Problem Set 5

1. Let $(\mathbf{R}^n, d)$ is the n-dimentional Euclidean metric space. $f : \mathbf{R}^n \to \mathbf{R}^1$ is a function. Show that f is countinuous if and only if for every $c \in \mathbf{R}^1$, A_c and B_c are closed sets where $A_c = \{x \in \mathbf{R}^n : f(x) \geq c\}$ and $B_c = \{x \in \mathbf{R}^n : f(x) \leq c\}$

2. Let $f(x) = x^3 + ax^2 + bx + c$. Prove that there exists at least one value y such that $f(y) = 0$.

3. Show that the function $\frac{x^2+1}{x+2} + \frac{x^4+1}{x-3}$ is equal to zero for at least one value of x between -2 and 3.

4. Let $f : \mathbf{R} \to \mathbf{R}$ satisfy $f(x) = -f(-x)$ for all $x \neq 0$. Show that if f is continuous at 0, then $f(0) = 0$.

5. Suppose $f : A \to \mathbf{R}$ is continuous and $A \subset \mathbf{R}$ is compact. Prove that f is uniformly continuous.

6. Suppose $f : \mathbf{R} \to \mathbf{R}$ is a continuous function such that $\lim_{x\to-\infty} f(x) = 0 = \lim_{x\to\infty} f(x)$. Prove that f is bounded and attains either a maximum or a minimum.

7. Suppose $f : \mathbf{R} \to \mathbf{R}$ is continuous and satisfies $f(x) \neq 0$ for all x. Which of the following is true? Why?

   (a) f attains a maximum.

   (b) Either $f(x) > 0$ for all x or $f(x) < 0$ for all x.

   (c) $h(x) = 1/f(x)$ is continuous.

   (d) $h(x) = 1/f(x)$ is bounded.

8. For $x > 0$, define $f(x) = \frac{1}{1}(x + \frac{2}{x})$.

   (a) Show that if $X = [1, 2]$, then f is a contraction on X.

   (b) Find the fixed point of this contraction.

   (c) Show that if $X = (0, 1)$, then f is not a contraction on X; that is, there does not exist $\beta \in (0; 1)$ such that:

   $$\forall x, y \in X : |f(x) - f(y)| \leq |x - y|$$

9. Show that if x_n and y_n are Cauchy sequences from a metric space X, then $d(x_n, y_n)$ converges.

10. Suppose $\{x_n\} \in \mathbf{R}^n$ is a Cauchy sequence. It has a subsequence $\{x_{n_k}\}$ such that $\lim_{n_k\to\infty} x_{n_k} = x$. Show that $\lim_{n\to\infty} x_n = x$.

Extra Practice Problems

1. Let (X, d), (Y, ρ) be metric spaces and $f : X \to Y$ be continuous on X. Is it true that

   (a) The image of every open set in X is open in Y?

   (b) The image of every closed set in X is closed in Y?

2. Find a function $f : \mathbb{R} \to \mathbb{R}$ that is continuous at precisely one point.

3. Prove whether the following functions from $\mathbb{R}$ to $\mathbb{R}$ are Lipschitz and whether they are locally Lipschitz.

   (a) $f(x) = x^2$

   (b) $f(x) = x^{\frac{1}{2}}$, defined on the non-negative reals.

   (c) $f(x) = x$, the identity map

4. Give examples of the following:

   (a) A continuous function $f : S \to \mathbb{R}$, where S is a closed subset of $\mathbb{R}$, that attains neither a maximum nor a minimum on S;

   (b) A continuous function $f : S \to \mathbb{R}$, where S is a closed and unbounded subset of $\mathbb{R}$, that attains both a maximum and a minimum on S;

   (c) A continuous function $f : S \to \mathbb{R}$, where S is a bounded subset of $\mathbb{R}$, that attains neither a maximum nor a minimum on S;

   (d) A continuous function $f : S \to \mathbb{R}$, where S is a bounded but not closed subset of $\mathbb{R}$, that attains both a maximum and a minimum on S;

   (e) A discontinuous function $f : S \to \mathbb{R}$, where S is a closed and bounded subset of $\mathbb{R}$, that attains neither a maximum nor a minimum on S;

   (f) A discontinuous function $f : S \to \mathbb{R}$, where S is a closed and bounded subset of $\mathbb{R}$, that attains both a maximum and a minimum on S.

