Lecture 4

Econ 2001

2015 August 13

Lecture 4 Outline

1. Open and Closed Set
2. Continuity

Announcements:
- Tomorrow: first test at 3pm, in WWPH 4716. The exam will last an hour.
- Tomorrow: recitation at 1pm, in WWPH 4716.

Open and Closed Sets

- A set is open if at any point we can find a neighborhood of that point contained in the set.

Definition

Let (X, d) be a metric space. A set $A \subset X$ is open if

$$\forall x \in A \; \exists \varepsilon > 0 \qquad \text{such that} \qquad B_\varepsilon(x) \subset A$$

- Remember that $B_\varepsilon(x) = \{y \in X : d(y, x) < \varepsilon\}$... so openness depends on X.

Definition

A set $C \subset X$ is closed if $X \setminus C$ is open.

Draw Pictures

Open and Closed Sets: Examples

Open Interval in **R**

(a, b) is open in **R** (with the usual Euclidean metric).

- Given $x \in (a, b)$, $a < x < b$. Let
$$\varepsilon = \min\{x - a, b - x\} > 0$$
- Then
$$y \in B_\varepsilon(x) \Rightarrow y \in (x - \varepsilon, x + \varepsilon) \subset (x - (x - a), x + (b - x)) = (a, b)$$
- Hence $B_\varepsilon(x) \subset (a, b)$, so (a, b) is open.

- ε must depend on x; in particular, ε gets smaller as x nears the boundaries of the interval.

Closed Interval in **R**

$[a, b]$ is closed in **R** (with the usual Euclidean metric).

- $\mathbf{R} \setminus [a, b] = (-\infty, a) \cup (b, \infty)$ is the union of two open sets, which must be open (prove this later).

Open and Closed Sets: Examples

An open ball is always an open set

Suppose $y \in B_\varepsilon(x)$. Then $d(x,y) < \varepsilon$.

- Let
$$\delta = \varepsilon - d(x,y) > 0$$
- If $d(z,y) < \delta$, then
$$\begin{aligned} d(z,x) &\leq d(z,y) + d(y,x) \\ &< \delta + d(x,y) \\ &= \varepsilon - d(x,y) + d(x,y) \\ &= \varepsilon \end{aligned}$$
- Hence $B_\delta(y) \subset B_\epsilon(x)$, so $B_\varepsilon(x)$ is open.

- This is very usefuly since it holds for any X, d.

Open and Closed Sets: Examples

Openness and closedness depend on the underlying metric space

In the metric space $X = [0, 1]$ (with standard metric), $[0, 1]$ is open.

- Since $[0, 1]$ is the underlying metric space,

$$B_\varepsilon(0) = \{x \in X : d(x, 0) < \varepsilon\} = \{x \in [0, 1] : |x - 0| < \varepsilon\} = [0, \varepsilon)$$

Most sets are neither open nor closed

$[0, 1] \cup (2, 3)$ is neither open nor closed.

An open set may consist of a single point

If $X = \mathbf{N}$ and $d(m, n) = |m - n|$, then

$$B_{1/2}(1) = \{m \in \mathbf{N} : |m - 1| < 1/2\} = \{1\}$$

- Since 1 is the only element of the set $\{1\}$ and $B_{1/2}(1) = \{1\} \subset \{1\}$, the set $\{1\}$ is open.

Open and Closed Sets: Examples

In any metric space (X, d), $\emptyset$ and X are both open... and closed

- To see that $\emptyset$ is open, note that the statement

$$\forall x \in \emptyset \; \exists \varepsilon > 0 \qquad \text{such that} \qquad B_\varepsilon(x) \subset \emptyset$$

  is vacuously true since there aren't any $x \in \emptyset$ (any statement about points in the empty set is true).
- To see that X is open, note that $B_\varepsilon(x) = \{z \in X : d(z, x) < \varepsilon\}$ is trivially contained in X.
- Since $\emptyset$ is open, X is closed; since X is open, $\emptyset$ is closed.

