ECON 2001
Exam 2
19 August 2011

This is a closed book exam. Please read the entire exam before starting.
You have 60 minutes to answer all FOUR questions.
Start each question on a new page.

Question 1

Let $x = (4, 4, 0)$, $y = (1, 1, 1)$, and $z = (1, 1, 0)$.

1. Find an equation of the plane that passes through the points x, y, and z.
2. Find an equation of a line that passes through x and is orthogonal to the plane that you found in the first part.
3. Find an equation of a plane that passes through $(0, 1, 1)$ and does not intersect the plane that you found in the first part.

Solutions

1. Two directions in the plane are $(0, 0, 1)$ and $(3, 3, 0)$. So a normal to the plane is $(1, -1, 0)$.
   Hence an equation of the plane is: $(1, -1, 0) \cdot (u_1 - 1, u_2 - 1, u_3) = 0$ or $u_1 - u_2 = 0$.
2. You need a line through x with direction $(1, -1, 0)$. You can write this as $u = (4, 4, 0) + t(1, -1, 0)$.
3. The plane must have the same normal as the plane in the first part, so its equation is: $(1, -1, 0) \cdot (u_1, u_2 - 1, u_3 - 1) = 0$ or $u_1 - u_2 = -1$.

Question 2

Consider the following two by two matrices

$$A = \begin{pmatrix} 4 & 1 \\ 1 & -2 \end{pmatrix}, B = \begin{pmatrix} 3 & 0 \\ 1 & 3 \end{pmatrix}$$

and answer the following questions:

1. Is each matrix diagonalizable? Justify your answer (but you do not need to diagonalize them).
2. Find each matrix eigenvalues and at least one corresponding eigenvector.
3. Is the quadratic form $x^t Ax$ positive (semi-) definite, negative (semi-) definite, or indefinite? Explain why the same question for $x^t Bx$ does not make sense.

Solutions

1. A is symmetric, hence it is diagonalizable.
   B has only one distinct eigenvalue, so it is not diagonalizable (see below).
2. A's characteristic equation is $\det \begin{pmatrix} 4 & 1 \\ 1 & -2 \end{pmatrix} = 0$ that can be rewritten as $(4-\lambda)(-2-\lambda) - 1 = 0$; the solutions of this equation are $1 \pm \sqrt{10}$; associated eigenvectors are $(1, -3+\sqrt{10})$ and $(3-\sqrt{10}, 1)$.
   B is triangular and therefore its eigenvalues are the diagonal elements; the diagonal elements are equal, so there is only one distinct eigenvalue which is equal to 3. The eigenvectors are multiples of $(0, 1)$.
3. From above, we know that A's eigenvector have opposite signs, so the corresponding quadratic form is indefinite; another way to see that is to note that $\det A = (4)(-2) - 1 = -9 < 0$.
   B is not symmetric, hence $x^t Bx$ is not a quadratic form.

Question 3

Suppose f and g are linear functions from $\mathbb{R}^n$ to $\mathbb{R}^k$ and from $\mathbb{R}^k$ to $\mathbb{R}^m$ respectively. Prove that $h = g(f())$ is also linear.

HINT: l is linear if and only if (i) for all x, y, $l(x+y) = l(x) + l(y)$ and (ii) for all scalars λ, $l(\lambda x) = \lambda l(x)$.

Solution

Verify (i):

$$h(x+y) = g(f(x+y)) = g(f(x) + f(y)) = g(f(x)) + g(f(y)) = h(x) + h(y)$$

Verify (ii):

$$h(\lambda x) = g(f(\lambda x)) = g(\lambda f(x)) = \lambda g(f(x)) = \lambda h(x)$$

In both cases, the first and last equality follow from the definition of h while the middle ones follows from the linearity of f and g respectively.

Question 4

A matrix A is *idempotent* if $AA = A$. Prove that the eigenvalues of an idempotent matrix are all either zero or one..

HINT: You do not need to compute eigenvalues.

Suppose that λ is an eigenvalue of A. Then there is a non-zero eigenvector x, such that $Ax = \lambda x$. Hence

$$\begin{aligned}
\lambda x &= Ax \\
&= AAx && \text{because } A \text{ is idempotent} \\
&= A(Ax) \\
&= A(\lambda x) && \text{by the first equality} \\
&= \lambda(Ax) && \text{by scalar multiplication} \\
&= \lambda(\lambda x) && \text{by the first equality again} \\
&= \lambda^2 x
\end{aligned}$$

Hence

$$\lambda^2 x = \lambda x$$

or

$$0 = \lambda^2 x - \lambda x = (\lambda - 1)\lambda x$$

Since x is an eigenvector, it is nonzero, and therefore the quadratic equation above is solved by $\lambda = 0$ and $\lambda = 1$.