $(u^m, u^+) \in \Omega^m \times \Omega^+$ is treated analogously.

Accurate Riemann Solver

Acurate Riemann Solver is the self-similar solution described in Proposition 2.1, with every rarefaction wave $(w, \mathcal{R}_k(w)(\epsilon))$ replaced by a piecewise constant rarefaction fan:

$$u(t,x) = \mathcal{R}_k(w)(s\tilde{\epsilon}) \quad \text{for } \frac{x}{t} \in \Big(\lambda_k(\mathcal{R}_k(w)(s\tilde{\epsilon})),\ \lambda_k(\mathcal{R}_k(w)((s+1)\tilde{\epsilon}))\Big) \qquad \forall s : 0 \ldots N-1$$

where $N = [\epsilon/\delta] + 1$, $\tilde{\epsilon} = \epsilon/N$.

Simplified Riemann Solver

Case 1. Let $k > i$ be the family of a small (physical) wave of strength ϵ_k^{in}, impinging from the left a large shock of the i-th family, as in Figure 3.1.

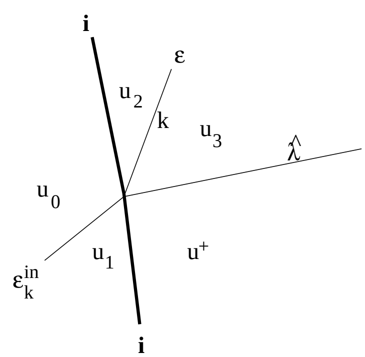

Figure 3.1

We solve the Riemann problem (u_0, u^+) in the following way:

$$\begin{cases} u_0 & \text{for } x/t < \Lambda^i(u_0, u_2) \\ u_2 & \text{for } x/t \in (\Lambda^i(u_0,u_2), \lambda_k(u_2,u_3)) \\ u_3 = \Psi_k(u_2, \epsilon) & \text{for } x/t \in (\lambda_k(u_2,u_3), \hat{\lambda}) \\ u^+ & \text{for } x/t > \hat{\lambda}. \end{cases}$$

Here the outgoing strength

$$\epsilon = \epsilon_k^{in} \cdot \frac{\partial \epsilon_k^{out}}{\partial \epsilon_k^{in}}_{|\epsilon_k^{in}=0}$$

in the solution given in Proposition 2.1, and

$$\lambda_k(u_2, u_3) = \begin{cases} \lambda_k(u_2) & \text{if } \epsilon > 0 \text{ and } k\text{-field is genuinely nonlinear} \\ \dfrac{\langle f(u_2) - f(u_3), u_2 - u_3 \rangle}{|u_2 - u_3|^2} & \text{otherwise} \end{cases}$$

is the Rankine-Hugoniot speed of the shock (or contact discontinuity) joining the states $u_2, u_3 \in \Omega^m$ if the wave ϵ was a shock or contact discontinuity itself, or a single discontinuity approximation of the rarefaction wave between the states (u_2, u_3) if the original wave ϵ is an approximated rarefaction.

The middle state u_2 is defined as follows. By (2.7)(ii), after possibly permuting the coordinates in $\mathbf{R}^n$, the matrix

$$\frac{\partial \Psi^i}{\partial((u^m)_1, \dots, (u^m)_{n-1})} (u_0^-, u_0^m)$$

is invertible (here $\{(u^m)_s\}_{s=1}^n$ are the components of the point $u^m \in \mathbf{R}^n$), so by implicit function theorem there exists a smooth function $\varphi^i : \Omega^- \times (a,b) \longrightarrow \mathbf{R}^{n-1}$ such that $\varphi^i(u_0^-, (u_0^m)_1) = \big((u_0^m)_2, \dots, (u_0^m)_n\big)$ and $\Psi^i(u^-, ((u^m)_1, \varphi^i(u^-, (u^m)_1))) = 0$. Set $u_2 = ((u^+)_1, \varphi^i(u_0, (u^+)_1)$. Then $\Psi^i(u_0, u_2) = 0$.

If the small k-wave hits the large i-shock from the right ($k < i$), we construct the approximate solver in the analogous way, letting the k-wave pass through the i-wave changing its strength by an appropriate factor, and create a new non-physical wave travelling with speed $\hat{\lambda}$.

Case 2. A big i-shock is hit by a small wave of the same family or by a non-physical discontinuity. This case is treated entirely the same as in [BJ].

Define the strength of a non-physical wave as the distance between its right and left states. We will also adopt the notation that the non-physical waves belong to a $(n+1)$th characteristic family. Moreover:

define the strength of any large wave of i-th or j-th characteristic family to be equal to some fixed number $B \leq 1$ (bigger than all strengths of small waves). (3.1)

Proposition 3.1. *(Interaction estimates)*

(i) *Let the Riemann problem $(u^-, u^m) \in \Omega^- \times \Omega^m$ be given. The sum of the strengths of small waves generated by the Accurate Riemann Solver is estimated:*

$$\sum_{k:1\dots n, k\neq i} |\epsilon_k^{out}| = O(1) \cdot (|u_0^- - u^-| + |u_0^m - u^m|).$$

(ii) *Assume the large i-wave interacts with a small (possibly non-physical) wave having strength ϵ^{in}. Then all the outgoing small waves generated by Accurate or Simplified Riemann Solvers have their strengths estimated:*

$$\sum_{k:1\ldots n+1, k\neq i} |\epsilon_k^{out}| = O(1) \cdot |\epsilon^{in}|.$$

Here and in the sequel, with the Landau symbol $O(1)$ we denote a quantity whose absolute value is uniformly bounded quantity, depending only on the system (1.1) or the a-priori bounds on the initial data such as diameters of sets Ω^-, Ω^m or the constant δ_0 appearing in (2.12). The proof of Proposition 3.1 follows using Taylor expansions.

Glimm's functionals

Once the steps of the algorithm have been defined (as they are entirely the same as in [BJ], we omit the details), one needs to prove that it generates an approximate solution, defined globally in time. To this end, we will derive suitable a priori bounds on the Total Variation of approximate profiles, bounds on the global number of wave fronts and the total strength of all non-physical waves. As in the case of only weak shocks present [B1] [BJ] [R], this will be done by introducing a suitable wave interaction potential [G].

Let $u(t,x)$ be a piecewise constant approximate solution, generated by the wave front tracking algorithm. At a fixed time $t \geq 0$, $u(t,\cdot)$ is piecewise constant and its jump points are located at the intersections of the wave fronts in u with the line $\{(t,x);\ x \in \mathbf{R}\}$. If $t > 0$ then precisely two jumps are large: the first belonging to the i-th, second to the j-th characteristic family.

Definition 3.2. *(Approaching waves)*

(i) *We say that two small (possibly non-physical) fronts α and β, located at points $x_\alpha < x_\beta$ and belonging to the characteristic families $k_\alpha, k_\beta \in \{1 \ldots n+1\}$ respectively, approach each other iff the following two conditions hold simultaneously:*
   - *x_α and x_β both lay in one of the three intervals (two of them unbounded) into which $\mathbf{R}$ is partitioned by the locations of large i and j-shocks. In other words: the states joined by the fronts under consideration both belong to the same set Ω^-, Ω^m or Ω^+.*
   - *Either $k_\alpha < k_\beta$ or else $k_\alpha = k_\beta$ and at least one of the waves is a genuinely nonlinear shock.*

   In this case we write: $(\alpha, \beta) \in \mathcal{A}$.

(ii) *We say that a small wave α of the characteristic family $k_\alpha \in \{1, \ldots, n+1\}$ located at x_α is approaching a large shock of family $k_\beta \in \{i, j\}$, located at a point x_β iff either $k_\alpha \leq k_\beta$ and $x_\alpha > x_\beta$ or $k_\alpha \geq k_\beta$ and $x_\alpha < x_\beta$. We then write: $\alpha \in \mathcal{A}_{k_\beta}$.*

We adopt the following notation. Assume that we are given three sets of positive numbers $\{w_k^-\}_{k=1}^{n+1}, \{w_k^m\}_{k=1}^{n+1}$ and $\{w_k^+\}_{k=1}^{n+1}$. For a small wave of family $k \in \{1, \ldots, n+1\}$ and strength ϵ_k, that connects two states u_1 and u_2 we define its weighted strength as:

$$b_k = \begin{cases} w_k^- \cdot \epsilon_k & \text{if } u_1, u_2 \in \Omega^- \\ w_k^m \cdot \epsilon_k & \text{if } u_1, u_2 \in \Omega^m \\ w_k^+ \cdot \epsilon_k & \text{if } u_1, u_2 \in \Omega^+. \end{cases} \tag{3.2}$$

b_k can be interpreted as strength of the wave under consideration, computed along the reparametrized curve $\Psi_k(u_1, \cdot)$. The reparametrization ratio is equal to the weight w_k^- in Ω^-, w_k^m in Ω^m, w_k^- if $u_1 \in \Omega^+$.

Let x_α, $\alpha : 1 \dots N$ be the locations of the fronts in $u(t, \cdot)$. By ϵ_α (b_α) we denote the strength (weighted strength) of the wave front at x_α.

Definition 3.3. *Let $t > 0$. The total (weighted) strength of waves in $u(t, \cdot)$ is defined by:*

$$V(t) = \sum_\alpha |b_\alpha|,$$

where the summation ranges over all small wave fronts of all families. The (weighted) wave interaction potentials:

$$Q_{\mathcal{A}}(t) = \sum_{(\alpha,\beta)\in\mathcal{A}} |b_\alpha b_\beta|,$$

$$Q_i(t) = \sum_{\alpha\in\mathcal{A}_i} |b_\alpha|, \quad Q_j(t) = \sum_{\alpha\in\mathcal{A}_j} |b_\alpha|,$$

$$Q(t) = \kappa Q_{\mathcal{A}}(t) + Q_i(t) + Q_j(t).$$

The Glimm functional:

$$\Gamma(t) = V(t) + \tilde{\kappa} Q(t) + |u^*(t) - u_0^m|,$$

where $\kappa, \tilde{\kappa} > 0$ are constants to be specified later. The vector $u^*(t)$ is the right state of the first left (i-th) large shock, at the time t.

Note that V and Q (and thus Γ) are constant between any pair of subsequent interaction times. On the other hand, across an interaction time both Q and Γ decrease, as shown in the next Proposition.

Proposition 3.4. *Assume that the Finiteness Condition holds. There exist weights $\{w_k^-\}, \{w_k^m\}, \{w_k^+\}$ in (3.2), constants $\kappa, \tilde{\kappa} > 0$ and $\delta > 0$ such that the following holds. Let $u(0, \cdot) : \mathbf{R} \longrightarrow \mathbf{R}^n$ be such that:*

- $\lim_{x\to-\infty} u(0, x) = u_0^-$, $\quad \lim_{x\to\infty} u(0, x) = u_0^+$,
- *there exist points $x^i < x^j$ in $\mathbf{R}$ such that*

$$u(0, x) \in \begin{cases} \Omega^- & \text{for } x < x^i \\ \Omega^m & \text{for } x^i < x < x^j \\ \Omega^+ & \text{for } x > x^j. \end{cases} \tag{3.3}$$

If $T.V.(u(0, \cdot) - \tilde{u}) < \delta$ (for some $\tilde{u}$ as in (2.12)), then for any $t > 0$ when two wave fronts of families α and β interact we have:

(i)

$$\Delta Q(t) = Q(t+) - Q(t-) \leq \begin{cases} -c|b_\alpha \cdot b_\beta| & \textit{if both waves are small} \\ -c|b_\alpha| & \textit{if } \alpha \textit{ wave is small and } \beta \textit{ is a large shock.} \end{cases}$$

(ii) *The same estimate as in (i) above holds for* $\Delta\Gamma(t) = \Gamma(t+) - \Gamma(t-)$.