Open and Closed Sets: Results

Theorem

Let (X, d) be a metric space. Then

1. *$\emptyset$ and X are both open and closed.*
2. *The union of an arbitrary (finite, countable, or uncountable) collection of open sets is open.*
3. *The intersection of a finite collection of open sets is open.*

Proof.

1. Already done.
2. Suppose $\{A_\lambda\}_{\lambda \in \Lambda}$ is a collection of open sets.

$$\begin{aligned} x \in \bigcup_{\lambda \in \Lambda} A_\lambda \quad &\Rightarrow \quad \exists \lambda_0 \in \Lambda \text{ such that } x \in A_{\lambda_0} \\ &\Rightarrow \quad \exists \varepsilon > 0 \text{ such that } B_\varepsilon(x) \subset A_{\lambda_0} \subset \bigcup_{\lambda \in \Lambda} A_\lambda \end{aligned}$$

so $\cup_{\lambda \in \Lambda} A_\lambda$ is open. □

Prove that the intersection of a finite collection of open sets is open.

Proof.

Suppose $A_1, \ldots, A_n \subset X$ are open sets.

- If $x \in \cap_{i=1}^n A_i$, then
$$x \in A_1,\ x \in A_2, \ldots,\ x \in A_n$$
- so
$$\exists \varepsilon_1 > 0, \ldots, \varepsilon_n > 0 \text{ such that } B_{\varepsilon_1}(x) \subset A_1, \ldots, B_{\varepsilon_n}(x) \subset A_n$$
- Let
$$\varepsilon = \min\{\varepsilon_1, \ldots, \varepsilon_n\} > 0$$
- Then
$$B_\varepsilon(x) \subset B_{\varepsilon_1}(x) \subset A_1, \ldots, B_\varepsilon(x) \subset B_{\varepsilon_n}(x) \subset A_n \text{ hence } B_\varepsilon(x) \subset \bigcap_{i=1}^n A_i$$
which proves that $\cap_{i=1}^n A_i$ is open. □

- This needs finite intersection as the infimum of an infinite set of positive numbers could be zero and the proof would fail.
- The intersection of an infinite collection of open sets need not be open.

Interior, Closure, Exterior and Boundary

Let (X, d) be a metric space and $A \subset X$.

Definition

The interior of A, denoted $\mathrm{int} A$, is the largest open set contained in A (alternatively, the union of all open sets contained in A).

Definition

The closure of A, denoted $\bar{A}$, is the smallest closed set containing A (alternatively, the intersection of all closed sets containing A).

Definition

The exterior of A, denoted $\mathrm{ext}\, A$, is the largest open set contained in $X \setminus A$.

- Note that $\mathrm{ext}\, A = \mathrm{int} X \setminus A$.

Definition

The boundary of A, denoted ∂A is equal to $\overline{(X \setminus A)} \cap \bar{A}$.

Interior, Closure, Exterior and Boundary

Let (X, d) be a metric space and $A \subset X$.

FACTS

- A point is interior if and only if it has an open ball that is a subset of the set
$$x \in \text{int}A \qquad \Leftrightarrow \qquad \exists \varepsilon > 0, B_\varepsilon(x) \subset A$$
- A point is in the closure if and only if any open ball around it intersects the set
$$x \in \bar{A} \qquad \Leftrightarrow \qquad \forall \varepsilon > 0, B_\varepsilon(x) \cap A \neq \varnothing$$
- A point is exterior if and only if an open ball around it is entirely outside the set
$$x \in \text{ext}A \qquad \Leftrightarrow \qquad \exists \varepsilon > 0, B_\varepsilon(x) \subset X \setminus A$$
- A point is on the boundary if any open ball around it intersects the set and intersects the outside of the set
$$x \in \partial A \Leftrightarrow \forall \varepsilon > 0, \quad \begin{array}{c} B_\varepsilon(x) \cap A \neq \varnothing \\ \text{and} \\ B_\varepsilon(x) \cap X \setminus A \neq \varnothing \end{array}$$

