Problem Set 10

1. Given the following functions $f : \mathbf{R}^3 \to \mathbf{R}^2$, calculate $(Df)(x)$, the derivative of f at $x \in \mathbf{R}^3$.

   (a) $f(x_1, x_2, x_3) = \begin{pmatrix} x_1x_2 + x_3^2 \\ x_2^2x_3 \end{pmatrix}$

   (b) $f(x_1, x_2, x_3) = \begin{pmatrix} x_1^2 + 2x_1x_2 - x_3 \\ x_2x_3 + x_3^2 \end{pmatrix}$

2. Let $f : \mathbf{R}^2 \to \mathbf{R}$ be defined by setting $f(0,0) = 0$ and $f(x,y) = \frac{xy}{x^2+y^2}$ if $(x,y) \neq (0,0)$. Is f differentiable at $(0,0)$? Is f continuous at $(0,0)$?

3. Let $f : \mathbf{R}^2 \to \mathbf{R}$ be defined by $f(x_1, x_2) = x_1x_2$. Show that the Jacobian is equal to the average of the partial derivatives at $(0,0)$.

4. Let $f : \mathbf{R}^2 \to \mathbf{R}$ be defined by $f(x_1, x_2) = |x_1|^{\frac{1}{2}}|x_2|^{\frac{1}{2}}$. Do the partial derivatives exist at $(0,0)$? Is this function differentiable at $(0,0)$?

5. Let $f : \mathbf{R} \to \mathbf{R}$ be defined by

   $$f(x) = \begin{cases} x \text{ if } x \text{ is rational} \\ 0 \text{ if } x \text{ is irrational} \end{cases}$$

   Find the set of points for which f is continuous and show that this function is nowhere differentiable.

6. Let $f : \mathbf{R}^2 \to \mathbf{R}$ be defined by $f(x_1, x_2) = x_1^3x_2 - x_2^2$.

   (a) Graph the zero-value level set:

   $$\{x_1, x_2 \in \mathbf{R} \times \mathbf{R} : f(x_1, x_2) = 0\}$$

   (b) Determine the equation of the line tangent to this level set at the point $(1,1)$.

   (c) Find the equation of the tangent hyperplane to the graph of $y = f(x_1, x_2)$ at $(x_1, x_2, y) = (2, 1, 7)$.

7. Let $f : \mathbf{R}^2 \to \mathbf{R}$ be defined by $f(x_1, x_2, x_3) = 3x_1^2 + 2x_2 + x_1x_2$.

   (a) Graph the zero-value level set:

   (b) Compute $\nabla f(\cdot)$, the gradient of f.

   (c) Find the equation of the plane tangent to the graph of f at the point $(3, -10, 17)$.

8. Let $f : \mathbf{R}^2 \to \mathbf{R}$ be defined by $f(x_1, x_2) = x_1^3x_2 - x_2^2$.

   (a) Graph the zero-value level set:

   $$\{x_1, x_2 \in \mathbf{R} \times \mathbf{R} : f(x_1, x_2) = 0\}$$

   (b) Determine the equation of the line tangent to this level set at the point $(1,1)$.

(c) Find the equation of the tangent hyperplane to the graph of $y = f(x_1, x_2)$ at $(x_1, x_2, y) = (2, 1, 7)$.

9. Evaluate the following derivative. $Dh(2, 1)$, for $h = f \circ g$ where $f : \mathbf{R}^2 \to \mathbf{R}^2$ and $g : \mathbf{R}^2 \to \mathbf{R}^2$ are defined by $f(y) = \begin{pmatrix} y_1^2 + 3 \\ y_1 y_2 \end{pmatrix}$ and $g(x) = \begin{pmatrix} x_1 + x_2 \\ x_1 - x_2 \end{pmatrix}$.

10. Suppose $f : \mathbf{R}^2 \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}^2$ are defined by $f(x) = x_1^2(3x_1 + x_2^2)$ and $g(t) = \begin{pmatrix} t^2 \\ e^t \end{pmatrix}$.

    (a) Compute $Df(x)$ and $Dg(t)$.

    (b) Is $f \circ g$ differentiable over $\mathbf{R}$? (Briefly explain why you know.)

    (c) Compute the first derivative of $f \circ g$ using the chain rule.

11. Let $f(x) = x^2$ and $g(y_1; y_2) = y_1 + log[(y_2)^2 + 3] + y_1 y_2$.

    (a) Find the partial derivatives of $g(\cdot)$.

    (b) Find the partial derivatives of $f \circ g$ directly and by using the chain rule.

    (c) 3 Which of the functions f, g, and $f \circ g$ are homogeneous?

    (d) Find the equation of a plane tangent to the graph of $g(\cdot)$ at the point $(y, y) = (1, 1)$.

    (e) If $F(x, y, z) = f[u(x, y, z), v(x, y, z)]$ evaluate $\frac{\partial F}{\partial x}$ at the point (x_0, y_0, z_0) if $u(x, y, z) = (x^2 - 4y)e^x$, $v(x, y, z) = xyz$, $f(u, v) = log(u + v + 2)$ and $(x, y, z) = (2, 1, 0)$.

12. Consider the function $f(x, y) = x^3 y + y^2$ defined on $\mathbf{R}^2$.

    (a) Graph $\{f(x, y) : f(x, y) = 0\}$.

    (b) Find an equation of the hyperplane tangent to the graph of $f(x, y) = z$ at the point $(x, y, z) = (0, 2, 4)$.

    (c) Find the equation of some line that lies in the tangent hyperplane that you found in part 2.

    (d) Compute the directional derivative of the function $f(\cdot)$ in the direction $v = (3/5, 4/5)$.

13. Consider the function $g(x, y) = x^2 + y^2$.

    (a) Graph $\{f(x, y) : f(x, y) = 8 \text{ and } (x, y) \geq 0\}$.

    (b) Find an equation of a line tangent to the curve that you drew in part 1 at the point $(x, y) = (2, 2)$.

    (c) Let $f(t) = (t, t^2)$. Use the chain rule to compute all partial derivatives of $f \circ g$ at $(x, y) = (2, 2)$.