The number c *is some small positive, uniform constant.*

Proof. For $k : 1 \dots n$ set $w_k^m = w_k$ from the Finiteness Condition. Let $t > 0$ be a fixed time of interaction of two waves (one of them possibly large or non-physical). By standard estimates in [Sm] and Proposition (3.1)(ii) we receive the following estimates on the change in Q across the time t, in terms of the strengths $\epsilon_\alpha, \epsilon_\beta$ of interacting waves:

	$\Delta Q_{\mathcal{A}} \leq$	$\Delta Q_i \leq$	$\Delta Q_j \leq$
case I: ϵ_α small ϵ_β small	$-\lvert b_\alpha b_\beta\rvert +$ $O(1)V(t-)\lvert b_\alpha b_\beta\rvert$	$O(1)\lvert b_\alpha b_\beta\rvert$	$O(1)\lvert b_\alpha b_\beta\rvert$
case II: $\beta = i$ ϵ_α small ϵ_β large	$O(1)V(t-)\lvert b_\alpha\rvert$	$-\lvert b_\alpha\rvert$	$\begin{cases} \sum_{k:j\dots n} \lvert b_k^{out}\rvert & \text{if ARS} \\ \lvert b_{n+1}^{out}\rvert & \text{if SRS} \end{cases}$
case III: $\beta = j$ ϵ_α small ϵ_β large	$O(1)V(t-)\lvert b_\alpha\rvert$	$\begin{cases} \sum_{k:1\dots i} \lvert b_k^{out}\rvert & \text{if ARS} \\ 0 & \text{if SRS} \end{cases}$	$-\lvert b_\alpha\rvert$

Here $\{b_k^{out}\}$ are, as usual, the reparametrised strengths of the outgoing waves, in the interaction under consideration. For clarity, denote the biggest of the uniform constants playing role in the above estimates by C.

Then, if $V(t-) \leq 1/2C$ and $\kappa \geq 6C$, one sees that (i) in case I is satisfied with $c = C$. To treat cases II and III note that if we set w_k^- for $k \geq i$ and w_k^+ for $k \leq j$ to be big enough (relatively to the other weights) then in view of Proposition 3.1(ii) we get (i) with $c = 1/4$ in the following two cases:

- Case II when a small α wave interacts with the large i shock from the left.
- Case III when a small α wave interacts with the large j shock from the right.

provided that $V(t-) \leq 1/2\kappa C$.

To treat the remaining cases, the Finiteness Condition must be used. Fix w_{n+1}^m so small that (2.9) holds with $k = n+1$ and $w_k = w_{n+1}^m$. This is possible by Proposition 3.1(ii). Let θ be as in Finiteness Condition (2.8) (2.9). By (2.8) and (2.9), if

$$V(t-) \leq \frac{1-\theta}{2\kappa C},$$

then (i) is satisfied with the constant $c = (1-\theta)/2 > 0$. Also, note that

$$\Delta V(t) = V(t+) - V(t-) \leq \begin{cases} C|b_\alpha b_\beta| & \text{in case I} \\ C|b_\alpha| & \text{in cases II and III,} \end{cases}$$

by [B1] and Proposition 3.1(ii).

Let us now estimate the third summand in the definition of $\Gamma(t)$. Obviously, in cases I and III $u^*(t-) = u^*(t+)$, so the component $|u^*(t) - u_0^m|$ does not change across the interaction time t. In case II $|u^*(t+) - u^*(t-)| = O(1)|b_\alpha|$ by Proposition 3.1(ii). Thus, if $\tilde{\kappa}$ is big enough, we get both estimates (i) and (ii), provided that

$$V(t-) \leq \tilde{\delta} = \min\left\{\frac{1}{2C}, \frac{1}{2\kappa C}, \frac{1-\theta}{2\kappa C}\right\}.$$

Note now that:

$$\begin{aligned} V(t-) &\leq \Gamma(t-) \leq \Gamma(0+) = V(0+) + \tilde{\kappa}Q(0+) + |u^*(0+) - u_0^m| \\ &\leq C_1 \cdot T.V.(u(0+,\cdot) - \widetilde{u}) + \tilde{\kappa}\Big\{\kappa C_1 \cdot [T.V.(u(0+,\cdot) - \widetilde{u})]^2 \\ &\qquad\qquad + 2C_1 \cdot T.V.(u(0+,\cdot) - \widetilde{u})\Big\}, \end{aligned} \tag{3.4}$$

where C_1 is a uniform positive constant depending on the curvature of $\{\Psi_k(u,\cdot\}_{k=1}^n$ as well as on their parametrisation, given by the weights $\{w_k^-, w_k^m, w_k^+\}_{k=1}^n$. By Proposition 3.1(i) $T.V.(u(0+,\cdot)-\tilde{u}) = O(1)\cdot T.V.(u(0,\cdot)-\tilde{u})$ thus in view of (3.4), if the constant δ is small enough, the inequality $T.V.(u(0,\cdot)-\tilde{u}) < \delta$ implies $V(t-) < \tilde{\delta}$ and the result follows. ■

As in the case without the presence of large waves [BJ], Proposition 3.4 results in the following assertions. If $u(0,\cdot)$ satisfies the assumptions of Proposition 3.4, then our wave front tracking algorithm generates a piecewise constant approximate solution (that has finitely many discontinuity lines) $u(t,\cdot)$ for all $t \in [0,\infty)$. Moreover, the functional Γ computed for $u(t,\cdot)$ is nonincreasing in time, and in particular (by (3.4)) we get:

$$\begin{aligned} &\Gamma(t) \leq \Gamma(0+), \\ &T.V.(u(t,\cdot) - \hat{u}) = O(1)\cdot\Gamma(t) = O(1)\cdot T.V.(u(t,\cdot) - \widetilde{u}) \end{aligned} \tag{3.5}$$

for some $\hat{u}$ as $\widetilde{u}$ in (2.12).

The total strength of all non-physical waves occuring at any fixed time $t > 0$ is of the order $O(1)(\rho + \delta)$.

Following [BLY] below we gather all the main properties of the wave front tracking approximate solutions.

Theorem 3.5. *Assume that a piecewise constant function $u(0,\cdot)$ satisfies the assumptions of Proposition 3.4. Fix $\epsilon > 0$. Then for some parameters $\rho, \delta > 0$ the corresponding wave front tracking algorithm produces the function $u : [0,\infty)\times\mathbf{R} \longrightarrow \mathbf{R}^n$, such that:*

(i) *u is piecewise constant function, with discontinuities occuring along finitely many lines in the $t-x$ plane. Only finitely many interactions take place, each involving exactly two incoming fronts. Jumps can be of four types: small shocks (or contact discontinuities), rarefactions, non-physical waves and large shocks, denoted as $\mathcal{J} = \mathcal{S}\cup\mathcal{R}\cup\mathcal{NP}\cup\mathcal{LS}$.*

(ii) *Along each shock (or contact discontinuity) $x = x_\alpha$, $\alpha\in\mathcal{S}$, its left and right states satisfy $u(t,x_\alpha+) = \Psi_{k_\alpha}(u(t,x_\alpha-),\epsilon_\alpha)$ for some $k_\alpha : 1\ldots n$ and the corresponding wave strength ϵ_α. If the k_α characteristic family is genuinely nonlinear, then $\epsilon_\alpha < 0$. Moreover, the speed of the shock satisfies:*

$$|\dot{x}_\alpha - \lambda_{k_\alpha}(u(t,x_\alpha-), u(t,x_\alpha+))| \leq \epsilon. \tag{3.6}$$

(iii) *Along each rarefaction front* $x = x_\alpha$, $\alpha \in \mathcal{R}$, *one has* $u(t, x_\alpha+) = \Psi_{k_\alpha}(u(t, x_\alpha-), \epsilon_\alpha)$ *for some genuinely nonlinear family* k_α *and the corresponding wave strength* $\epsilon_\alpha \in (0, \epsilon)$. *Moreover:*

$$|\dot{x}_\alpha - \lambda_{k_\alpha}(u(t, x_\alpha+)| \leq \epsilon. \tag{3.7}$$

(iv) *Every non-physical front* $x = x_\alpha$, $\alpha \in \mathcal{NP}$, *has the same speed* $\dot{x}_\alpha = \hat{\lambda}$, *where* $\hat{\lambda}$ *is a fixed constant strictly greater than all characterictic speeds. The total strength of all non-physical waves in* $u(t, \cdot)$ *remains uniformly small:*

$$\sum_{\alpha \in \mathcal{NP}} |u(t, x_\alpha-) - u(t, x_\alpha+)| \leq \epsilon \quad \forall t \geq 0. \tag{3.8}$$

(v) *The two large shocks,* $x = x_\alpha$, $\alpha \in \mathcal{LS}$, *belonging to the families* i *and* j *satisfy* $\Psi^{k_\alpha}(u(t, x_\alpha-), u(t, x_\alpha+)) = 0$, $k_\alpha \in \{i, j\}$ *and travel with the exact speed* $\dot{x}_\alpha = \Lambda^{k_\alpha}(u(t, x_\alpha-), u(t, x_\alpha+))$.

The function u *as above will be called an* ϵ*-approximate solution of (1.1).*

4. The Lyapunov functional

As announced in Sections 1 and 2, the trajectories of the semigroup $\mathcal{S}$ of solutions to (1.1) will be constructed by means of the wave front tracking algorithm described in Section 3, and a usual limiting process, to be described in Section 6. To give some more insight in the actual behaviour of the limiting process (in particular, it is going to appear that any sequence of approximate wave front tracking solutions $u_\epsilon(\cdot, \cdot)$ converges to a weak solution of the original Cauchy problem (1.1) (1.2), when $u_\epsilon(0, \cdot)$ converges to $\bar{u}$), and show that $\mathcal{S}$ is continuous (thus giving the positive answer to question B on Section 1), we follow the approach of [BLY] [LY1] [LY2] based on the construction of a suitable Lyapunov functional with the following properties:

$$\frac{1}{C} \cdot \parallel u(t, \cdot) - v(t, \cdot) \parallel_{\mathrm{L}^1} \leq \Phi(u(t, \cdot), v(t, \cdot)) \leq C \cdot \parallel u(t, \cdot) - v(t, \cdot) \parallel_{\mathrm{L}^1}, \tag{4.1}$$

$$\Phi(u(t, \cdot), v(t, \cdot)) - \Phi(u(s, \cdot), v(s, \cdot)) = O(1) \cdot \epsilon \cdot (t - s) \quad \forall t > s \geq 0, \tag{4.2}$$

satisfied for any two ϵ-approximate solutions u and v.

The formula (4.1) claims that Φ is equivalent to the L^1 distance within the set of piecewise constant functions with (3.3). The formula (4.2) says that Φ is 'almost decreasing' in time. These two formulas imply in particular that the flow of piecewise constant ϵ-approximate solutions is 'almost Lipschitz' and the error is of the order ϵ. As we will see in Section 6, this guarantees that the exact flow $\mathcal{S}$ is Lipschitz continuous, as announced in Theorem B.

Let $u, v : [0, \infty) \times \mathbf{R} \longrightarrow \mathbf{R}^n$ be two ϵ-approximate solutions, with the properties given in Theorem 3.5. Fix a time $t > 0$ and consider a space point $x \in \mathbf{R}$, which is not a discontinuity point of the functions $u = u(t, \cdot)$, $v = v(t, \cdot)$. We define the scalar quantities $\{q_k(x)\}_{k=1}^n$, as the weighted strengths of the corresponding shock waves in the jump $(u(x), v(x))$. More precisely, we consider the Riemann data (z^-, z^+), where:

$$(z^-, z^+) = \begin{cases} (u(x), v(x)) & \text{if } (u(x), v(x)) \in (\Omega^- \times \Omega^-) \cup (\Omega^- \times \Omega^m) \cup (\Omega^- \times \Omega^+) \\ & \qquad \cup (\Omega^m \times \Omega^m) \cup (\Omega^m \times \Omega^+) \cup (\Omega^+ \times \Omega^+) \\ (v(x), u(x)) & \text{if } (u(x), v(x)) \in (\Omega^m \times \Omega^-) \cup (\Omega^+ \times \Omega^-) \cup (\Omega^+ \times \Omega^m). \end{cases} \tag{4.3}$$

By Proposition 2.1, Proposition 2.4 and Remark 2.3 the above Riemann problem (z^-, z^+) has a unique self-similar solution, following the shock curves $\mathcal{S}_k$. The weighted strengths of the waves in this solution are to be called $q_k(x)$. They are defined as in formula (3.2), with weights $\{\tilde{w}_k\}$ given in Stability Condition, replacing weights $\{w_k\}$ from Finiteness Condition. In particular, if for example $(u(x), v(x)) \in (\Omega^- \times \Omega^-)$, then for every $k : 1 \dots n$ we have $q_k(x) = \tilde{w}_k^- \cdot \epsilon_k(x)$ where the strengths $\{\epsilon_k(x)\}_{k=1}^n$ are implicitely defined by:

$$v(x) = \mathcal{S}_n(\dots, \mathcal{S}_1(u(x), \epsilon_1(x)), \dots, \epsilon_n(x)).$$

Note that due to (4.3), the locations of large shocks in u and v divide $\mathbf{R}$ into five intervals (two of them unbounded) where the distance between $u(x)$ and $v(x)$ is computed along shocks in possibly different 'directions': either from $u(x)$ to $v(x)$ or from $v(x)$ to $u(x)$ (see Figure 4.1). In Figure 4.1 we depict all possible configurations of the positions of large shocks in u and v and give names to their distinct 'types'. This notation is going to be used in Section 5, where we treat different cases to receive bounds on the 'local' increase of Φ, as we describe below.