Interior, Closure, Exterior and Boundary

Interior, Closure, Exterior and Boundary Example

Let $A = [0,1] \cup (2,3)$. Then

$$\operatorname{int} A = (0,1) \cup (2,3)$$

$$\bar{A} = [0,1] \cup [2,3]$$

$$\begin{aligned} \operatorname{ext} A &= \operatorname{int}(X \setminus A) \\ &= \operatorname{int}\,((-\infty,0) \cup (1,2] \cup [3,+\infty)) \\ &= (-\infty,0) \cup (1,2) \cup (3,+\infty) \end{aligned}$$

$$\begin{aligned} \partial A &= \overline{(X \setminus A)} \cap \bar{A} \\ &= ((-\infty,0] \cup [1,2] \cup [3,+\infty)) \cap ([0,1] \cup [2,3]) \\ &= \{0,1,2,3\} \end{aligned}$$

Sequences and Closed Sets

- We can characterize closedness also using sequences: a set is closed if it contains the limit of any convergent sequence within it, and a set that contains the limit of any sequence within it must be closed.

Theorem

A set A in a metric space (X, d) is closed if and only if

$$\begin{array}{c} \{x_n\} \subset A \\ \textit{and} \\ x_n \to x \in X \end{array} \Rightarrow x \in A$$

- We will prove the two directions in turn.

A set is closed if it contains the limit of any convergent sequence within it.

Proof.

Let A be closed. Then $X \setminus A$ is open.

- Consider a convergent sequence $x_n \to x \in X$, with $x_n \in A$ for all n.
- We need to show that $x \in A$. Suppose not.
- If $x \notin A$, then $x \in X \setminus A$, so there is some $\varepsilon > 0$ such that $B_\varepsilon(x) \subset X \setminus A$ (by the definition of open set).
- Since $x_n \to x$, there exists $N(\varepsilon)$ such that

$$\begin{aligned} n > N(\varepsilon) \quad &\Rightarrow \quad x_n \in B_\varepsilon(x) \\ &\Rightarrow \quad x_n \in X \setminus A \\ &\Rightarrow \quad x_n \notin A \end{aligned}$$

- This is a contradiction. Therefore,

$$\{x_n\} \subset A, x_n \to x \in X \Rightarrow x \in A$$

as desired. □

A set that contains the limit of any sequence within it must be closed

Proof.

Suppose $\begin{matrix} \{x_n\} \subset A \\ \text{and} \\ x_n \to x \in X \end{matrix} \Rightarrow x \in A$.

We need to show that A is closed (or equivalently, that $X \setminus A$ is open).

- Suppose not: $X \setminus A$ is not open. Hence, there exists $x \in X \setminus A$ such that for every $\varepsilon > 0$,
$$B_\varepsilon(x) \not\subseteq X \setminus A$$
- So there exists $y \in B_\varepsilon(x)$ such that $y \notin X \setminus A$. Then $y \in A$, hence
$$B_\varepsilon(x) \bigcap A \neq \emptyset$$
- Construct a sequence $\{x_n\}$ as follows: for each n, choose
$$x_n \in B_{\frac{1}{n}}(x) \cap A$$
- Given $\varepsilon > 0$, by the Archimedean Property we can find $N(\varepsilon)$ such that $N(\varepsilon) > \frac{1}{\varepsilon}$, so $n > N(\varepsilon) \Rightarrow \frac{1}{n} < \frac{1}{N(\varepsilon)} < \varepsilon$, therefore $x_n \to x$.
- Then $\{x_n\} \subseteq A$, $x_n \to x$, so $x \in A$, a contradiction. Therefore, $X \setminus A$ is open, and A is closed. □

Familiar (maybe) Terminology

Definitions

Let (X, d) be a metric space and $E \subset X$.