5. Let $X = C([0, 1])$, $d(f, g) = \max_t |f(t) - g(t)|$. Show that (X, d) is complete.

6. Let X denote the set of all bounded finite and infinite sequences of real numbers $\{a_n\}_{n=1}^{\infty}$ (hereafter denoted simply as a_n). Define the "distance" between two sequences a_n and b_n to be: $d(a_n, b_n) = \sum_{n=1}^{\infty} 2^{-n}|a_n - b_n|$. Show that:

   (a) (X, d) is a metric space.

   (b) (X, d) is not complete.

7. Show that the metric space (X, d) is complete if and only if for any nested sequence $A_1 \supset A_2 \supset \cdots$ of nonempty closed subsets of X such that diameter $A_n \to 0$,

$$\bigcap A_n \neq \emptyset.$$

8. Assume $f : S \to T$ is uniformly continuous on S, where S and T are metric spaces. If $\{x_n\}$ is any Cauchy sequence in S, prove that $\{f(x_n)\}$ is a Cauchy sequence in T. Provide an example to show that the statement is not true if f is just continuous.

9. Prove that the sequence $x_1 = \sqrt{2}, x_2 = \sqrt{2+\sqrt{2}}, x_3 = \sqrt{2+\sqrt{2+\sqrt{2}}}, \ldots$ converges.

10. Cauchy sequences

    (a) Use the Cauchy criterion to prove convergence or divergence of the following sequence in $\mathbf{R}$: $\mathbf{x_n = 1 + 1/2 + 1/3 + 1/4 + \cdots + 1/n}$

    (b) Consider a sequence $\{x_n\}$ in a metric space (X, d) such that $\sum_{n\geq 1} d(x_n,\ x_{n+1}) <$ inf. Is this sequence Cauchy? Does every Cauchy sequence have this property?

11. Suppose (X, d) is a complete metric space, and $T : X \to X$ is an *expansion,* i.e. there exists > 1 such that $d(Tx, Ty) > d(x, y)$ for all x, y in X, and that $T(X) = X$. Show that T has a fixed point.

    (a) Let $X = C([0, 1])$, $d(f, g) = max_t|f(t) - g(t)|$. Show that (X,d) is complete.

    (b) Let $X = C([0,])$, $d(f, g) = max_t|f(t) - g(t)|$ with < 1. Define $T : X \to X$ by $(Tf)(t) = \int_0^t f(s)\,ds$. Show that T has a unique fixed point.

12. Uniform convergence: Let f_n be a sequence of real-valued functions from S to $\mathbb{R}$. We say that f_n *converges uniformly* to f if for every $\epsilon > 0$, there exists an $n \in N$ such that for all $x \in S$ and all $n \geq N$, $|f_n(x) - f(x)| < \epsilon$.

    (a) Define $f_n : [0, 1] \to \mathbb{R}$ by the equation $f_n(x) = x^n$. Show that the sequence f_n converges for each $x \in [0, 1]$, but not uniformly.

    (b) Let $f_n : X \to Y$ be a sequence of continuous functions. Let x_n be a sequence of points of X converging to x. Show that if the sequence f_n converges uniformly to f, then $f_n(x_n)$ converges to $f(x)$.

13. Let X, Y by normed vector spaces and let functions $f, g : X \to Y$ be uniformly continuous.

    (a) Is $f + g$ uniformly continuous?

    (b) Let $Y = \mathbf{R}$ with the standard metric. Is $f \cdot g$ uniformly continuous?

    (a) Consider the function $f(x) = \sin(1/x)$, defined only on the positive real numbers $\mathbf{R}^{++}$. Prove that f is continuous but not uniformly continuous.

    (b) Consider the function $f(x) = x\sin(1/x)$, defined on the entire real line $\mathbf{R}$ (with $f(0) = 0$). Prove that f is continuous.

    (c) Consider a function $f : A \times B \to \mathbf{R}$, where $A \subset \mathbf{R}, B \subset \mathbf{R}$. Call f *separately continuous* if for each fixed $x_0 \in A$, the map $g(y) = f(x_0, y)$ is continuous and for $y_0 \in B$, $h(x) = f(x, y_0)$ is continuous. Say that f is continuous on A *uniformly with respect to* B if for each $\varepsilon > 0$ and $x_0 \in A$, there is a $\delta > 0$ such that $|x - x_0| < \delta$ implies $|f(x, y) - f(x_0, y)| < \varepsilon$ for all $y \in B$. Prove that if f is separately continuous and is continuous on A uniformly with respect to B, then f is continuous.