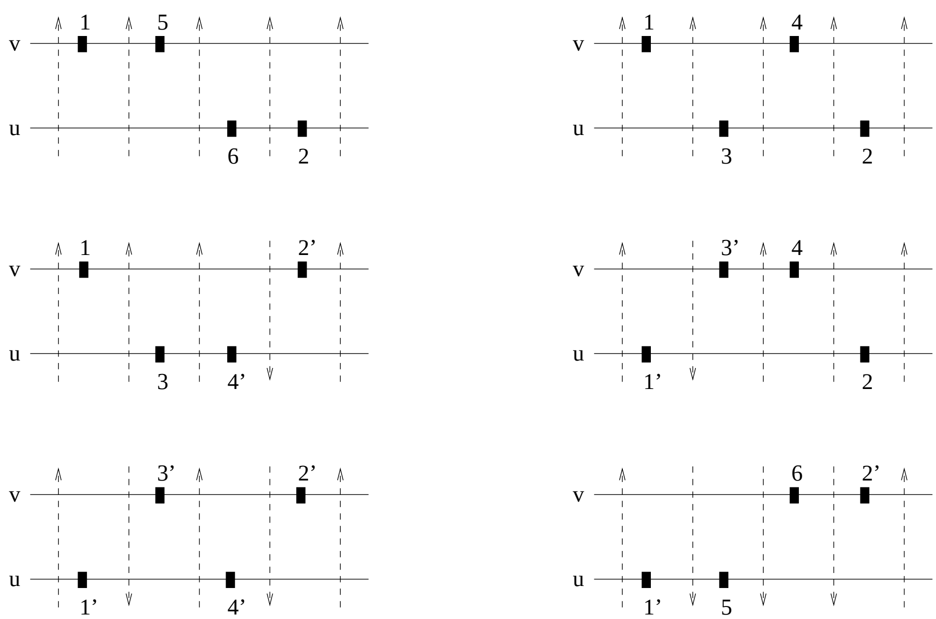

FIGURE 4.1

We define the functional:

$$\Phi(u,v) := \sum_{k=1}^{n} \int_{-\infty}^{\infty} |q_k(x)| W_k(x) dx, \tag{4.4}$$

where the weights are given by:

$$W_k(x) := 1 + \kappa_1 A_k(x) + \kappa_2 [Q(u) + Q(v)]. \tag{4.5}$$

The constants κ_1, κ_2 are to be defined later. Q is our Glimm's interaction potential, introduced in Definition 3.3. The amount of waves in u and v, which approach the wave $\epsilon_k(x)$ is defined in the following way (for x that is not a location of a jump in u or v):

$$A_k(x) = B_k(x) + \tag{4.6}$$
$$\begin{cases} C_k(x) & \text{if } k\text{-field is genuinely nonlinear and } k\text{-wave } q_k(x) \text{ is small} \\ F_k(x) & \text{if } k = i \text{ and } k\text{-wave is large } q_k(x) = B \\ G_k(x) & \text{if } k = j \text{ and } k\text{-wave is large } q_k(x) = B \\ 0 & \text{otherwise} \end{cases}$$

Here the notions 'small' or 'large' describe waves that connect states in the same or in distinct domains $\Omega^-, \Omega^m, \Omega^+$, respectively.

The summands in (4.6) are the following:

$$B_k(x) = \left[\sum_{\substack{\alpha \in \mathcal{J}, \\ x_\alpha < x,\ k < k_\alpha \le n}} + \sum_{\substack{\alpha \in \mathcal{J}, \\ x_\alpha > x,\ 1 \le k_\alpha < k}} \right] |\epsilon_\alpha|,$$

$$C_k(x) = \begin{cases} \left[\sum_{\substack{\alpha \in \mathcal{J}(u) \setminus \mathcal{LS}, \\ x_\alpha < x,\ k_\alpha = k}} + \sum_{\substack{\alpha \in \mathcal{J}(v) \setminus \mathcal{LS}, \\ x_\alpha > x,\ k_\alpha = k}} \right] |\epsilon_\alpha| & \text{if } q_k(x) < 0 \\ \\ \left[\sum_{\substack{\alpha \in \mathcal{J}(v) \setminus \mathcal{LS}, \\ x_\alpha < x,\ k_\alpha = k}} + \sum_{\substack{\alpha \in \mathcal{J}(u) \setminus \mathcal{LS}, \\ x_\alpha > x,\ k_\alpha = k}} \right] |\epsilon_\alpha| & \text{if } q_k(x) > 0 \end{cases}$$
$$+ \begin{cases} 2B & \text{if } k \in \{i,j\} \\ 0 & \text{otherwise,} \end{cases}$$

$$F_k(x) = \left[\sum_{\substack{\alpha \in \mathcal{J} \setminus \mathcal{LS},\ x_\alpha < x,\ k_\alpha = i, \\ \text{both states joined by } \alpha \\ \text{are located in } \Omega^-}} + \sum_{\substack{\alpha \in \mathcal{J} \setminus \mathcal{LS},\ x_\alpha > x,\ k_\alpha = i, \\ \text{both states joined by } \alpha \\ \text{are located in } \Omega^m}} \right] |\epsilon_\alpha|,$$

$$G_k(x) = \left[\sum_{\substack{\alpha \in \mathcal{J} \setminus \mathcal{LS},\ x_\alpha < x,\ k_\alpha = j, \\ \text{both states joined by } \alpha \\ \text{are located in } \Omega^m}} + \sum_{\substack{\alpha \in \mathcal{J} \setminus \mathcal{LS},\ x_\alpha > x,\ k_\alpha = j, \\ \text{both states joined by } \alpha \\ \text{are located in } \Omega^+}} \right] |\epsilon_\alpha|.$$

Here ϵ_α stands for the (nonweighted) strength of the wave $\alpha \in \mathcal{J}$, located at point x_α and belonging to the characteristic family k_α. $\mathcal{J} = \mathcal{J}(u) \cup \mathcal{J}(v)$, $\mathcal{LS} = \mathcal{LS}(u) \cup \mathcal{LS}(v)$ stand for the set of all waves (in u and v) and the set of all large shocks (in u and v) respectively, as introduced in Theorem 3.5.

We comment briefly on the formula (4.6). The presence of the first summand simply says that any wave of a faster family, located to the left of x and any wave of a slower family, located to the right of x, approaches the wave under consideration. Only physical waves are involved, no matter if they are large or small.

The first term of the summand C is identical to the one in the corresponding definition of $A_k(x)$ in [BLY] and it accounts for waves in u and v of the same k-th family. Only small physical waves are considered. The second term (containing $2B$ or 0) mirrors the convention that a small i (j) wave is always approached by any large i (j) wave in u or v, no matter where it is located. This convention is justified by assumed Lax stability of large shocks ((2.2) (2.4) and Proposition 2.1(iii)).

The last two summands in F and G are also connected with the Lax stability of large shocks and say that a large k shock is approached by small k waves in $\mathcal{J}(u) \cup \mathcal{J}(v)$ if they have bigger speed and are located to the left of x or have smaller speed and are located to the right of x.

We now examine how the functional Φ evolves in time. For $k : 1 \dots n$ call $\lambda_k(x)$ the speed of the k-wave $\epsilon_k(x)$ in the solution of the Riemann problem (4.3) (along the shock waves). At a time t which is not the interaction time of the waves in $u(t, \cdot) = u(t)$ or $v(t, \cdot) = v(t)$ a direct computation yields:

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\ \Phi(u(t), v(t)) \\ &\quad = \sum_{\alpha\in\mathcal{J}} \sum_{k=1}^{n} \{|q_k(x_\alpha-)|W_k(x_\alpha-) - |q_k(x_\alpha+)|W_k(x_\alpha+)\} \cdot \dot{x}_\alpha \\ &\quad = \sum_{\alpha\in\mathcal{J}} \sum_{k=1}^{n} \{|q_k(x_\alpha+)|W_k(x_\alpha+)(\lambda_k(x_\alpha+) - \dot{x}_\alpha) - \\ &\qquad\qquad |q_k(x_\alpha-)|W_k(x_\alpha-)(\lambda_k(x_\alpha-) - \dot{x}_\alpha)\} \cdot \dot{x}_\alpha, \end{aligned} \tag{4.7}$$

where $\dot{x}_\alpha$ is the speed of the discontinuity at the $\alpha \in \mathcal{J}$ wave. We introduce the notation:

$$E_{\alpha,k} = |q_k^{\alpha+}|W_k^{\alpha+}(\lambda_k^{\alpha+} - \dot{x}_\alpha) - |q_k^{\alpha-}|W_k^{\alpha-}(\lambda_k^{\alpha-} - \dot{x}_\alpha), \tag{4.8}$$

where $q_k^{\alpha+} = q_k(x_\alpha+)$, $\lambda_k^{\alpha+} = \lambda_k(x_\alpha+)$ and so on. Then (4.7) becomes:

$$\frac{\mathrm{d}}{\mathrm{d}t}\ \Phi(u(t), v(t)) = \sum_{\alpha\in\mathcal{J}} \sum_{k=1}^{n} E_{\alpha,k}. \tag{4.9}$$

Our main goal will be to establish the bounds:

$$\sum_{k=1}^{n} E_{\alpha,k} = O(1) \cdot |\epsilon_\alpha| \qquad \forall \alpha \in \mathcal{C}, \tag{4.10}$$

$$\sum_{k=1}^{n} E_{\alpha,k} \leq 0 \qquad \forall \alpha \in \mathcal{LS}, \tag{4.11}$$

$$\sum_{k=1}^{n} E_{\alpha,k} = O(1) \cdot\ \epsilon\ |\epsilon_\alpha| \qquad \forall \alpha \in \mathcal{R} \cup \mathcal{S}, \tag{4.12}$$

If t is an interaction time of two fronts in u or v then all weights $W_k(x)$ decrease across time t. (4.13)

We remark that the quantities denoted by the Landau symbol $O(1)$ are also independent of the constants κ_1, κ_2.

From (4.10) (4.11) (4.12), recalling (3.8) and the uniform bound on the total strenghs of waves (3.5) we get:

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\Phi(u(t),v(t)) = O(1)\cdot\epsilon.$$

Integrating the above formula in t and combining it with (4.13) (that in turn yields the decrease of Φ across the interaction times) we prove (4.2).

In the remaining part of this Section we briefly discuss the easy to obtain estimates (4.1) and (4.13). To prove (4.1) note that once κ_1 and κ_2 are set, the weights $W_k(x)$ satisfy the bounds:

$$1 \le W_k(x) \le W, \tag{4.14}$$

for some uniform constant W. Since one can also assume that:

$$\frac{1}{W}\,|v(x)-u(x)| \le \sum_{k=1}^{n} |q_k(x)| \le W\,|v(x)-u(x)|,$$

by (4.4), (4.1) becomes obvious. For completeness, let us state at this point that no matter how big κ_1 and κ_2 are, we can always assume that

$$1+\kappa_1 A_k(x)+\kappa_2(Q(u)+Q(v)) \le 2 \tag{4.15}$$

by shrinking the sets Ω^-, Ω^m, Ω^+ and uniformly adjusting the weights w_k.

To get (4.13) it is enough to recall Proposition 3.4(i) and Proposition 3.1(ii). One sees that the result holds if $\kappa_2 >> \kappa_1$ in (4.5).

5. Stability of approximate solutions

The goal of this Section is to prove the estimates (4.10) (4.11) and (4.12). We will treat them separately. Since always a particular, α-wave is under consideration, no ambiguity arises if we drop α in the subscripts and superscripts of the formulae (4.8) – (4.11). Recall that by q_k we denote the weighted strengths of k-family waves, while ϵ_k stands for their corresponding unweighted strengths, $q_k = \tilde{w}_k \cdot \epsilon_k$.