- A point x is a limit point of E if every $B_\varepsilon(x)$ contains a point $y \neq x$ such that $y \in E$.
- If $x \in E$ and x is not a limit point of E, then x is called an isolated point of E.
- E is dense in X if every point of X is a limit point of E, or a point of E (or both).

Results

- E is closed if every limit point of E is a point of E.
- E is open if every point of E is an interior point of E.

Limits of Functions in Metric Spaces

- Yesterday we defined the limit of a sequence, and now we extend those ideas to functions from one metric space to another.
- For functions from reals to reals: $f : (c, d) \to \mathbb{R}$, y is the limit of f at x_0 if

for each $\varepsilon > 0$ there is a $\delta(\varepsilon) > 0$ such that $0 < |x - x_0| < \delta(\varepsilon) \Rightarrow |f(x) - y| < \varepsilon$

- We extend this to metric spaces by replacing each absolute value with a metric.

Definition

Let (X, d) and (Y, ρ) be metric spaces with $A \subset X$, $f : A \to Y$, and $\mathbf{x}_0 \in A$. We say that f has limit $\mathbf{y}_0$ as $\mathbf{x} \to \mathbf{x}_0$ if

$$\forall \varepsilon > 0 \ \exists \delta(\varepsilon) > 0 \text{ such that } 0 < d(\mathbf{x}, \mathbf{x}_0) < \delta(\varepsilon) \Rightarrow \rho(f(\mathbf{x}), \mathbf{y}_0) < \varepsilon$$

Notation

When f has limit $\mathbf{y}_0$ as $\mathbf{x}$ goes to $\mathbf{x}_0$ we write

$$f(\mathbf{x}) \to \mathbf{y}_0 \text{ as } \mathbf{x} \to \mathbf{x}_0 \qquad \text{or} \qquad \lim_{\mathbf{x} \to \mathbf{x}_0} f(\mathbf{x}) = \mathbf{y}_0$$

- Notice the different metrics.

Limits of Functions and Sequences

- Standard results like the uniqueness of limits theorem hold. Also, the results we saw on sequences about sums, multiplication by a scalar, products, and division extend to functions (when appropriate).

- The limit of a function can also be characterized using sequences.

Theorem

Let (X, d) and (Y, ρ) be metric spaces, $f : X \to Y$, and $\mathbf{x}_0 \in X$. Then

$$\lim_{\mathbf{x} \to \mathbf{x}_0} f(\mathbf{x}) = \mathbf{y}_0$$

if and only if

for each $\{\mathbf{x}_n\} \to \mathbf{x}_0$ *in* (X, d) *with* $\mathbf{x}_n \neq \mathbf{x}_0$	*the sequence* $\{f(\mathbf{x}_n)\}$ *converges to* $\mathbf{y}_0$ *in* (Y, ρ)

- For every sequence that converges in the domain, the corresponding sequence given by the function converges to in the range.
- Prove these results as part of Problem Set 4.

Continuity in Metric Spaces

- For functions from reals to reals: $f : (a, b) \longrightarrow \mathbb{R}$ is *continuous at* x_0 means for each $\varepsilon > 0$ there is a $\delta(\varepsilon) > 0$ such that $|x - x_0| < \delta(\varepsilon) \Rightarrow |f(x) - f(x_0)| < \varepsilon$ which is easy to generalize.

