Problem Set 3

1. Which of the following functions from $\mathbb{R} \times \mathbb{R}$ is a metric on $\mathbb{R}$?

   (a) $d(x, y) = (x - y)^2$

   (b) $d(x, y) = |x - 2y|$

2. Show that if d is the metric associated with a norm on the normed vector space V then for any $x, y, z \in V$ and $k \in \mathbb{R}$
$$d(x + z, y + z) = d(x, y)$$
and
$$d(kx, ky) = kd(x, y)$$
Why do we need V to be a vector space?

3. Prove the Cauchy-Schwarz Inequality: if $v, w \in \mathbf{R}^n$, then
$$\left(\sum_{i=1}^{n} v_i w_i\right)^2 \leq \left(\sum_{i=1}^{n} v_i^2\right)\left(\sum_{i=1}^{n} w_i^2\right)$$

4. Suppose $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbf{R}^n$; define $\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{n} x_i y_i$ (the inner product), and $\|\mathbf{x}\| = (\mathbf{x} \cdot \mathbf{x})^{\frac{1}{2}} = (\sum_{i=1}^{n} x_i^2)^{\frac{1}{2}}$ (standard norm). Show that

   (a) $\|\mathbf{x}\| \geq 0$.

   (b) $\|\mathbf{x}\| = 0$ if and only if $\mathbf{x} = 0$.

   (c) $\|\alpha\mathbf{x}\| = |\alpha| \, \|\mathbf{x}\|$.

   (d) $\|\mathbf{x} \cdot \mathbf{y}\| \leq \|\mathbf{x}\| \, \|\mathbf{y}\|$.

   (e) $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$.

   (f) $\|\mathbf{x} - \mathbf{z}\| \leq \|\mathbf{x} - \mathbf{y}\| + \|\mathbf{y} - \mathbf{z}\|$.

5. Let $\{a_n\}$ and $\{b_n\}$ be sequences such that
$$\lim_{n\to\infty} a_n = a \qquad \text{and} \qquad \lim_{n\to\infty} b_n = b,$$
Show that

   (a) $\lim_{n\to\infty}(a_n + b_n) = a + b$

   (b) $\lim_{n\to\infty} ca_n = ca$, where $c \in \mathbb{R}$

   (c) $\lim_{n\to\infty} a_n b_n = ab$

   (d) $\lim_{n\to\infty}(a_n)^k = a^k$

   (e) $\lim_{n\to\infty} \frac{1}{a_n} = \frac{1}{a}$ provided $a_n \neq 0$ $(n = 1, 2, 3, \dots)$, and $a \neq 0$

(f) $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}$ provided $b_n \neq 0$ ($n = 1, 2, 3, \dots$), and $b \neq 0$

6. Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers such that $a_n \leq b_n \leq c_n$, for all $n \in \mathbb{N}$, and assume that $a_n \to a$, and $c_n \to a$. Show that $b_n \to a$.

7. Show that if $0 \leq a_n \leq b_n$ for $n \geq N$, where N is some fixed number, and if $b_n \to 0$, then $a_n \to 0$.

8. Prove the Rising Sun Lemma

Extra Practice Problems

1. Show that $C([0,1])$, i.e. set of continuous functions from $[0,1]$ to $\mathbb{R}$ is

   (a) a vector space

   (b) a normed space with norm $||f|| = \max_{t\in[0,1]} |f(t)|$

   (c) Is this normed space complete?

   (d) Is the set of all polynomials open in this space? closed? (Hint: this fact may be useful: for any bounded continuous function f from $[0,1]$ to $\mathbb{R}$, and every positive ϵ, there is a polynomial p such that $||f - p|| < \epsilon$.

   (e) Is the set of continuous functions $x(t)$ for which $|x(t)| < 1$ for every t is closed or open in this space?

2. Let $\{x_n\}$ be a sequence of real numbers. Show that $x = lim_{n\to\infty} x_n$ if and only if

$$x = x_1 + \sum_{s=1}^{\infty} (x_{s+1} - x_s).$$

3. Let $x_1 = 1.5$ and $x_{n+1} = \sqrt{1 + x_n}$ for $n \in \mathbb{N}$.

   (a) Prove that $x_n < 2$ for $n \in \mathbb{N}$.

   (b) Prove that the sequence $\{x_n\}$ is increasing.

   (c) Prove that $\{x_n\}$ is convergent and compute its limit.

4. Construct a sequence of real numbers that

   (a) does not have any cluster points

   (b) is unbounded and has at least one cluster point

   (c) has every real number as a cluster point (Hint: use fact that rational numbers are dense in $\mathbb{R}$ i.e. for every $x \in \mathbb{R}$, and every positive ϵ, there exists $m, n \in \mathbb{Q}$ such that $\left\| x - \frac{m}{n} \right\| < \epsilon$)

5. Show that $\limsup x_n = \infty$ if and only if given $M > 0$ and $n \in \mathbb{N}$, $\exists k \geq n$ with $x_k > M$.

6. Show that

$$\limsup x_n + \liminf y_n \leq \limsup(x_n + y_n) \leq \limsup x_n + \limsup y_n,$$

provided the right and the left sides are not of the form $\infty - \infty$.