Problem Set 11

1. Prove that if $g(\cdot)$ is homogeneous of degree one, then $g(x, y)/(x + y)$ is homogeneous of degree zero. [Add any assumptions (i.e. regularity conditions) needed to prove this result.]

2. Prove that if $g(\cdot)$ is differentiable and homogeneous of degree n, then all of the partial derivatives of $g(\cdot)$ are homogeneous of degree $n - 1$. [Add any regularity conditions needed to prove this result.]

3. Suppose that the equation $f(x, y, z) = 0$ can be solved for each of the three variables as a differentiable function of the other two. Denote by $X(y, z)$ the solution to $f(X(y, z), y, z) = 0$. Similarly define $Y(x, z)$ and $Z(x, y)$. Show that
$$\frac{\partial X}{\partial y}\frac{\partial Y}{\partial z}\frac{\partial Z}{\partial x} = -1$$

4. A twice continuously differentiable function $f(\cdot)$ from the real line to itself satisfies the following conditions: $f(1) = -1$; $f(2) = 3$; $f(3) = f(4) = 2$. Find the minimum number of times that $f(x) = 0$, $f'(x) = 0$, and $f''(x) = 0$.

5. Suppose that $f(\cdot)$ is a continuous real-valued function defined on the open interval $(-1, 1)$. Further suppose that $f(x)$ exists for all $x \neq 0$ and that $\lim_{x\to 0} f'(x)$ exists. Prove that $f(\cdot)$ is di erentiable at $x = 0$ and that $f'(\cdot)$ is continuous at $x = 0$.

6. Consider the following functions:
$$f(x) = 3x^3 + 1 \qquad f(x) = x\log(x + 1) \qquad f(x) = e^{(1+x^2)} \qquad f(x) = 3x - 1$$
and answer for each the following questions
   (a) Graph $f(\cdot)$.
   (b) Find the equation of the line tangent to the graph of f at $x = 1$
   (c) Find a second-order Taylor expansion of f around $x = 0$

7. Write the second-order Taylor expansion for $f(x, y) = 11y^2 + 7x^2 - (4y - 9)x + (2x + 3)y - 6$ around the point (x_0, y_0). Then, choose some value for (x_0, y_0) and explain how the approximation depends on your choice.

8. Write the second-order Taylor expansion of $f(x, y) = y\log(xy) + e^{xy}$ around the point (1, 1) through the quadratic term and write the error estimate.

9. Find the third-degree Taylor approximation to $f(x_1, x_2) = (x_1 + x_2)^2$ at $(0, 0)$ and $(1, 1)$.

10. Find the third-degree Taylor approximation to $f(x, y, z) = x^3y^2z$ at $(-1, 0, 1)$.