Definition

Let (X, d) and (Y, ρ) be metric spaces. A function $f : X \to Y$ is continuous at a point $\mathbf{x}_0 \in X$ if

$$\forall\varepsilon > 0\ \exists\delta(\mathbf{x}_0, \varepsilon) > 0 \text{ such that } d(\mathbf{x}, \mathbf{x}_0) < \delta(\mathbf{x}_0, \varepsilon) \Rightarrow \rho(f(\mathbf{x}), f(\mathbf{x}_0)) < \varepsilon$$

- Notice that different metrics are used when appropriate.
- Continuity at $\mathbf{x}_0$ requires:
  - $f(\mathbf{x}_0)$ is defined;
  - and either
    - $\mathbf{x}_0$ is an isolated point of X (i.e. $\exists\varepsilon > 0$ such that $B_\varepsilon(\mathbf{x}_0) = \{\mathbf{x}_0\}$); or
    - $\lim_{\mathbf{x}\longrightarrow\mathbf{x}_0} f(\mathbf{x})$ exists and equals $f(\mathbf{x}_0)$

Definition

f is continuous if it is continuous at every element of its domain.

- $\delta(\mathbf{x}_0, \varepsilon)$ means that δ can depend on $\mathbf{x}_0$. What if it does not? Later.

Continuity in Metric Spaces

- Remember, given $f : X \to Y$ and $A \subset Y$ the inverse image is a subset of X defined as:

$$f^{-1}(A) = \{\mathbf{x} \in X : f(\mathbf{x}) \in A\} \subset X$$

- The inverse image is used to provide a characterization of continuous functions.

Theorem

Let (X, d) and (Y, ρ) be metric spaces, and $f : X \to Y$. Then

f is continuous

if and only if

$f^{-1}(A)$ is open in X for every $A \subset Y$ such that A is open in Y

- Continuity is equivalent to the fact that the inverse image of every open set in the range is an open set in the domain.

NOTE: an equivalent statement of the theorem

f is continuous if and only if $f^{-1}(C)$ is closed in X for every closed $C \subset Y$.

f is continuous $\Rightarrow f^{-1}(A)$ is open in X for every $A \subset Y$ such that A is open in Y.

Proof.

Suppose f is continuous.
Given $A \subset Y$, with A open, we must show that $f^{-1}(A)$ is open in X.

- Suppose $\mathbf{x}_0 \in f^{-1}(A)$. Let $\mathbf{y}_0 = f(\mathbf{x}_0) \in A$.
- Since A is open, we can find $\varepsilon > 0$ such that $B_\varepsilon(\mathbf{y}_0) \subset A$.
- Since f is continuous, there exists $\delta > 0$ such that

$$\begin{aligned} d(\mathbf{x}, \mathbf{x}_0) < \delta \quad &\Rightarrow \quad \rho(f(\mathbf{x}), f(\mathbf{x}_0)) < \varepsilon \\ &\Rightarrow \quad f(\mathbf{x}) \in B_\varepsilon(\mathbf{y}_0) \\ &\Rightarrow \quad f(\mathbf{x}) \in A \\ &\Rightarrow \quad \mathbf{x} \in f^{-1}(A) \end{aligned}$$

- So $B_\delta(\mathbf{x}_0) \subset f^{-1}(A)$, and therefore $f^{-1}(A)$ is open. □

The inverse image of any open set in the range is an open set in the domain $\Rightarrow f$ is continuous.

Proof.

Suppose

$$f^{-1}(A) \text{ is open in } X \text{ for each } A \subset Y \text{ such that } A \text{ is open in } Y$$

We need to show that f is continuous.

- Let $\mathbf{x}_0 \in X$, with $\varepsilon > 0$, and let $A = B_\varepsilon(f(\mathbf{x}_0))$.
- A is an open ball, hence an open set, so $f^{-1}(A)$ is open in X.
- $\mathbf{x}_0 \in f^{-1}(A)$, so there exists $\delta > 0$ such that $B_\delta(\mathbf{x}_0) \subset f^{-1}(A)$.

$$\begin{aligned} d(\mathbf{x}, \mathbf{x}_0) < \delta \quad &\Rightarrow \quad \mathbf{x} \in B_\delta(\mathbf{x}_0) \\ &\Rightarrow \quad \mathbf{x} \in f^{-1}(A) \\ &\Rightarrow \quad f(\mathbf{x}) \in A \text{ which is nothing but } B_\varepsilon(f(\mathbf{x}_0)) \\ &\Rightarrow \quad \rho(f(\mathbf{x}), f(\mathbf{x}_0)) < \varepsilon \end{aligned}$$