Case of non-physical waves – the estimate (4.10)

This estimate is obtained exactly as the corresponding one in [BLY]. Since the weighted wave vector $(q_1, \dots, q_n)$, solving the Riemann discontinuity (z^-, z^+), as described in Section 4, depends Lipschitz continuously on both z^- and z^+, using the notation of Section 4 we get:

$$\begin{aligned} q_k^+ - q_k^- &= O(1)\cdot|\epsilon_\alpha|,\\ \lambda_k^+ - \lambda_k^- &= O(1)\cdot|\epsilon_\alpha|. \end{aligned} \tag{5.1}$$

By the finite propagation speed of the system, (4.14) and (5.1), we see that:

$$\begin{aligned}\sum_{k=1}^n E_k = \sum_{k=1}^n \big[W_k^+(|q_k^+| - |q_k^-|)(\lambda_k^+ - \dot{x}_\alpha) + \\ (W_k^+ - W_k^-)|q_k^-|(\lambda_k^+ - \dot{x}_\alpha) + W_k^-|q_k^-|(\lambda_k^+ - \lambda_k^-)\big] \\ = O(1)\cdot|\epsilon_\alpha| + O(1)\cdot(W_k^+ - W_k^-)|q_k^-|.\end{aligned} \tag{5.2}$$

Note that if $q_k^+q_k^- > 0$ (case $q_k^+ = q_k^- = B$ included) then by definition (4.6) $W_k^+ = W_k^-$. On the other hand if $q_k^+q_k^- \leq 0$ then by (5.1) $|q_k^-| = O(1)|\epsilon_\alpha|$. In both cases (5.2) implies (4.10).

Case of large shocks – the estimate (4.11)

Before presenting the appropriate computations, we collect below some straightforward estimates that will be used in the sequel.

Lemma 5.1. *Consider the wave pattern as in Figure 5.2 b). Then*

(i) $|v^- - \mathcal{S}_{i-1}(\epsilon_{i-1}^+ \dots (\mathcal{S}_1(\epsilon_1^+, v^-)\dots)| + |v^+ - \widetilde{\mathcal{S}}_{i+1}(-\epsilon_{i+1}^+ \dots (\widetilde{\mathcal{S}}_n(-\epsilon_n^+, v^+)\dots)| = O(1)\cdot\sum_{k\geq i}|\epsilon_k^-|$.

(ii) $|\lambda_i^+ - \dot{x}_\alpha| = O(1)\cdot\sum_{k\geq i}|\epsilon_k^-|$.

(iii) $\sum_{k<i}|\epsilon_k^-| = O(1)\cdot\left[\sum_{k<i}|\epsilon_k^+| + \sum_{k\geq i}|\epsilon_k^-|\right]$.

(iv) $\sum_{k>i}|\epsilon_k^+| = O(1)\cdot\sum_{k\geq i}|\epsilon_k^-|$.

Proof. The analysis performed for the definition of the Simplified Riemann Solver (Case 1) in Section 3 has proved that the 'basic' shock stability condition (2.7) implies that for a fixed left state $u^- \in \Omega^-$ there exists a smooth curve $\mathcal{BS}_i(\cdot, u^-)$ in Ω^m of the right states that can be connected to u^- by an admissible big i-shock.

Using the notation as in Figure 5.2 b), define the mapping:

$$\begin{aligned}G(u^-, \epsilon_1^-, \dots, \epsilon_n^-, \epsilon) &= \mathcal{S}_n(\epsilon_n^-, \dots \mathcal{S}_1(\epsilon_1^-, u^-)\dots) - \mathcal{S}_{i-1}(\epsilon_{i-1}^+, \dots \mathcal{S}_1(\epsilon_1^-, u^-)\dots) \\ &= v^- - \mathcal{S}_{i-1}(\epsilon_{i-1}^+ \dots \mathcal{S}_1(\epsilon_1^+, v^-)\dots),\end{aligned}$$

where $(\epsilon_1^+ \dots \epsilon_{i-1}^+, \epsilon_{i+1}^+ \dots \epsilon_n^+)$ constitute the usual wave vector solving the Riemann data $(u^-, \mathcal{BS}_i(\epsilon, v^-)$, as in Proposition 2.1. Obviously

$$G(u^-, \epsilon_1^-, \dots, \epsilon_{i-1}^-, 0\dots 0, \epsilon) = 0$$

and thus, by Lipschitz continuity of G we get $|v^- - \mathcal{S}_{i-1}(\epsilon_{i-1}^+ \dots (\mathcal{S}_1(\epsilon_1^+, v^-)\dots)| = O(1)\cdot\sum_{k\geq i}|\epsilon_k^-|$. The estimate $|v^+ - \widetilde{\mathcal{S}}_{i+1}(-\epsilon_{i+1}^+ \dots (\widetilde{\mathcal{S}}_n(-\epsilon_n^+, v^+)\dots)| = O(1)\cdot\sum_{k\geq i}|\epsilon_k^-|$ is proved in the same way.

The estimate (ii) is an immediate consequence of (i) and Proposition 2.1(iii). (iii) and (iv) are proved exactly the same as (i). ■

Remark 5.2. Obviously, the analogous estimates hold with the wave pattern as in Figures 5.4 b), 5.6 b) and 5.8 b).

Lemma 5.3. *Consider the wave scheme as in Figure 5.10 b). Then:*

(i) $|\mathcal{S}_{i-1}(\epsilon^-_{i-1},\dots(\mathcal{S}_1(\epsilon^-_1,u^-)\dots) - \mathcal{S}_{i-1}(\epsilon^+_{i-1},\dots(\mathcal{S}_1(\epsilon^+_1,u^+)\dots)| +$
$|\widetilde{\mathcal{S}}_{i+1}(-\epsilon^-_{i+1},\dots(\widetilde{\mathcal{S}}_n(-\epsilon^-_n,v^-)\dots) - \widetilde{\mathcal{S}}_{i+1}(-\epsilon^+_{i+1},\dots(\widetilde{\mathcal{S}}_{j-1}(-\epsilon^+_{j-1},w)\dots)| +$
$|v^- - w| + |v^+ - \widetilde{\mathcal{S}}_{j+1}(-\epsilon^+_{j+1},\dots(\widetilde{\mathcal{S}}_n(-\epsilon^+_n,v^+)\dots)| =$
$O(1)\cdot\sum_{k\geq j}|\epsilon^-_k|$.

(ii) $|\lambda^-_i - \lambda^+_i| + |\lambda^+_j - \dot{x}_\alpha| = O(1)\cdot\sum_{k\geq j}|\epsilon^-_k|$.

Proof. The statement (i) is proved exactly as (i) in Lemma 5.1. (ii) follows from (i) and Proposition 2.4. ■

Remark 5.4. The analogous estimates hold for Figure 5.11 b).

The next lemma is a consequence of stability conditions (2.10) and (2.11).

Lemma 5.5. (i) *Consider the wave scheme as in Figure 5.6 b). There exists a constant* $\gamma\in(0,1)$ *such that:*

$$\sum_{k\geq j}|q^+_k||\lambda^+_k - \dot{x}_\alpha| \leq \sum_{k\geq j}|q^-_k||\lambda^-_k - \dot{x}_\alpha| + \gamma\cdot\sum_{k\leq i}|q^+_k||\lambda^+_k - \dot{x}_\alpha|.$$

(ii) *In the situation as in Figure 5.8 b), there exists a constant* $\gamma\in(0,1)$ *such that:*

$$\sum_{k\leq i}|q^-_k||\lambda^-_k - \dot{x}_\alpha| \leq \sum_{k\leq i}|q^+_k||\lambda^+_k - \dot{x}_\alpha| + \gamma\cdot\sum_{k\geq j}|q^-_k||\lambda^-_k - \dot{x}_\alpha|.$$

Proof. We prove only the statement (i), the other one being entirely similar.

We prove that

$$\sum_{k\geq j}|q^-_k(\lambda^-_k - \dot{x}_\alpha) - q^+_k(\lambda^+_k - \dot{x}_\alpha)| \leq \gamma\cdot\sum_{k\leq i}|q^+_k||\lambda^+_k - \dot{x}_\alpha| \tag{5.3}$$

that in turn implies (i). We first show how to obtain the inequality (5.3) in case when $q^+_k = 0$ for every $k\neq s$, for any fixed index $s\leq i$. By (2.2) (2.4) the formula (5.3) is then equivalent to:

$$\sum_{k\geq j}\frac{1}{|q^+_s|}\cdot\left|q^-_k\ \frac{(\lambda^-_k - \dot{x}_\alpha)}{(\lambda^+_k - \dot{x}_\alpha)}\right| \leq \gamma \tag{5.4}$$

(compare Figure 5.1).

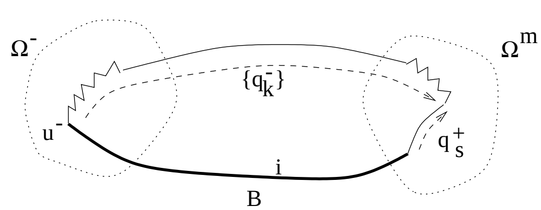

FIGURE 5.1

Following the notation of the proof of Lemma 5.1, we define the C^2 mapping

$$(u^-,\epsilon,q^+_s)\mapsto\left\{q^-_k\cdot\frac{(\lambda^-_k - \dot{x}_\alpha)}{(\lambda^+_k - \dot{x}_\alpha)}\right\}_{k=j\dots n},$$

where the quantities q_k^- (λ_k^-) stand for the weighted strengths (and corresponding speeds) of the usual wave vector solving the Riemann data

$$(u^-, \mathcal{S}_s(q_s^+/\tilde{w}_s^m, \mathcal{BS}_i(\epsilon, u^-)))$$

as in Proposition 2.1. $\dot{x}_\alpha$ is the speed of the big shock joining u^- and $\mathcal{BS}_i(\epsilon, u^-)$. We have:

$$\begin{aligned}
&\sum_{k\geq j} \frac{1}{|q_s^+|}\cdot\left|q_k^- \ \frac{(\lambda_k^- - \dot{x}_\alpha)}{(\lambda_k^+ - \dot{x}_\alpha)}\right| \\
&\qquad = \sum_{k\geq j} \frac{1}{|q_s^+|}\cdot\left|\int_0^{q_s^+} \frac{\partial}{\partial q_s^+}_{|(u^-,\epsilon,\theta)} \ q_k^- \ \frac{(\lambda_k^- - \dot{x}_\alpha)}{(\lambda_k^+ - \dot{x}_\alpha)} \,\mathrm{d}\theta\right| \\
&\qquad \leq \sum_{k\geq j}\left[\left|\frac{\partial}{\partial q_s^+}_{|(u_0^-,0,0)} \ q_k^- \ \frac{(\lambda_k^- - \dot{x}_\alpha)}{(\lambda_k^+ - \dot{x}_\alpha)}\right| + O(1)\cdot\epsilon\cdot|q_s^+|\right] \leq \gamma,
\end{aligned}$$

for any $\gamma \in (\Theta, 1)$, if only ϵ and $|q_s^+|$ are small enough. The last inequality follows from (2.10) and (2.14) (2.2).

