Problem Set 9

1. Prove that if $T \in L(X, Y)$ is invertible, then $T^{-1} \in L(Y, X)$.

2. Define the coordinate representation of $\mathbf{x}$ with respect to the basis V defined in the lecture notes as $crd_V(\mathbf{x})$ is an isomorphism.

3. Given a linear transformation $T : X \to Y$, prove that T is one-to-one if and only if $\ker T = \{0\}$.

4. $T : V \to V$ is a linear transformation, where V is a finite dimensional vector space. Prove that T is invertible if and only if $\ker T = \{0\}$.

5. $T : X \to Y$ is a linear transformation, where X and Y are finite dimensional vector spaces. Prove the following statements:

   (a) If T is invertible, then $\dim(X) = \dim(Y)$.

   (b) If $\dim(X) < \dim(Y)$, then T cannot be onto.

   (c) If $\dim(X) > \dim(Y)$, then T cannot be one-to-one.

6. Show that if $(x_1, x_2, ..., x_n)$ spans X and $T \in L(X, Y)$ is onto, then $(Tx_1, Tx_2, ..., Tx_n)$ spans Y.

7. Show that the coordinate representation of $\mathbf{x}$ with respect to the basis V defined in class (crd_V) is an imorphism.

8. $T : M_{2\times 3} \to M_{2\times 2}$ defined by

$$T\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} = \begin{pmatrix} 2a_{11} - a_{12} & a_{13} + 2a_{12} \\ 0 & 0 \end{pmatrix}$$

   Determine Ker(T), $\dim(Ker(T))$ and Rank(T). Is T one-to-one, onto, or neither?

9. Recall that a reflection across the x-axis can be achieved with the transformation $(x, y) \to (x, -y)$. Derive a transformation, T, which reflects a point across the line $y = 3x$.

   (a) Calculate the action of T on the points $(1, 3)$ and $(-3, 1)$.

   (b) Write the matrix representation of T using these two vectors as bases.

   (c) Write the matrix representation of T in the standard basis.

10. Derive a transformation, $T : R^2 \to R^2$, which reflects a point across the line $y = 5x$.

   (a) First, calculate the action of T on the points $(1, 5)$ and $(-5, 1)$.

   (b) Next, write the matrix representation of T using these two vectors as a basis.

   (c) Find S and S^{-1}, the matrices that changes coordinates under this basis to standard coordinates and back again.

(d) Write the matrix representation of T in the standard basis.

11. The sets $x + 2y + z = 4$ and $3x + y + 2z = 3$ intersect in a straight line.

    (a) Find the equation of the line of intersection.

    (b) Find the equation of the plane perpendicular to the line you found above and the point $(0, 0, 0)$.

12. Let $w = (1, 4, 0)$ and $v = (1, 0, 2)$.

    (a) Find the equation of the line that passes through the point w in the direction v.

    (b) Find the equation of a hyperplane that contains the point w and contains the line you found above.

    (c) Find an equation of a line that is contained in the hyperplane that you found in part (2), contains the point w, and is orthogonal to the line you found in part (1).