- Thus, we have shown that f is continuous at $\mathbf{x}_0$.
- Since $\mathbf{x}_0$ is an arbitrary point in X, f is continuous. □

Algebra of Continuity

The composition of continuous functions is continuous:

Theorem

Let (X, d_X), (Y, d_Y) and (Z, d_Z) be metric spaces. If $f : X \to Y$ and $g : Y \to Z$ are continuous, then $g \circ f : X \to Z$ is continuous.

Proof.

Suppose $A \subset Z$ is open.

- g is continuous, thus $g^{-1}(A)$ is open in Y; f is continuous, thus $f^{-1}(g^{-1}(A))$ is open in X.
- I claim that
  $$f^{-1}(g^{-1}(A)) = (g \circ f)^{-1}(A)$$
  - Observe
    $$\begin{aligned} \mathbf{x} \in f^{-1}(g^{-1}(A)) &\Leftrightarrow f(\mathbf{x}) \in g^{-1}(A) \\ &\Leftrightarrow g(f(\mathbf{x})) \in A \\ &\Leftrightarrow (g \circ f)(\mathbf{x}) \in A \\ &\Leftrightarrow \mathbf{x} \in (g \circ f)^{-1}(A) \end{aligned}$$
    which establishes the claim.
- This shows that $(g \circ f)^{-1}(A)$ is open in X, so $g \circ f$ is continuous. □

Uniform Continuity

- The definition of continuity at a point allows for δ to depend on that point.
- When it does not, the function is, for lack of a better expression, more continuous.

Definition (Uniform Continuity)

Let (X, d) and (Y, ρ) be metric spaces and $f : X \to Y$. We say f is uniformly continuous if

$$\forall \varepsilon > 0\ \exists \delta(\varepsilon) > 0 \text{ such that } \forall \mathbf{x}_0 \in X,\ d(\mathbf{x}, \mathbf{x}_0) < \delta(\varepsilon) \Rightarrow \rho(f(\mathbf{x}), f(\mathbf{x}_0)) < \varepsilon$$

- Contrast with the earlier definition of continuity at a point $\mathbf{x}_0$: $\forall \varepsilon > 0\ \exists \delta(\mathbf{x}_0, \varepsilon) > 0$ such that $d(\mathbf{x}, \mathbf{x}_0) < \delta(\mathbf{x}_0, \varepsilon) \Rightarrow \rho(f(\mathbf{x}), f(\mathbf{x}_0)) < \varepsilon$. This allows δ to depend on $\mathbf{x}_0$ and ε.
- Uniform continuity requires δ to depend only on ε. It is more restrictive than continunity.

Uniform Continuity: Example

A continuous function that is not uniformly continuous

Consider $f : (0,1] \to \mathbf{R}$ defined as $f(x) = \frac{1}{x}$.
f is continuous (check this) but not uniformly continuous.

- Fix $\varepsilon > 0$ and $x_0 \in (0,1]$. Let $x = \frac{x_0}{1+\varepsilon x_0}$. Clearly, $x < x_0$. Then

$$\frac{1}{x} - \frac{1}{x_0} > 0$$

$$|f(x) - f(x_0)| = \left|\frac{1}{x} - \frac{1}{x_0}\right| = \frac{1}{x} - \frac{1}{x_0} = \frac{1+\varepsilon x_0}{x_0} - \frac{1}{x_0} = \frac{\varepsilon x_0}{x_0} = \varepsilon$$

- Thus, in the definition of continuity $\delta(x_0, \varepsilon)$ must be chosen small enough so that

$$\delta(x_0, \varepsilon) \leq x_0 - \frac{x_0}{1+\varepsilon x_0} = \frac{\varepsilon (x_0)^2}{1+\varepsilon x_0} < \varepsilon (x_0)^2$$

  which converges to zero as $x_0 \to 0$.
- Hence there is no $\delta(\varepsilon) > 0$ that will work for all $x_0 \in (0,1]$.

