Econ 2001   Summer 2015

Problem Set 4

1. State whether the following are open, closed, both, or neither:

   (a) The interval $(0, 1)$ as a subset of $\mathbb{R}$.

   (b) The interval $(0, 1)$ as a subset of $\mathbb{R}^2$.... well, the interval $(0, 1)$ is NOT a subset of $\mathbb{R}^2$, how about: the interval $(0, 1)$ imbedded in $\mathbb{R}^2$ as the subset $\{(x, 0) \mid x \in (0, 1)\}$.

   (c) $\mathbb{R}$ as a subset of $\mathbb{R}$.

   (d) $\mathbb{R}$ imbedded in $\mathbb{R}^2$ as the subset $\{(x, 0) \mid x \in \mathbb{R}\}$.

   (e) $\{(x, y, z) \mid 0 \leq x + y \leq 1, z = 0\}$ as a subset of $\mathbb{R}^3$

   (f) $\{(x, y) \mid 0 < x + y < 1\}\}$ as a subset of $\mathbb{R}^2$

2. Prove the following statements:

   (a) If A is closed in Y and Y is closed in X, then A is closed in X.

   (b) If U is open in X and A is closed in X then $U \setminus A$ is open in X and $A \setminus U$ is closed in X.

3. Find the boundary and the interior of each of the following subsets of $\mathbb{R}^2$:

   (a) $\{(x, y) : y = 0\}$.

   (b) $\{(x, y) : x > 0 \text{ and } y \neq 0\}$.

   (c) $\{(x, y) : x + y = 0\}$.

   (d) $\{(x, y) : x - y = 0 \text{ and } x \neq y\}$.

4. Identify the set of interior points, limit points, isolated points, and boundary points of the following sets. Assume the metric is Euclidean unless indicated otherwise (no proofs necessary):

   (a) $\{1, 1/2, 1/3, 1/4, ...\} \cup \{-1, -1/2, -1/3, -1/4, ...\} \cup \{0\} \subset \mathbf{R}$ (i.e. the ambient space is $\mathbf{R}$)

   (b) $\mathbf{N} \subset \mathbf{R}$

   (c) $\mathbf{N} \subset \mathbf{R}$, discrete metric ($d(x, y) = 0$ iff $x = y$, and $d(x, y) = 1$ if $x \neq y$))

   (d) $\mathbf{Q} \subset \mathbf{R}$

   (e) $\mathbf{Q} \subset \mathbf{R}$, discrete metric

   (f) $\{x \in \mathbf{Q} : \mathbf{x} < \boldsymbol{\pi}\} \subset \mathbf{R}$

   (g) $\{x \in \mathbf{Q} : \mathbf{x} < \boldsymbol{\pi}\} \subset \mathbf{Q}$

5. Closed sets and boundary points.

   (a) Consider a set A in an arbitrary metric space (X, d). Show that a point $x \notin A$ is a limit point of A iff it is a boundary point of A.

(b) Prove that a set A in an arbitrary metric space (X, d) is closed iff it contains all its boundary points.

6. Let (X, d) be a metric space and $E \subset X$. Show the following:

   (a) E is closed if every limit point of E is a point of E.

   (b) E is open if every point of E is an interior point of E.

7. Show that in any normed vector space the norm is a continuous function from X to $\mathbf{R}$. (Hint: Use the triangle inequality.)

8. Let (X, d) be a metric spaces and $(Y, ||\cdot||)$ a normed vector space. Suppose $f : X \to Y$ and $g : X \to Y$ are both continuous at x_0. Show that

   (a) $f + g$ is continuous at x_0

   (b) λf with λ scalar is continuous at x_0

   (c) fg is continuous at x_0 when $Y = \mathbf{R}$ and $||\cdot||$ is the usual norm

   (d) f/g is continuous at x_0 when $Y = \mathbf{R}$ and $||\cdot||$ is the usual norm, provided g is not zero in some open ball around x_0.

9. Let $g : \mathbf{R} \to \mathbf{R}$ be a continuous function, then the functions $H, G : \mathbf{R}^2 \to \mathbf{R}$ defined by

$$G(x, y) = g(x) \text{ and } H(x, y) = g(y)$$

   are also continuous.

10. Show that the function $f : \mathbf{R}^2 \to \mathbf{R}$ defined by $f(x, y) = x^2 - y^2$ is continuous.

11. Use the "pre-image of a closed set is closed" definition of continuity to show that the following subsets of $\mathbb{R}^2$ are closed:

   (a) $\{(x, y) : xy = 1\}$

   (b) $\{(x, y) : x^2 + y^2 = 1\}$

   (c) $\{(x, y) : x^2 + y^2 \leq 1\}$

   (d) $\{(x, y) \in \mathbb{R}^2 \mid y \geq x^2\}$.

