Econ 2001   Summer 2015

Problem Set 2

1. Show that $\mathbf{R}$ and $\mathbf{Q}$ are fields.

2. Show that if F is a field and x, y, z are any three elements of A, the following hold:

   (a) Cancellation law: If $x + y = x + z$ then $y = z$.
   (b) If $x + y = x$ then $x = 0$.
   (c) If $x + y = 0$ then $y = -x$.
   (d) $-(-x) = x$.
   (e) If $x \neq 0$ and $xy = xz$ then $y = z$.
   (f) If $x \neq 0$ and $xy = x$ then $y = 1$.
   (g) If $x \neq 0$ and $xy = 1$ then $y = \frac{1}{x}$.
   (h) If $x \neq 0$ then $\frac{1}{\frac{1}{x}} = x$.
   (i) $0x = 0$.
   (j) If $x \neq 0$ and $y \neq 0$ then $xy \neq 0$.
   (k) $(-x)y = -(xy) = x(-y)$.
   (l) $(-x)(-y) = (xy)$.

3. The following statements are true in every ordered field F.

   (a) If $x > 0$ then $-x < 0$, and vice versa.
   (b) If $x > 0$ and $y < z$ then $xy < xz$.
   (c) If $x < 0$ and $y < z$ then $xy > xz$.
   (d) If $x \neq 0$ then $x^2 > 0$. In particular, $1 > 0$.
   (e) If $0 < x < y$ then $0 < \frac{1}{y} < \frac{1}{x}$.

4. Show that $\mathbf{R}$ and $\mathbf{R}^n$ are vector spaces.

5. Show that the set of all functions $f : \mathbf{R} \to \mathbf{R}$ is a vector space.

6. Show that the set of all quadratic functions (functions of the form $ax^2 + bx + c$ where $a, b, c \in \mathbb{R}$) is a subspace.

7. Prove that for every real number $x > 0$ and every integer $n > 0$ there is a unique positive real number y such that $y^n = x$ (this defines $\sqrt[n]{x}$ or $x^{\frac{1}{n}}$).

8. Prove that if $x \in \mathbf{R}$, $y \in \mathbf{R}$, and $x > 0$, then there is a positive integer n such that $nx > y$. Archimedean Property.

9. Prove that if $x \in \mathbf{R}$, $y \in \mathbf{R}$, and $x < y$, then there exists a $p \in \mathbf{Q}$ such that $x < p < y$. THE RATIONALS ARE DENSE IN THE REALS.

10. For each $n \in N$, assume we are given a closed interval

$$I_n = [a_n, b_n] = \{x \in \mathbb{R} : a_n \leq x \leq b_n\}.$$

Assume also that each I_n contains I_{n+1} (i.e. $I_1 \supset I_2 \supset I_3 \supset \cdots$). Show that $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

11. Given a set $X \subseteq \mathbf{R}$, prove that if $\min X$ exists it is equal to $\inf X$.

Extra Practice Problems

1. Determine which of the following are vector spaces over $\mathbb{R}$. Verify the properties or explain why they are not satisfied.

   (a) The set of polynomials of degree n.

   (b) The set of 2×2 matrices with entries from $\mathbb{R}$.

   (c) The set of *invertible* 2×2 matrices with entries from $\mathbb{R}$.

   (d) The set of continuous functions from $[0, 1]$ to *itself*.

2. Prove the following statement or falsify it with a counterexample: A vector space cannot be written as the union of two proper subspaces.

3. Which of the following are vector spaces over the field $\mathbb{R}$?

   (a) The set of all convergent real sequences

   (b) The set of all divergent real sequences

   (c) The set of all bounded functions from $[0, 1]$ to $\mathbb{R}$.

4. Let X and Y be non-empty bounded sets of positive reals and XY is a set of all possible products xy, such that $x \in X$, $y \in Y$. Show that XY is a bounded set, and moreover $\sup XY = \sup X \cdot \sup Y$, $\inf XY = \inf X \cdot \inf Y$.

5. Let X and Y be non-empty sets of $\mathbb{R}$, such that

   (a) for any $x \in X$ and any $y \in Y$ we have $x \leq y$

   (b) for any $\epsilon > 0$ there is $x_\epsilon \in X$ and $y_\epsilon \in Y$, such that $y_\epsilon - x_\epsilon < \epsilon$
   Show that $\sup X = \inf Y$.