Having (5.4) established, we turn now to prove (5.3). Consider the C^2 mapping

$$G(u^-, \epsilon, q_1^+ \dots q_n^+) = \left\{q_k^-(\lambda_k^- - \dot{x}_\alpha) - q_k^+(\lambda_k^+ - \dot{x}_\alpha)\right\}_{k=j\dots n},$$

defined by Figure 5.6 b) (ϵ is such that $u^+ = \mathcal{BS}_i(\epsilon, u^-)$). Note that $G(u^-, \epsilon, 0\dots 0) = 0$ and

$$\frac{\partial G}{\partial q_s^+}(u^-, \epsilon, 0\dots 0) = 0$$

for every $s > i$. Fix $s \leq i$. We get by (5.4):

$$\begin{aligned}
\sum_{k\geq j}\left|\frac{\partial G^k}{\partial q_s^+}(u^-, \epsilon, 0\dots 0)\right| &= \lim_{b_s^+\to 0}\sum_{k\geq j}\left|\frac{q_k^-(\lambda_k^- - \dot{x}_\alpha)}{q_s^+}\right| \\
&\leq \gamma\cdot\lim_{b_s^+\to 0}|\lambda_s^+ - \dot{x}_\alpha| = \gamma\cdot|\lambda_s(u^+) - \dot{x}_\alpha|.
\end{aligned} \tag{5.5}$$

Using the Taylor expansion, in view of (5.5) we get:

$$\begin{aligned}
&\sum_{k\geq j}\left|G^k(u^-, \epsilon, q_1^+, \dots, q_n^+)\right| \\
&\qquad = \sum_{k\geq j}\left|\sum_{s\leq i}\frac{\partial G^k}{\partial q_s^+}(u^-, \epsilon, 0\dots 0)\cdot q_s^+ + O(1)\cdot\left(\sum_{r=1}^n |q_r^+|\right)^2\right| \\
&\qquad \leq \gamma\cdot\sum_{s\leq i}|\lambda_s(u^+) - \dot{x}_\alpha||q_s^+| + O(1)\cdot\left(\sum_{r=1}^n |q_r^+|\right)^2.
\end{aligned} \tag{5.6}$$

Obviously, increasing γ a little bit, (5.6) implies (i), if only Ω^m is small enough. ■

Referring to Figure 4.1, we are going to consider separately different configurations of the positions of the large i and j shocks in the profiles of $u(t,\cdot)$ and $v(t,\cdot)$.

Case 1. – Figure 5.2

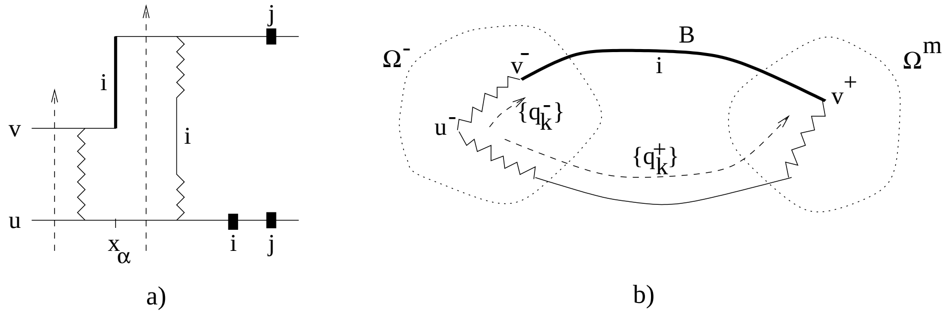

FIGURE 5.2

By Lemma (5.1)(ii) (2.14) and Definition 4.6, the following estimate holds:

$$\begin{aligned} E_i &= B \cdot W_i^+(\lambda_i^+ - \dot{x}_\alpha) - |q_i^-| \; W_i^-(\lambda_i^- - \dot{x}_\alpha) \\ &\leq O(1) \cdot B \sum_{k \geq i} |q_k^-| - 2B\kappa_1 |q_i^-| \cdot |\lambda_i^- - \dot{x}_\alpha|. \end{aligned} \tag{5.7}$$

By (2.14) (2.2), definitions (4.6) and (4.15), if δ_0 from Theorems A and B is small enough (which implies that the sum of strengths of the small waves is small) we get:

$$\begin{aligned} \sum_{k<i} E_k &= \sum_{k<i} \Big[|q_k^+|(\lambda_k^+ - \dot{x}_\alpha)(W_k^+ - W_k^-) \\ &\qquad + W_k^- \big(|q_k^+|(\lambda_k^+ - \dot{x}_\alpha) - |q_k^-|(\lambda_k^- - \dot{x}_\alpha) \big) \Big] \\ &\leq \sum_{k<i} \Big[-\frac{1}{2} |q_k^+| |\lambda_k^+ - \dot{x}_\alpha| (W_k^+ - W_k^-) \\ &\qquad - \frac{1}{2} |q_k^+| \; c(W_k^+ - W_k^-) + W_k^- |q_k^-| \cdot |\lambda_k^- - \dot{x}_\alpha| \Big] \\ &\leq \sum_{k<i} \Big[-\frac{1}{2} |q_k^+| |\lambda_k^+ - \dot{x}_\alpha| \cdot \kappa_1 B + O(1) \cdot |q_k^-| \cdot |\lambda_k^- - \dot{x}_\alpha| \Big], \end{aligned} \tag{5.8}$$

$$\begin{aligned} \sum_{k>i} E_k &= \sum_{k>i} \Big[|q_k^-|(\lambda_k^- - \dot{x}_\alpha)(W_k^+ - W_k^-) \\ &\qquad + W_k^+ \big(|q_k^+|(\lambda_k^+ - \dot{x}_\alpha) - |q_k^-|(\lambda_k^- - \dot{x}_\alpha) \big) \Big] \\ &\leq \sum_{k>i} \Big[-\frac{1}{2} |q_k^-| |\lambda_k^- - \dot{x}_\alpha| (W_k^- - W_k^+) \\ &\qquad - \frac{1}{2} |q_k^-| \; c(W_k^+ - W_k^-) + W_k^+ |q_k^+| \cdot |\lambda_k^+ - \dot{x}_\alpha| \Big] \\ &\leq \sum_{k>i} \Big[-\frac{1}{2} |q_k^-| |\lambda_k^- - \dot{x}_\alpha| \cdot \kappa_1 B + (O(1) + 3\kappa_1 B) \cdot |q_k^+| \cdot |\lambda_k^+ - \dot{x}_\alpha| \Big], \end{aligned} \tag{5.9}$$

Summing (5.7) (5.8) (5.9) and using Lemma 5.1(iii) we get:

$$\begin{aligned}\sum_{k=1}^n E_k \leq -\frac{\kappa_1 B}{4}\Bigg[\sum_{k<i}|q_k^+|\cdot|\lambda_k^+ - \dot{x}_\alpha| + \sum_{k\geq i}|q_k^-|\cdot|\lambda_k^- - \dot{x}_\alpha|\Bigg]& \\ + O(1)\cdot\kappa_1 B\sum_{k>i}|q_k^+|\cdot|\lambda_k^+ - \dot{x}_\alpha|& \\ \leq -\kappa_1 B\ c\sum_{k=1}^n |q_k^-|\cdot|\lambda_k^- - \dot{x}_\alpha| + O(1)\cdot\kappa_1 B\sum_{k>i}|q_k^+|\cdot|\lambda_k^+ - \dot{x}_\alpha|,&\end{aligned} \tag{5.10}$$

without loss of generality if c is small and κ_1 big enough.

Clearly, in view of Lemma 5.1(iv), the formula (5.10) implies (4.11) provided that the weights $\tilde{w}_k^-$ for $k \geq i$ are big enough (relatively to other weights).

Case 1'. – Figure 5.3

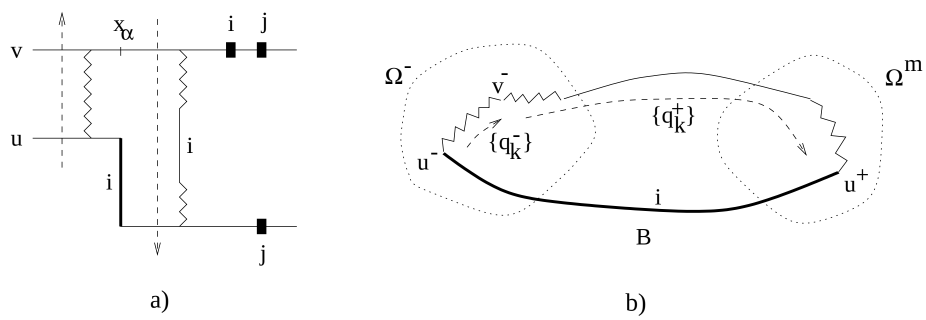

Figure 5.3

Call $\left\{\widetilde{q_k^-}\right\}_{k=1}^n$ the (weighted) wave vector solving the Riemann data (v^-, u^-). Standard computations show that:

$$\widetilde{q_k^-} - q_k^- = O(1)\cdot\left(\sum_{s=1}^n |q_s^-|\right)^2,$$

$$\widetilde{\lambda}_k^- - \lambda_k^- = O(1)\cdot\sum_{s=1}^n |q_s^-|$$

(where $\widetilde{\lambda}_k^-$ is the Rankine-Hugoniot speed of the wave $\widetilde{q_k^-}$).

Since replacing q_k^- with $\widetilde{q_k^-}$ we obtain precisely the situation considered in Case 1, the above estimates provide us with (4.11) in the present Case.

Case 2. – Figure 5.4

Analogously to the treatment of Case 1, recalling Remark 5.2, we estimate the terms in $\sum_{k=1}^n E_k$:

$$\begin{aligned}E_j &= -B\cdot W_j^-(\lambda_j^+ - \dot{x}_\alpha) + |q_j^+|\ W_j^+(\lambda_j^+ - \dot{x}_\alpha) \\ &\leq O(1)\cdot B\sum_{k\leq j}|q_k^+| - 2B\kappa_1|q_j^+|\cdot|\lambda_j^+ - \dot{x}_\alpha|.\end{aligned} \tag{5.11}$$

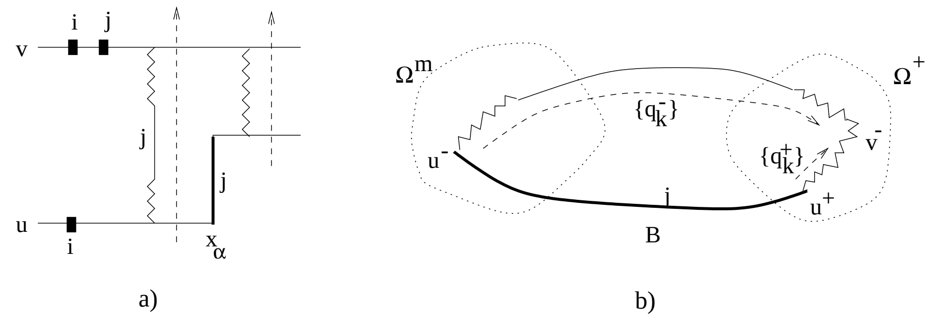

FIGURE 5.4

$$\sum_{k>j} E_k \leq \sum_{k>j} \Big[-\frac{1}{2}|q_k^-||\lambda_k^- - \dot{x}_\alpha| \cdot \kappa_1 B + O(1) \cdot |q_k^+| \cdot |\lambda_k^+ - \dot{x}_\alpha| \Big], \tag{5.12}$$

$$\sum_{k<j} E_k = \sum_{k<j} \Big[-\frac{1}{2}|q_k^+||\lambda_k^+ - \dot{x}_\alpha| \cdot \kappa_1 B + (O(1) + 3\kappa_1 B) \cdot |q_k^-| \cdot |\lambda_k^- - \dot{x}_\alpha| \Big], \tag{5.13}$$

Summing (5.11) (5.12) (5.13) together, one sees that:

$$\sum_{k=1}^{n} E_k \leq -\kappa_1 B \ c \sum_{k=1}^{n} |q_k^+| \cdot |\lambda_k^+ - \dot{x}_\alpha| + O(1) \cdot \kappa_1 B \sum_{k<j} |q_k^-| \cdot |\lambda_k^- - \dot{x}_\alpha|.$$

As before, this implies (4.11) provided that the weights $\tilde{w}_k^+$ for $k \leq j$ are big enough.

CASE 2'. – Figure 5.5

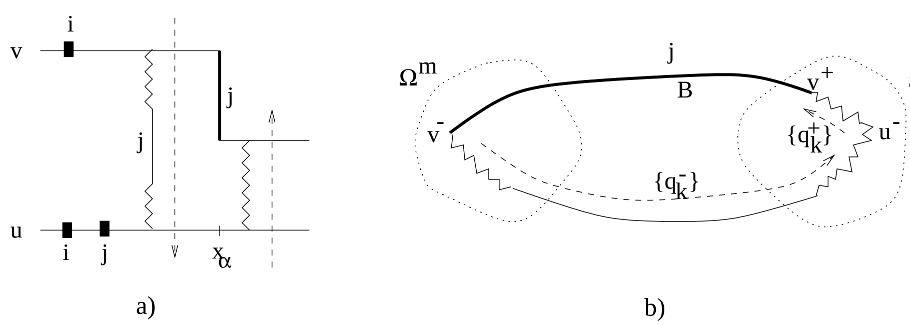

FIGURE 5.5

This case reduces to Case 2 precisely in the way as Case 1' reduces to Case 1.

