Econ 2001 Summer 2015

Problem Set 8

1. Show that if λ is an eigenvalue of A then λ^k is an eigenvalue of A^k and λ^{-1} is an eigenvalue of A^{-1}.

2. Let A, B be $n \times n$ matrices. Prove that AB has the same eigenvalues as BA.

3. Let A and B denote symmetric $n \times n$ matrices, such that $AB = BA$. Prove that A and B have a common eigenvector. (note that the claim that every eigenvector of AB is also an eigenvector of BA is false).

4. Prove that the eigenvalues of a triangular matrix (upper or lower triangular) are the entries on the diagonal.

5. Show that A and A^t have the same characteristic polynomials and hence the same eigenvalues.

6. Square matrices A and B are similar means that $A = P^{-1}BP$, for some invertible matrix P.

   (a) Which matrices are similar to the identity matrix? to zero matrix?

   (b) Would your answers to part (a) suggest that a matrix of the form cI for some scalar c is similar only to itself? Or, that a diagonal matrix is similar only to itself?

   (c) Show that if $A - \lambda I$ and B are similar matrices then A and $B + \lambda I$ are also similar.

   (d) Show that similar matrices have the same determinant.

   (e) Show that matrices A and B have the same characteristic polynomials and hence the same eigenvalues

   (f) Show that if A is similar to B and A is non-singular then B is non-singular and A^{-1} and B^{-1} are similar.

   (g) Show that if A and B are similar and λ is a scalar then $A - \lambda I$ and $B - \lambda I$ are similar.

7. Show that similar matrices have the same trace, by following the steps below.

   (a) For any $n \times n$ matrices A and B $tr(AB) = tr(BA)$ where $tr(M)$ denotes the trace of matrix M.

   (b) If A and B are similar $n \times n$ matrices, then $tr(A) = tr(B)$.

8. Identify which of the following matrices are diagonalizable and provide the diagonalization. If the diagonalization does not exist, prove it.

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}, \text{and } C = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

9. For each of the following quadratic forms:

$$f(x, y) = 5x^2 + 2xy + 5y^2, \;\; g(x, y) = 4xy$$

   find a symmetric matrix A such that the form equals $(x\ y)A\binom{x}{y}$

10. Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

   (a) Find the eigenvalues of A.

   (b) Find a maximal cardinality set of linearly independent eigenvectors for A. Associate these eigenvectors with the eigenvalues you found above.

   (c) Is A diagonalizable?

   (d) If A diagonalizable, find a matrix P such that $A = PDP^{-1}$, where D is diagonal.

   (e) State whether the quadratic form $Q(x) = x^t Ax$ is positive (semi-) definite, negative (semi-) definite, or indefinite.

11. Suppose A is an $n \times n$ positive semidefinite matrix. Is $B^T AB$ positive semidefinite for an arbitrary $n \times m$ matrix B ? Suppose A is positive definite; is $B^T AB$ in this case positive definite? What assumptions (if any) can you make to get these conclusions?

12. Let V be the subspace of $\Re^5$ defined by $V = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 : x_1 = 3x_2, x_3 = 7x_4\}$. Find a basis for V.

13. Suppose $\dim X = n \in \mathbf{N}$, $V \subset X$, and $|V| = n$. Prove the following statements.

   (a) If V is linearly independent, then V spans X, so V is a basis.

   (b) If V spans X, then V is linearly independent, so V is a basis.

14. Let X and Y be two vector spaces. We say $T : X \to Y$ is a *linear transformation* if

$$T(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 T(x_1) + \alpha_2 T(x_2) \quad \forall x_1, x_2 \in X, \alpha_1, \alpha_2 \in \mathbf{R}$$

   Let $L(X, Y)$ denote the set of all linear transformations from X to Y. Prove that $L(X, Y)$ is a vector space.