Extra Practice Problems

1. State whether the following are open, closed, both, or neither:

   (a) $\{\frac{1}{n} \mid n \in \mathbb{N}\}$ as a subset of $\mathbb{R}$

   (b) $\{\frac{1}{n} \mid n \in \mathbb{N}\}$ as a subset of the interval $(0, \infty)$

   (c) $\mathbb{Z}$ in the topology on $\mathbb{R}$ induced by the usual metric;

(d) $\{1/n \mid n \in \mathbb{N}\}$ in the topology on $\mathbb{R}$ induced by the usual metric;

(e) $\mathbb{Q}$ in the topology on $\mathbb{R}$ induced by the usual metric.

2. Give an example of a metric space X and a non-trivial set in X which is both open and closed.

3. Let A be a subset of a metric space. Prove that $\text{int}(\text{int}(A)) = \text{int}(A)$.

4. Let $A_1, A_2, \ldots$ be subsets of a metric space, and $B = \bigcup_{n \in \mathbb{N}} A_i$.

   (a) Prove that $\overline{B} \supset \bigcup_{n \in \mathbb{N}} \overline{A_i}$.

   (b) Give an example where the inclusion in (a) is proper.
   (Compare this result with De La Fuente's Theorem 4.3(iii) in chapter 2. Again, weird things can happen when you perform infinitely many operations.)

5. Let A, B, and A_α denote subsets of a space X and let A' denote the set of limit points of A. Determine whether the following equations hold; if an equality fails, determine whether one of the inclusions $\subset$ or $\supset$ holds.

   (a) $\overline{A \cap B} = \bar{A} \cap \bar{B}$.

   (b) $\overline{\cap A_\alpha} = \cap \bar{A}$.

   (c) $\overline{A \setminus B} = \bar{A} \setminus \bar{B}$.

   (d) $(A \cup B)' = A' \cup B'$.

   (e) $(A \cap B)' = A' \cap B'$.

6. Let (X, d), (Y, ρ) be metric spaces and $f : X \to Y$ be continuous on X. Is it true that

   (a) The image of every open set in X is open in Y?

   (b) The image of every closed set in X is closed in Y?

7. Find a function $f : \mathbb{R} \to \mathbb{R}$ that is continuous at precisely one point.

8. Prove whether the following functions from $\mathbb{R}$ to $\mathbb{R}$ are Lipschitz and whether they are locally Lipschitz.

   (a) $f(x) = x^2$

   (b) $f(x) = x^{\frac{1}{2}}$, defined on the non-negative reals.

   (c) $f(x) = x$, the identity map.

9. Uniform convergence: Let f_n be a sequence of real-valued functions from S to $\mathbb{R}$. We say that f_n *converges uniformly* to f if for every $\epsilon > 0$, there exists an $n \in N$ such that for all $x \in S$ and all $n \geq N$, $|f_n(x) - f(x)| < \epsilon$.

   (a) Define $f_n : [0, 1] \to \mathbb{R}$ by the equation $f_n(x) = x^n$. Show that the sequence f_n converges for each $x \in [0, 1]$, but not uniformly.

(b) Let $f_n : X \to Y$ be a sequence of continuous functions. Let x_n be a sequence of points of X converging to x. Show that if the sequence f_n converges uniformly to f, then $f_n(x_n)$ converges to $f(x)$.

10. Let X, Y by normed vector spaces and let functions $f, g : X \to Y$ be uniformly continuous.

(a) Is $f + g$ uniformly continuous?

(b) Let $Y = \mathbf{R}$ with the standard metric. Is $f \cdot g$ uniformly continuous?

(a) Consider the function $f(x) = \sin(1/x)$, defined only on the positive real numbers $\mathbf{R}^{++}$. Prove that f is continuous but not uniformly continuous.

(b) Consider the function $f(x) = x \sin(1/x)$, defined on the entire real line $\mathbf{R}$ (with $f(0) = 0$). Prove that f is continuous.

(c) Consider a function $f : A \times B \to \mathbf{R}$, where $A \subset \mathbf{R}, B \subset \mathbf{R}$. Call f *separately continuous* if for each fixed $x_0 \in A$, the map $g(y) = f(x_0, y)$ is continuous and for $y_0 \in B$, $h(x) = f(x, y_0)$ is continuous. Say that f is continuous on A *uniformly with respect to* B if for each $\varepsilon > 0$ and $x_0 \in A$, there is a $\delta > 0$ such that $|x - x_0| < \delta$ implies $|f(x, y) - f(x_0, y)| < \varepsilon$ for all $y \in B$. Prove that if f is separately continuous and is continuous on A uniformly with respect to B, then f is continuous.