CASE 3. – Figure 5.6

With the same remarks as in Case 1, we get the following estimates:

$$\begin{aligned} E_i &= -B \cdot W_i^-(\lambda_i^- - \dot{x}_\alpha) + |q_i^+| \ W_i^+(\lambda_i^+ - \dot{x}_\alpha) \\ &\leq O(1) \cdot B \sum_{k \leq i} |q_k^+| - 2B\kappa_1 |q_i^+| \cdot |\lambda_i^+ - \dot{x}_\alpha|, \end{aligned} \tag{5.14}$$

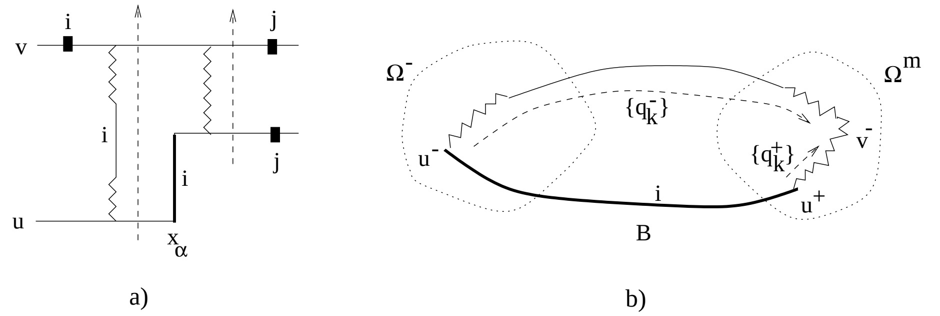

FIGURE 5.6

$$\begin{aligned}\sum_{k<i} E_k = \sum_{k<i} \Big[& |q_k^+|(\lambda_k^+ - \dot{x}_\alpha)(W_k^+ - W_k^-) \\ & + W_k^- \big(|q_k^+|(\lambda_k^+ - \dot{x}_\alpha) - |q_k^-|(\lambda_k^- - \dot{x}_\alpha)\big)\Big] \\ \leq \sum_{k<i} \Big[& - B\kappa_1 |q_k^+||\lambda_k^+ - \dot{x}_\alpha| + O(1) \cdot \big(|q_k^+| + |q_k^-|\big) \\ & - B\kappa_1 |q_k^+| \cdot |\lambda_k^+ - \dot{x}_\alpha| + B\kappa_1 |q_k^-| \cdot |\lambda_k^- - \dot{x}_\alpha|\Big],\end{aligned} \tag{5.15}$$

$$\begin{aligned}\sum_{i<k<j} E_k = \sum_{i<k<j} \Big[& |q_k^-|(\lambda_k^- - \dot{x}_\alpha)(W_k^+ - W_k^-) \\ & + W_k^+ \big(|q_k^+|(\lambda_k^+ - \dot{x}_\alpha) - |q_k^-|(\lambda_k^- - \dot{x}_\alpha)\big)\Big] \\ \leq \sum_{i<k<j} \Big[& - |q_k^-||\lambda_k^- - \dot{x}_\alpha| \cdot \kappa_1 B + O(1) \cdot \big(|q_k^-| + |q_k^+|\big)\Big],\end{aligned} \tag{5.16}$$

$$\begin{aligned}\sum_{k\geq j} E_k = \sum_{k\geq j} \Big[& |q_k^-|(\lambda_k^- - \dot{x}_\alpha)(W_k^+ - W_k^-) \\ & + W_k^+ \big(|q_k^+|(\lambda_k^+ - \dot{x}_\alpha) - |q_k^-|(\lambda_k^- - \dot{x}_\alpha)\big)\Big] \\ \leq \sum_{k\geq j} \Big[& - |q_k^-||\lambda_k^- - \dot{x}_\alpha| \cdot \kappa_1 B + O(1) \cdot \big(|q_k^-| + |q_k^+|\big) \\ & + 2B\kappa_1 |q_k^+| \cdot |\lambda_k^+ - \dot{x}_\alpha| - 2B\kappa_1 |q_k^-| \cdot |\lambda_k^- - \dot{x}_\alpha|\Big].\end{aligned} \tag{5.17}$$

Adding (5.14) – (5.17) we get:

$$\begin{aligned}
\sum_{k=1}^{n} E_k \leq & \Big[-\frac{1-\gamma}{2}\ B\kappa_1 \sum_{k\leq i} |q_k^+| \cdot |\lambda_k^+ - \dot{x}_\alpha| + O(1)\cdot B \sum_{k\leq i} |q_k^+| \Big] \\
& + \Big[-(1-\gamma)\ B\kappa_1 \sum_{k\leq i} |q_k^+| \cdot |\lambda_k^+ - \dot{x}_\alpha| \\
& \qquad + B\kappa_1 \sum_{k<i} |q_k^-| \cdot |\lambda_k^- - \dot{x}_\alpha| \Big] \\
& + \Big[-\frac{1-\gamma}{2}\ B\kappa_1 \sum_{k\leq i} |q_k^+| \cdot |\lambda_k^+ - \dot{x}_\alpha| - B\kappa_1 \sum_{k>i} |q_k^-| \cdot |\lambda_k^- - \dot{x}_\alpha| \\
& \qquad + O(1)\cdot \sum_{k\neq i} \big(|q_k^-| + |q_k^+|\big) \Big] \\
& + \Big[-2\gamma\ B\kappa_1 \sum_{k\leq i} |q_k^+| \cdot |\lambda_k^+ - \dot{x}_\alpha| + 2B\kappa_1 \sum_{k\geq j} |q_k^+| \cdot |\lambda_k^+ - \dot{x}_\alpha| \\
& \qquad - 2B\kappa_1 \sum_{k\geq j} |q_k^-| \cdot |\lambda_k^- - \dot{x}_\alpha| \Big] \\
& = I_1 + I_2 + I_3 + I_4,
\end{aligned} \tag{5.18}$$

where γ is as in Lemma 5.7. To get (4.11) we show that each term I_1, I_2, I_3, I_4 in (5.18) is nonpositive, provided that κ_1 is big enough.

$I_1 \leq 0$ by (2.2) and (2.14). $I_2 \leq 0$ follows from Remark 5.2, provided that the weights $\tilde{w}_k^-$ for $k < i$ are small (relatively to other weights).

To estimate I_3 we use Remark 5.2 without loss of generality (c as usual stands for a uniform arbitrary small but positive constant) get:

$$\begin{aligned}
I_3 & \leq -c\ B\kappa_1 \left(\sum_{k\leq i} |q_k^+| + \sum_{k>i} |q_k^-| \right) + O(1)\cdot \sum_{k\neq i} |q_k^+| + O(1)\cdot \sum_{k\neq i} |q_k^-| \\
& \leq -c\ B\kappa_1 \left(\sum_{k\leq i} |q_k^+| + \sum_{k>i} |q_k^-| \right) + \Big[O(1)\cdot \sum_{k>i} |q_k^+| + O(1)\cdot \sum_{k<i} |q_k^-| \Big] \\
& \leq -c\ B\kappa_1 \sum_{k=1}^{n} |q_k^+| + O(1)\cdot \sum_{k=1}^{n} |q_k^+| \leq 0.
\end{aligned}$$

$I_4 \leq 0$ is an obvious consequence of Lemma 5.5(i).

Case 3'. – Figure 5.7
This case reduces to Case 3 as Case 1' reduces to Case 1.

Case 4. – Figure 5.8
We treat this case similarily to Case 3.

$$\begin{aligned}
E_j & = -B \cdot W_j^-(\lambda_j^+ - \dot{x}_\alpha) + |q_j^-|\ W_j^-(\lambda_j^- - \dot{x}_\alpha) \\
& \leq O(1)\cdot B \sum_{k\geq j} |q_k^-| - 2B\kappa_1 |q_j^-| \cdot |\lambda_j^- - \dot{x}_\alpha|,
\end{aligned} \tag{5.19}$$

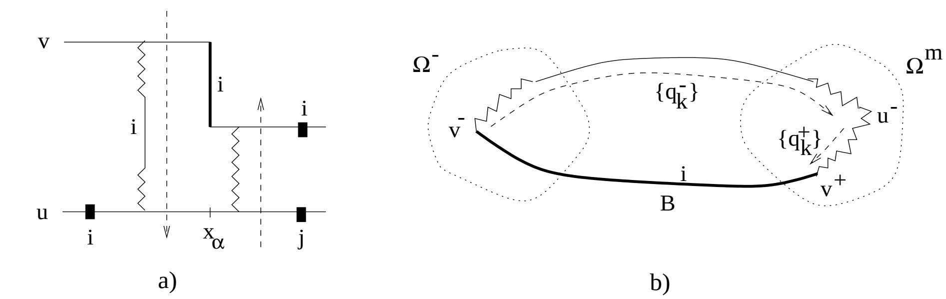

FIGURE 5.7

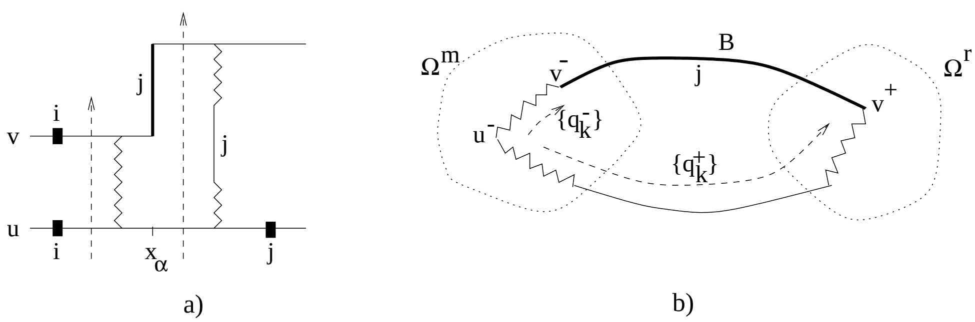

FIGURE 5.8

$$\sum_{k>j} E_k \leq \sum_{k>j} \Big[- B\kappa_1 |q_k^-||\lambda_k^- - \dot{x}_\alpha| + O(1) \cdot \big(|q_k^+| + |q_k^-|\big) \\ + B\kappa_1 |q_k^+| \cdot |\lambda_k^+ - \dot{x}_\alpha| - B\kappa_1 |q_k^-| \cdot |\lambda_k^- - \dot{x}_\alpha|\Big], \tag{5.20}$$

$$\sum_{i<k<j} E_k \leq \sum_{i<k<j} \Big[- |q_k^+||\lambda_k^+ - \dot{x}_\alpha| \cdot \kappa_1 B + O(1) \cdot \big(|q_k^-| + |q_k^+|\big)\Big], \tag{5.21}$$

$$\sum_{k\leq i} E_k \leq \sum_{k\leq i} \Big[- |q_k^+||\lambda_k^+ - \dot{x}_\alpha| \cdot \kappa_1 B + O(1) \cdot \big(|q_k^-| + |q_k^+|\big) \\ - 2B\kappa_1 |q_k^+| \cdot |\lambda_k^+ - \dot{x}_\alpha| + 2B\kappa_1 |q_k^-| \cdot |\lambda_k^- - \dot{x}_\alpha|\Big]. \tag{5.22}$$

Adding (5.19) – (5.22) we get:

$$\begin{aligned}\sum_{k=1}^{n} E_k \leq & \Big[- \frac{1-\gamma}{2} \ B\kappa_1 \sum_{k\geq j} |q_k^-| \cdot |\lambda_k^- - \dot{x}_\alpha| + O(1) \cdot B \sum_{k\geq j} |q_k^-| \Big] \\ & + \Big[-(1-\gamma) \ B\kappa_1 \sum_{k\geq j} |q_k^-| \cdot |\lambda_k^- - \dot{x}_\alpha| \\ & \qquad + B\kappa_1 \sum_{k>j} |q_k^+| \cdot |\lambda_k^+ - \dot{x}_\alpha| \Big] \\ & + \Big[- \frac{1-\gamma}{2} \ B\kappa_1 \sum_{k\geq j} |q_k^-| \cdot |\lambda_k^- - \dot{x}_\alpha| - B\kappa_1 \sum_{k<j} |q_k^+| \cdot |\lambda_k^+ - \dot{x}_\alpha| \\ & \qquad + O(1) \cdot \sum_{k\neq j} \big(|q_k^+| + |q_k^-| \big) \Big] \\ & + \Big[-2\gamma \ B\kappa_1 \sum_{k\geq j} |q_k^-| \cdot |\lambda_k^- - \dot{x}_\alpha| - 2B\kappa_1 \sum_{k\leq i} |q_k^+| \cdot |\lambda_k^+ - \dot{x}_\alpha| \\ & \qquad + 2B\kappa_1 \sum_{k\leq i} |q_k^-| \cdot |\lambda_k^- - \dot{x}_\alpha| \Big],\end{aligned} \tag{5.23}$$

where γ is as in Lemma 5.5.