Uniform Continuity: Example

An $f : [a, b] \to \mathbf{R}$ that has a bounded derivative is uniformly continuous on $[a, b]$. However, a function with an unbounded derivative can also be uniformly continuous.

A function with an unbounded derivative that is uniformly continuous

Consider $f : [0, 1] \to \mathbf{R}$ defined as $f(x) = \sqrt{x}$.

- f is continuous (verify this) and also uniformly continuous.
- Given $\varepsilon > 0$, let $\delta = \varepsilon^2$ (notice this does not depend on x).
- Then given any $x_0 \in [0, 1]$, $|x - x_0| < \delta$ implies (by the Fundamental Theorem of Calculus)

$$\begin{aligned} |f(x) - f(x_0)| &= \left| \int_{x_0}^{x} \frac{1}{2\sqrt{t}}\, dt \right| \\ &\leq \int_{0}^{|x-x_0|} \frac{1}{2\sqrt{t}}\, dt = \sqrt{|x - x_0|} \\ &< \sqrt{\delta} = \sqrt{\varepsilon^2} \\ &= \varepsilon \end{aligned}$$

- Thus, f is uniformly continuous on $[0, 1]$, even though $f'(x) \to \infty$ as $x \to 0$.

Lipschitz Continuity

Definition

Let X, Y be normed vector spaces. A function $f : X \to Y$ is Lipschitz on $E \subset X$ if

$$\exists K > 0 \text{ such that } \|f(\mathbf{x}) - f(\mathbf{z})\|_Y \leq K\|\mathbf{x} - \mathbf{z}\|_X \quad \forall \mathbf{x}, \mathbf{z} \in E$$

f is locally Lipschitz on $E \subset X$ if

$$\forall \mathbf{x}_0 \in E \; \exists \varepsilon > 0 \text{ such that } f \text{ is Lipschitz on } B_\varepsilon(\mathbf{x}_0) \cap E$$

- The function is locally Lipschitz if every point in its domain has a neighborhood on which the function is Lipschitz.

Remark

Lipschitz continuity is stronger than either continuity or uniform continuity:

$$\begin{aligned} \text{locally Lipschitz} &\Rightarrow \text{continuous} \\ \text{Lipschitz} &\Rightarrow \text{uniformly continuous} \end{aligned}$$

Every C^1 function is locally Lipschitz (a function $f : \mathbf{R}^m \to \mathbf{R}^n$ is said to be C^1 if all its first partial derivatives exist and are continuous).

Homeomorphisms

Definition

Let (X, d) and (Y, ρ) be metric spaces. A function $f : X \to Y$ is called a homeomorphism if it is one-to-one, onto, continuous, and its inverse function is continuous.

Note

Suppose that f is a homeomorphism and $U \subset X$, and let $g = f^{-1} : Y \to X$.

- Then
$$\mathbf{y} \in g^{-1}(U) \Leftrightarrow g(\mathbf{y}) \in U \Leftrightarrow \mathbf{y} \in f(U)$$
- and
$$U \text{ open in } X \Rightarrow g^{-1}(U) \text{ is open in } (f(X), \rho) \Rightarrow f(U) \text{ is open in } (f(X), \rho)$$
- Therefore (X, d) and $\left(f(X), \rho|_{f(X)}\right)$ are identical in terms of properties that can be characterized solely in terms of open sets.
- Such properties are called "topological properties."

Remark

Topological properties are invariant under homeomorphisms.

Tomorrow

We study more properties of functions from **R** to **R**.

1. Boudedness and Extreme Value Theorem
2. Intermediate Value Theorem and Fixed Points
3. Monotonicity
4. Complete Spaces and Cauchy Sequences
5. Contraction Mappings