Problem Set 7

1. This example shows that the correspondenceμ in Berge's theorem may not be lower hemicontinuous. Using the notation of that theorem, let $A = B = [0, 1]$, let $\varphi(a) = B$ for all a, and let $f(a, b) = b(a-\frac{1}{2})$. For all $a \in A$ let $h(a) = \max_{b\in\varphi(a)} f(a, b)$ and let $\mu(a) = \{b \in \varphi(a) : h(a) = f(a, b)\}$. Verify that the conditions for Berge's theorem are satisfied in this case. Compute μ and verify that it is upper but not lower hemicontinuous (a picture is enough).

2. Let A be an $m \times n$ matrix and let A^t denote its transpose.

   (a) Show that the matrix A^tA is well defined for all m and n.

   (b) Show that A^tA is symmetric.

3. Prove the following dot product properties for $x, y, z \in \mathbf{R}^n$ and $c \in \mathbf{R}$.

   The dot product is commutative: $x \cdot y = y \cdot x$.

   The dot product is distributive over vector addition: $x \cdot (y + z) = x \cdot y + x \cdot z$.

   The dot product is bilinear: $x \cdot (ry + z) = r(x \cdot y) + (x \cdot z)$.

   When multiplied by a scalar value, dot product satisfies: $(c_1x) \cdot (c_2y) = (c_1c_2)(x \cdot y)$

4. Prove that the set $\mathcal{M}_{m\times n}$ of all $m \times n$ real-valued matrices is a vector space.

5. Consider the vectors $x_1 = (1, 1, 0, 1)$, $x_2 = (2, 0, 0, 2)$, $x_3 = (1, 1, 1, 1)$, $x_4 = (1, 0, 1, 1)$. Denote by S the span of $\{x_1, x_2, x_3, x_4\}$.

   (a) Find the dimension of S. Confirm your answer by exhibiting a basis for this set.

   (b) Is $(1, 0, 0, 2)$ in S?

   (c) Is $(0, 1, 1, 0)$ in S?

6. Let S be the set spanned by a_1, a_2, a_3, a_4, where $a_1 = (1, 2, -2)$, $a_2 = (0, -1, 1)$, $a_3 = (0, 0, -5)$ and $a_4 = (3, 5, 10)$.

   (a) Select a basis from a_1, a_2, a_3, a_4.

   (b) Express the remaining vectors in a_1, a_2, a_3, a_4 as a linear combinations of the chosen basis.

7. A set of non-zero elements $\{x_1, ..., x_k\} \subset \mathbf{R}^n$ is said to be orthogonal if, for each $i, j = 1, ..., k$, if $i \neq j$, then $x_i \cdot x_j = 0$.

   (a) Prove that if the set $\{x_1, ..., x_k\}$ is orthogonal, then it is linearly independent.

   (b) Find an orthogonal basis for $\mathbf{R}^n$.

8. Find a basis and the dimension of the space for:

   (a) The set of solutions in R^3 to the following systems of linear equations

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_1 - 3x_2 + x_3 &= 0 \end{aligned}$$

   (b) The set of all $n \times n$ matrices having trace equal to zero (The trace of an $n \times n$ matrix M, denoted $tr(M)$, is the sum of the diagonal entries of M; that is, $tr(M) = M_{11} + M_{22} + ... + M_{nn.}$).

9. Show that if $X = \{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ is a set of linearly independent vectors in $\mathbf{R}^n$ that does not span it, then there exists $\mathbf{v} \in \mathbf{R}^n$ such that $X \cup \{\mathbf{v}\}$ is linearly independent.