Problem Set 12

1. Suppose: f is continuous for $x \geq 0$, $f'(x)$ exists for all $x > 0$, $f(0) = 0$, and f' is monotonically increasing. Let $g(x) = \frac{f(x)}{x}$, for $x > 0$. Show that $g(x)$ is monotonically increasing.

2. Consider the function $f : \mathbf{R}^2 \to \mathbf{R}$ defined by $f(x) = x_1 - 7x_1^2 + 9x_1x_2 - 3x_2^2$.

   (a) Calculate $\nabla f(x)$ and $D^2 f(x)$.

   (b) Write the two-equation system that defines a critical point x using the matrix-algebraic form $Ax + b = 0$, where A is a 2×2 matrix of constants, $x = (x_1, x_2)$, and b is a column vector of constants. Identify the matrix A and the vector b.

   (c) Identify whether x is a maximizer, a minimizer, or a saddle point.

3. For given data $(y_1, y_2, ..., y_n)$ and $(x_1, x_2, ..., x_n)$, find the values of a and b that minimize $\sum_{i=1}^{n}[y_i - (a + bx_i)]^2$, and prove your answer. Can you explain what have you just found in words? Solve the same problem when each x_i is a vector (b is thus also a vector).

4. Consider the function $f : \mathbf{R}^2 \to \mathbf{R}$ defined by $f(x) = 4x^3 + y^2 - 6xy + 6x$. Find its critical points and determine their nature.

5. Solve $\max xy + 2x + 5y - x^2 - y^2$. Verify first and second order conditions.

6. Consider the function $f(x, y) = x^3 - 3xe^{-y^2}$.

   (a) Compute $\nabla f(x, y)$.

   (b) Compute $D^2 f(x, y)$.

   (c) What is the function $g(x, y)$ that defines the tangent plane to the graph of f at the point $(x, y) = (2, 1)$? (Hint: g is the first-order Taylor polynomial of f at $(2, 1)$.)

   (d) Find the local minima and maxima of f. (Use necessary and sufficient conditions.)

7. Let $f : \mathbf{R}^2 \to \mathbf{R}^2$ be defined by the equation $f(x, y) = (x^2 - y^2, 2xy)$. Assuming that f is one-to-one and g is the inverse function, find $Dg(0, 1)$.

8. Let $f : R^3 \to R^3$ be defined by $f(x, y, z) = (x^2, y^2 - 1, z^2)$. Provide sufficient conditions for f to be locally invertible, and find all the points in R^3 where these conditions are satisfied. What is the Jacobian of f^{-1} at such points.

9. Let $E = \{(x, y) : x > 0,\ y > 0\}$ and set

$$f(x, y) = \left(\frac{x^2 + y^2}{x}, \frac{y}{x} \right) \quad \text{for } (x, y) \in E$$

   (a) Find the range of f. Show it is one-to-one from E onto its range.

   (b) Find a formula for $f^{-1}(s, t)$.

(c) Use the Inverse Function Theorem to compute $D(f^{-1})(f(x, y))$.

(d) Use the formula above to compute $D(f^{-1})(s, t)$ directly and verify that the Inverse Function Theorem holds.

10. Show that $f(x, y) = (\frac{x}{x^2+y^2}, \frac{x}{x^2+y^2})$ is locally invertible in a neighborhood of every point except the origin. Compute the inverse function explicitly.

11. Show that $f(x, y) = (e^x + e^y, e^x - e^y)$ and $g(x, y) = (x + y, x - 2y)$ are locally invertible everywhere.