Using the same arguments as in Case 3, one shows that each term in (5.23) is nonpositive, provided that the weights $\tilde{w}_k^+$ for $k > j$ are small (relatively to other weights). The nonpositivity of the last term in (5.23) follows from Lemma 5.5(ii).

Case 4'. – Figure 5.9

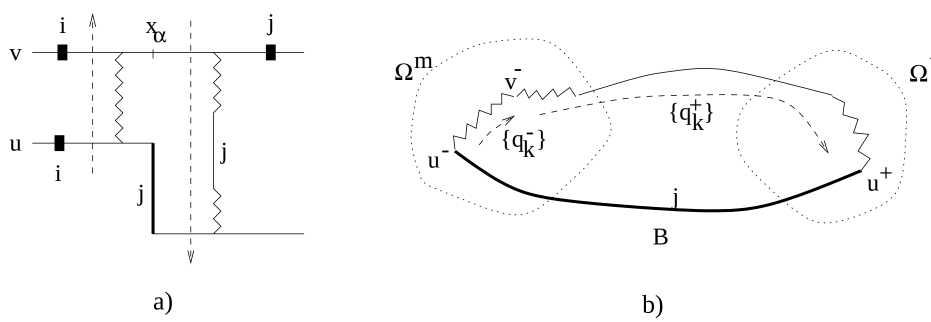

Figure 5.9

This case reduces to Case 4 as Case 1' reduces to Case 1.

Case 5. – Figure 5.10

Note that by Lemma 5.3 and definitions (4.5) (4.6)

$$\begin{aligned} E_i &= B(\lambda_i^+ - \dot{x}_\alpha)(W_i^+ - W_i^-) + B \cdot W_i^-(\lambda_i^+ - \lambda_i^-) \\ &\leq -B^2\kappa_1|\lambda_i^+ - \dot{x}_\alpha| + O(1) \cdot B \sum_{k\geq j} |q_k^-| \\ &\leq -B^2\kappa_1 c + O(1) \cdot B \sum_{k\geq j} |q_k^-|. \end{aligned} \tag{5.24}$$

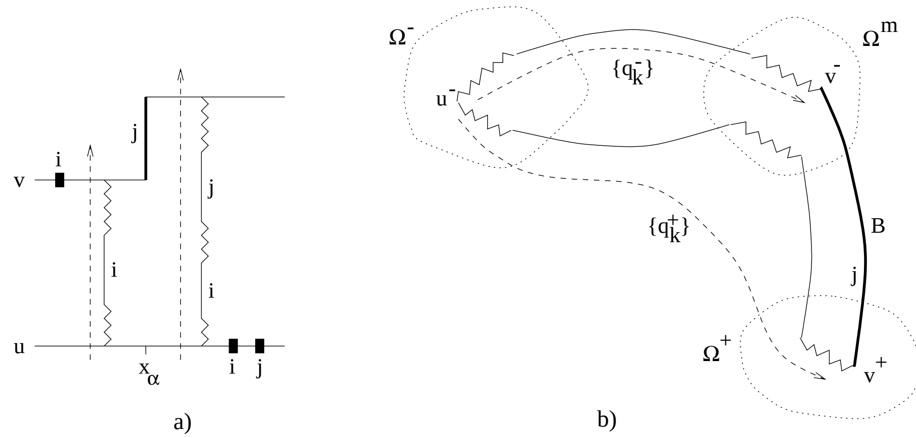

FIGURE 5.10

The first summand in (5.24) is a uniform negative constant. On the other hand, all the other E_k with $k \neq i$ contain only components that linearly depend on different sums of $|q_k^-|, |q_k^+|$ small. Thus if the sum of weighted strengths of all small waves in u and v is small enough (what is guaranteed by (3.5)), we get (4.11).

CASE 6. – Figure 5.11

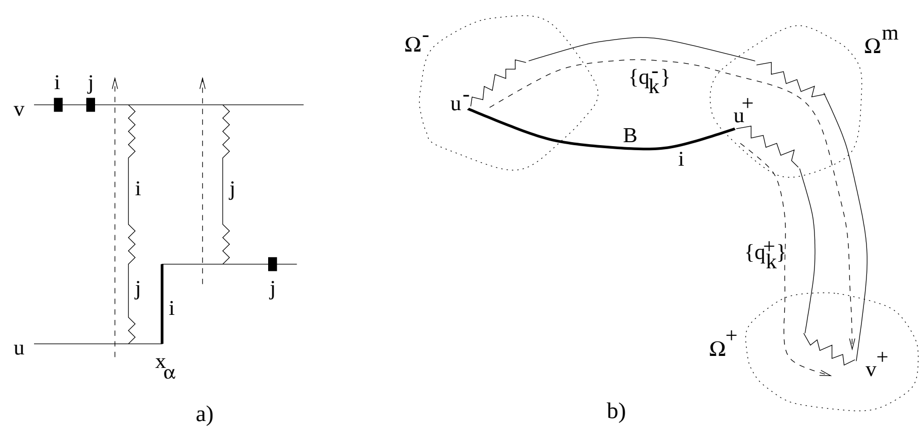

FIGURE 5.11

As in Case 5, we note the presence of a uniform negative term in E_j:

$$E_j = B(\lambda_j^- - \dot{x}_\alpha)(W_j^+ - W_j^-) + B \cdot W_j^+(\lambda_j^+ - \lambda_j^-) \leq -B^2\kappa_1 c + O(1) \cdot B \sum_{k \leq i} |q_k^+|,$$

that yields (4.11).

This ends the discussion of (4.11). For the convenience of the reader we recall now that the treatment of Cases 1 – 4 require the following (relative) sizes of the weights:

$$\begin{array}{c|c|c} & \tilde{w}_k^- & \tilde{w}_k^+ \\ \hline \text{big weight} & k \geq i & k \leq j \\ \hline \text{small weight} & k < i & k > j \end{array} \tag{5.25}$$

Case of small physical waves
– the estimate (4.12)

As in the proof of the estimate (4.11), we are going to consider different cases, according to the locations of the small physical wave α under consideration with respect to the locations of the large shocks. Examining Figure 4.1 a carefull reader can check that the discussed below four cases cover all possible configurations (without loss of generality we assume that the jump ϵ_α occures in v.)

Another remark is the following. Denote by $\dot{y}_\alpha$ the 'real' speed of the α wave under consideration, that is: $\dot{y}_\alpha = \lambda_{k_\alpha}(v^-, v^+)$ in case $\alpha \in \mathcal{S}$ or $\dot{y}_\alpha = \lambda_{k_\alpha}(v^+)$ in case $\alpha \in \mathcal{R}$. For $k : 1 \ldots n$ let's estimate the difference between E_k and a similar expression where $\dot{y}_\alpha$ replaces $\dot{x}_\alpha$:

$$\begin{aligned} E_k - \big[|q_k^+|W_k^+(\lambda_k^+ - \dot{y}_\alpha) - |q_k^-|W_k^-(\lambda_k^- - \dot{y}_\alpha)\big] \\ = (\dot{y}_\alpha - \dot{x}_\alpha)\big[|q_k^+|W_k^+ - |q_k^-|W_k^-\big] = O(1) \cdot \epsilon \cdot |\epsilon_\alpha|, \end{aligned} \tag{5.26}$$

by Theorem 3.5, which asserts that $|\dot{y}_\alpha - \dot{x}_\alpha| \leq \epsilon$. The term $|q_k^+|W_k^+ - |q_k^-|W_k^-$ is estimated by $O(1) \cdot |\epsilon_\alpha|$ exactly as in (5.2).

Below we will assume that $\dot{y}_\alpha = \dot{x}_\alpha$ and prove that under this hypothesis

$$\sum_{k=1}^{n} E_k \leq 0. \tag{5.27}$$

This together with (5.26) will yield (4.12).

Case A. – Figure 5.12

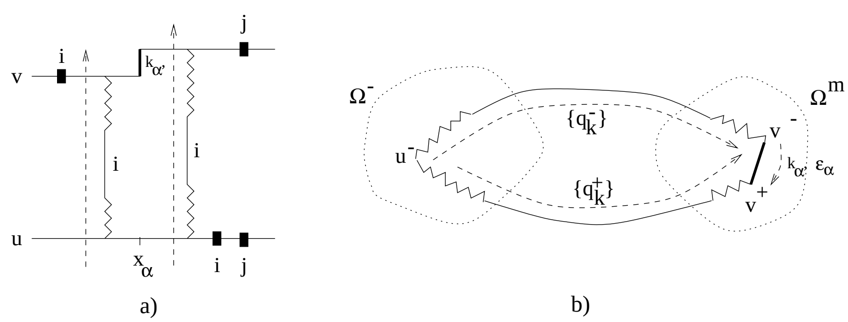

Figure 5.12

By definitions (4.5) (4.6), Proposition 2.1 and formula (2.2) we have:

$$\begin{aligned} E_i &= B \cdot \big[(W_i^+ - W_i^-)(\lambda_i^\pm - \dot{x}_\alpha) + W_i^\mp(\lambda_i^\pm - \lambda_i^\mp)\big] \\ &= B \cdot \big[- \kappa_1|\epsilon_\alpha||\lambda_i^\pm - \dot{x}_\alpha| + O(1) \cdot |\epsilon_\alpha|\big] \leq -B\kappa_1 c|\epsilon_\alpha| + O(1) \cdot B|\epsilon_\alpha|. \end{aligned} \tag{5.28}$$

(The choice of the upper or lower superindices depends on the family number k_α.)

If $k \notin \{i, k_\alpha\}$ and k-field is linearly degenerate or k-field is genuinely nonlinear but $q_k^+ \cdot q_k^- \geq 0$:

$$\begin{aligned} E_k &= |q_k^\pm|(W_k^+ - W_k^-)(\lambda_k^\pm - \dot{x}_\alpha) + W_k^\mp \left[|q_k^+|(\lambda_k^+ - \dot{x}_\alpha) - |q_k^-|(\lambda_k^- - \dot{x}_\alpha)\right] \\ &= -\kappa_1 |q_k^\pm||\epsilon_\alpha||\lambda_k^\pm - \dot{x}_\alpha| \\ &\qquad + W_k^\mp \cdot \left[(|q_k^+| - |q_k^-|)(\lambda_k^+ - \dot{x}_\alpha) + |q_k^-|(\lambda_k^+ - \lambda_k^-)\right] \\ &\leq \big(O(1) + 3B\kappa_1\big)\big(O(1) \cdot |q_k^+ - q_k^-| + O(1) \cdot |q_k^-||\epsilon_\alpha|\big). \end{aligned} \tag{5.29}$$

If $k \notin \{i, k_\alpha\}$ and k-field is genuinely nonlinear with $q_k^+ \cdot q_k^- < 0$ then the estimate is almost as above:

$$\begin{aligned} E_k &\leq -\kappa_1 |q_k^\pm||\epsilon_\alpha||\lambda_k^\pm - \dot{x}_\alpha| + O(1) \cdot |q_k^\pm||\lambda_k^\pm - \dot{x}_\alpha| \\ &\qquad + W_k^\mp \cdot \left[(|q_k^+| - |q_k^-|)(\lambda_k^+ - \dot{x}_\alpha) - |q_k^-|(\lambda_k^+ - \lambda_k^-)\right] \\ &\leq \big(O(1) + 3B\kappa_1\big)\big(O(1) \cdot |q_k^+ - q_k^-| + O(1) \cdot |q_k^-||\epsilon_\alpha|\big) + O(1) \cdot |\epsilon_\alpha|, \end{aligned} \tag{5.30}$$

because $|q_k^+| \leq |q_k^+| + |q_k^-| = |q_k^+ - q_k^-| = O(1) \cdot |\epsilon_\alpha|$.

If $k = k_\alpha \neq i$, then the above estimates (5.29) and (5.30) still hold, with the negative term $-\kappa_1 |q_k^\pm||\epsilon_\alpha||\lambda_k^\pm - \dot{x}_\alpha|$ replaced by $O(1) \cdot |\epsilon_\alpha|$. Thus also in this case:

$$E_k \leq \big(O(1) + 3B\kappa_1\big)\big(O(1) \cdot |q_k^+ - q_k^-| + O(1) \cdot |q_k^-||\epsilon_\alpha|\big) + O(1) \cdot |\epsilon_\alpha|, \tag{5.31}$$

Summing (5.28) – (5.31), we get by (3.1):

$$\begin{aligned} \sum_{k=1}^n E_k &\leq -B\kappa_1 c|\epsilon_\alpha| + O(1) \cdot |\epsilon_\alpha| \\ &\qquad + 3B\kappa_1 O(1) \cdot \left[\sum_{k\neq i} |q_k^+ - q_k^-| + \sum_{k \neq i} |q_k^-||\epsilon_\alpha|\right] \\ &= \left[-\frac{B\kappa_1 c}{2}|\epsilon_\alpha| + O(1) \cdot |\epsilon_\alpha|\right] \\ &\quad + \left[-\frac{B\kappa_1 c}{2}|\epsilon_\alpha| + 3B\kappa_1 O(1) \cdot \left[\sum_{k\neq i} |q_k^+ - q_k^-| + \sum_{k \neq i} |q_k^-||\epsilon_\alpha|\right]\right] \end{aligned} \tag{5.32}$$

If κ_1 is big enough then the first term in the right hand side of (5.32) is negative. The second term is also negative, if all weights w_k are sufficiently small. Thus we have proved (5.27).

Case B. – Figure 5.13

This case is treated exactly the same as Case A. The large negative term is given by E_j.

Case C. – Figure 5.14

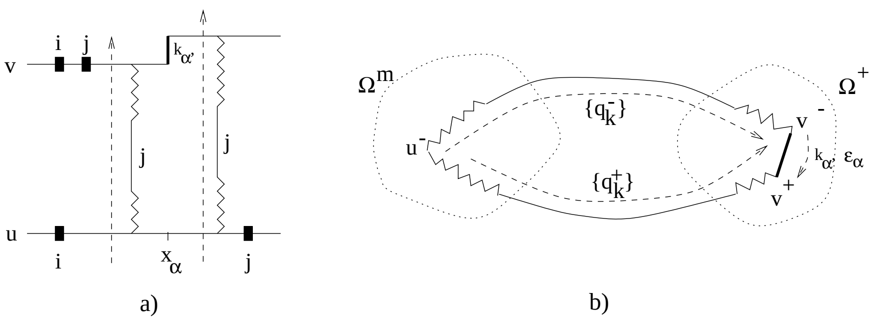

FIGURE 5.13

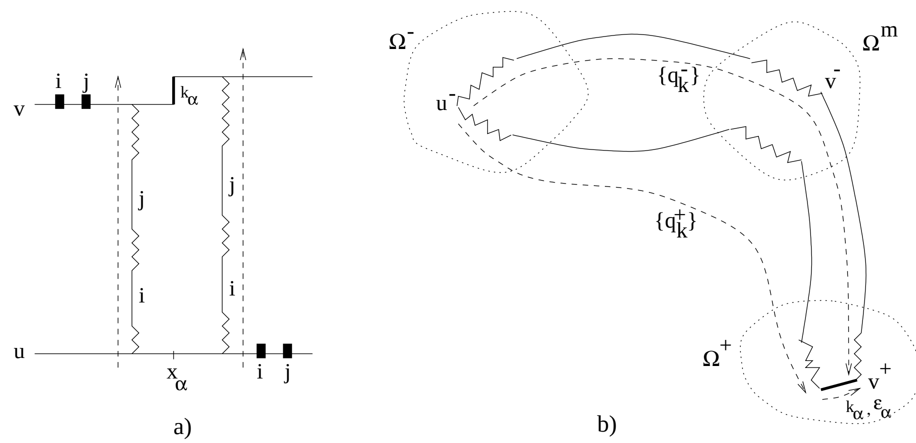

FIGURE 5.14

We have:

$$
\begin{aligned}
E_i + E_j &= B \cdot \big[(W_i^+ - W_i^-)(\lambda_i^\pm - \dot{x}_\alpha) + W_i^\mp(\lambda_i^\pm - \lambda_i^\mp)\big] \\
&\quad + B \cdot \big[(W_j^+ - W_j^-)(\lambda_j^\pm - \dot{x}_\alpha) + W_j^\mp(\lambda_j^\pm - \lambda_j^\mp)\big] \\
&= B \cdot \big[- \kappa_1 |\epsilon_\alpha| |\lambda_i^\pm - \dot{x}_\alpha| - \kappa_1 |\epsilon_\alpha| |\lambda_j^\pm - \dot{x}_\alpha| \\
&\qquad\qquad + (O(1) + B\kappa_1) \cdot O(1) \cdot |\epsilon_\alpha|\big] \\
&\leq B \cdot \Big[- \kappa_1 c |\epsilon_\alpha| - \frac{\kappa_1 c}{2} |\epsilon_\alpha| + O(1) \cdot |\epsilon_\alpha| \\
&\qquad\qquad - \frac{\kappa_1 c}{2} |\epsilon_\alpha| + O(1) \cdot B\kappa_1 |\epsilon_\alpha|\Big] \\
&\leq -B\kappa_1 c |\epsilon_\alpha|,
\end{aligned}
\tag{5.33}
$$

if only κ_1 is big enough and the weighted strength of large shocks B defined to be small enough (with respect to the uniform constants $O(1)$ of the system (1.1)).

The terms E_k for $k \notin \{i, j\}$ are estimated as in case A – it appeares that if κ_1 is big and the rescalings q_k/ϵ_k small, then the sum $\sum_{k \notin \{i,j\}} E_k$ is overtaken by the negative term $-B\kappa_1 c |\epsilon_\alpha|$ in (5.33). Thus (5.27) follows.

CASE D. – Figure 5.15

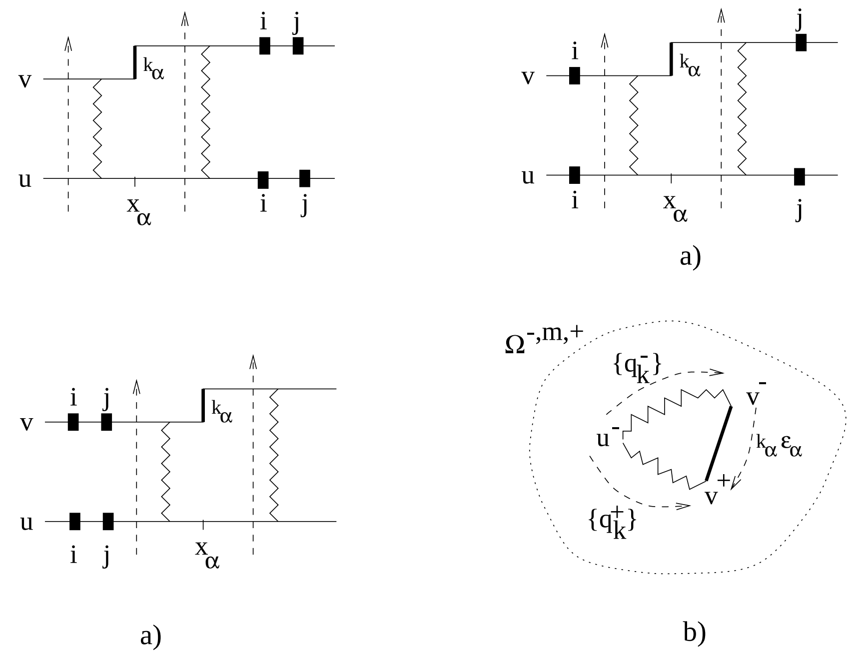

FIGURE 5.15

This case has been treated in [BLY]. If the constant B is small enough and κ_1 big (with respect to the uniform constants $O(1)$ in all the formulae), we get (5.27) as in [BLY].

6. PROOFS OF THE MAIN THEOREMS

The results of the analysis in Section 3 have been gathered in Theorem 3.5, showing the existence of a piecewise constant function that is an approximate solution to (1.1). The error, in the sence of the estimates (3.6) (3.7) (3.8) is given by a fixed parameter ϵ, arbitrarily small. Following the proof of consistency and compactness of the wave front tracking algorithm in case $u_0^- = u_0^+$ in [B1], Theorem A can be proved. To this end, take $\bar{u} \in \widetilde{\mathcal{D}}_{\delta_0}$, for δ_0 smaller than δ in Proposition 3.4. Given $\epsilon > 0$, fix a piecewise constant $\bar{u}_\epsilon \in \widetilde{\mathcal{D}}_{\delta_0}$, such that

$$\| \bar{u} - \bar{u}_\epsilon \|_{\mathrm{L}^1(\mathbf{R},\mathbf{R}^n)} < \epsilon.$$

Let u_ϵ be the ϵ-approximate solution of (1.1) with $u_\epsilon(0,\cdot) = \bar{u}_\epsilon$, as in Theorem 3.5. Leting $\epsilon \to 0$, one sees that it is possible to extract a sequence u_{ϵ_n} converging in $\mathrm{L}^1_{\mathrm{loc}}$ to a function $u(t,x)$. By the inequalities in Theorem 3.5, u must be a solution to (1.1) (1.2).

Towards the proof of Theorem B, define

$$\mathcal{D}_{\delta_0} = \mathrm{cl}\Bigg\{ u : \mathbf{R} \longrightarrow \mathbf{R}^n \text{ piecewise constant with:} \\ u - \widetilde{u} \in \mathrm{L}^1(\mathbf{R}, \mathbf{R}^n) \text{ and } \Gamma(u) \leq C \cdot \delta_0 \Bigg\}, \tag{6.1}$$

where cl denotes the closure in $\mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}, \mathbf{R}^n)$, $\widetilde{u}$ is as in (2.12), $C \geq 1$ is a constant such that

$$\frac{1}{C} \cdot T.V.(u - \widetilde{u}) \leq \Gamma(u) \leq C \cdot T.V.(u - \widetilde{u}), \tag{6.2}$$

and $\delta_0 < \delta/(2C^2)$, with δ as in Proposition 3.4. By (5.2)

$$\widetilde{\mathcal{D}}_{\delta_0} \subset \mathcal{D}_{\delta_0} \subset \widetilde{\mathcal{D}}_{\delta/2}.$$

Take $\bar{u} \in \mathcal{D}_{\delta_0}$. Let $\{u_\epsilon\}$ be any sequence of ϵ-approximate solutions of (1.1) such that

$$\Gamma(u_\epsilon(0, \cdot)) \leq C \cdot \delta_0, \quad \| \bar{u} - u_\epsilon(0, \cdot) \|_{\mathrm{L}^1(\mathbf{R}, \mathbf{R}^n)} < \epsilon,$$

and $\epsilon \to 0$. Using (4.1) and (4.2) it is possible to show that the sequence $\{u_\epsilon\}$ is Cauchy in $\mathrm{L}^1_{\mathrm{loc}}(\mathbf{R}, \mathbf{R}^n)$ and converges to the unique limit u.

We define $S(t, \bar{u}) = u(t, \cdot)$. By Theorem A, (iii) follows immediately. By construction we get (i). To prove (ii), fix $\bar{u}, \bar{w} \in \mathcal{D}_{\delta_0}$. Then for some ϵ-approximate solutions of (1.1), given by the wave front tracking algorithm, there holds:

$$\| \bar{u} - u_\epsilon(0, \cdot) \|_{\mathrm{L}^1} < \epsilon, \qquad \| \bar{w} - w_\epsilon(0, \cdot) \|_{\mathrm{L}^1} < \epsilon.$$

Using again (4.1) and (4.2) one gets:

$$\| u_\epsilon(t, \cdot) - w_\epsilon(t, \cdot) \|_{\mathrm{L}^1} = O(1) \cdot \| \bar{u}_\epsilon(0, \cdot) - \bar{w}_\epsilon(0, \cdot) \|_{\mathrm{L}^1} + O(1) \cdot t \cdot \epsilon.$$

Letting $\epsilon \to 0$, one gets the Lipschitz continuity of S with respect to initial data. The Lipschitz continuity of S with respect to time follows from the corresponding property satisfied uniformly by the ϵ-approximate solutions of (1.1)

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