Lecture 6

Econ 2001

2015 August 17

Lecture 6 Outline

1. Compactness
2. Correspondences
3. Continuity

In the first half, we talk about sets that are particularly useful, while in the second half, we look at point to set mappings.

Compactness

Definition

Given a metric space (X, d), a collection of sets

$$\mathcal{U} = \{U_\lambda : \lambda \in \Lambda\}$$

is an open cover of A if U_λ is open for all $\lambda \in \Lambda$ and

$$\cup_{\lambda \in \Lambda} U_\lambda \supset A$$

- Notice that Λ may be finite, countably infinite, or uncountable.

Definition

Given a metric space (X, d), a set A is compact if every open cover of A contains a finite subcover of A. In other words, if $\{U_\lambda : \lambda \in \Lambda\}$ is an open cover of A, there exist $n \in \mathbf{N}$ and $\lambda_1, \cdots, \lambda_n \in \Lambda$ such that

$$A \subset U_{\lambda_1} \cup \cdots \cup U_{\lambda_n}$$

- The definition does not say "A has a finite open cover" (that would be silly...).
- It requires that for any arbitrary open cover one can find a finite subcover of that given open cover.

Compactnes: Example

$(0,1]$ is not compact

To see this, let

$$\mathcal{U} = \left\{ U_m = \left(\frac{1}{m}, 2 \right) : m \in \mathbf{N} \right\}$$

- This is an open cover of $(0,1]$ since
$$\cup_{m \in \mathbf{N}} U_m = (0,2) \supset (0,1]$$
- For $(0,1]$ to be compact we need to find a finite subcover of $\mathcal{U}$.
- Take any finite collection $\{U_{m_1}, \ldots, U_{m_n}\}$ of elements $\mathcal{U}$, and let
$$m = \max\{m_1, \ldots, m_n\}$$
- Then
$$\cup_{i=1}^{n} U_{m_i} = U_m = \left(\frac{1}{m}, 2 \right) \not\supset (0,1]$$
- So $(0,1]$ is not compact.

- What about $[0,1]$? This argument doesn't work, so maybe that is compact...

Compactnes: Example

$[0, \infty)$ is closed but not compact

To see that $[0, \infty)$ is not compact, consider the open cover

$$\mathcal{U} = \{U_m = (-1, m) : m \in \mathbf{N}\}$$

- Compact requires that $\mathcal{U}$ must have a finite subcover.
- Take any finite subset of $\mathcal{U}$

$$\{U_{m_1}, \ldots, U_{m_n}\}$$

- As before, let

$$m = \max\{m_1, \ldots, m_n\}$$

- Then

$$U_{m_1} \cup \cdots \cup U_{m_n} = U_m = (-1, m) \not\supseteq [0, \infty)$$

Closed vs. Compact

Theorem

Every closed subset A of a compact metric space (X, d) is compact.

Proof.

Let $\{U_\lambda : \lambda \in \Lambda\}$ be an open cover of A.

- To use the compactness of X, we need an open cover of X.
- Claim: $U'_\lambda = U_\lambda \cup (X \setminus A)$ is an open cover of X.
  - Since A is closed, $X \setminus A$ is open; since U_λ is open, so is U'_λ.
  - For any $x \in X$, either $x \in A$ or $x \in X \setminus A$.
    - If $x \in A$, $\exists \lambda \in \Lambda$ s.t. $x \in U_\lambda \subset U'_\lambda$. (since $\{U_\lambda : \lambda \in \Lambda\}$ is an open cover of A.)
    - If $x \in X \setminus A$, then $\forall \lambda \in \Lambda$, $x \in U'_\lambda$. (by the definition of U'_λ.)
  - Either way, $x \in U'_\lambda$ and $X \subset \cup_{\lambda \in \Lambda} U'_\lambda$; thus $\{U'_\lambda : \lambda \in \Lambda\}$ is an open cover of X.
- Since X is compact, there is a finite subcover of $\{U'_\lambda : \lambda \in \Lambda\}$

$$\exists \lambda_1, \ldots, \lambda_n \in \Lambda \text{ such that } X \subset U'_{\lambda_1} \cup \cdots \cup U'_{\lambda_n}$$

- Then:

$$a \in A \Rightarrow a \in X \Rightarrow a \in U'_{\lambda_i} \text{ for some } i \Rightarrow a \in U_{\lambda_i} \cup (X \setminus A) \Rightarrow a \in U_{\lambda_i} \text{ so}$$

$$A \subset U_{\lambda_1} \cup \cdots \cup U_{\lambda_n}$$

- Thus A is compact. □

Compactnesss and Closedness

Although closed $\not\Rightarrow$ compact (see earlier example), the converse is true:

Theorem

If A is a compact subset of the metric space (X, d), then A is closed.

Proof.

By contradiction: suppose A is not closed. Then $X \setminus A$ is not open; so there is $x \in X \setminus A$ such that, for every $\varepsilon > 0$, $A \cap B_\varepsilon(x) \neq \emptyset$, and hence $A \cap B_\varepsilon[x] \neq \emptyset$.

- For $n \in \mathbf{N}$, let $$U_n = X \setminus B_{\frac{1}{n}}[x]$$
- Each U_n is open, and since $x \notin A$: $\cup_{n\in\mathbf{N}} U_n = X \setminus \{x\} \supset A$.
- Therefore, $\{U_n : n \in \mathbf{N}\}$ is an open cover of A.
- Since A is compact, there is a finite subcover $\{U_{n_1}, \dots, U_{n_k}\}$.
- Let $n = \max\{n_1, \dots, n_k\}$. Then $$U_n = X \setminus B_{\frac{1}{n}}[x] \supset \underbrace{X \setminus B_{\frac{1}{n_j}}[x]}_{j=1,\dots,k} = \cup_{j=1}^{k} U_{n_j} \supset A$$
- But $A \cap B_{\frac{1}{n}}[x] \neq \emptyset$, so $A \not\subseteq X \setminus B_{\frac{1}{n}}[x] = U_n$, a contradiction.
- Thus A is closed. $\square$

Sequential Compactness

Definition

A set A in a metric space (X, d) is sequentially compact if every sequence of elements of A contains a convergent subsequence whose limit lies in A.

- This is an alternative characterization of compactness.

Theorem

A set A in a metric space (X, d) is compact if and only if it is sequentially compact.

- Proof? Not obvious.

Easy Heine-Borel Theorem for R

Theorem

If $A \subset \mathbf{E}^1$, then A is compact if and only if A is closed and bounded.

Proof.

Let A be a closed, bounded subset of $\mathbf{R}$.
Then $A \subset [a, b]$ for some interval $[a, b]$.

- Let $\{x_n\}$ be a sequence of elements of $[a, b]$.
- By the Bolzano-Weierstrass Theorem, $\{x_n\}$ contains a convergent subsequence; let $x \in \mathbf{R}$ denote the limit.
- Since $[a, b]$ is closed, $x \in [a, b]$.
- Thus, $[a, b]$ is sequentially compact, hence compact. Since A is a closed subset of $[a, b]$ it is also compact.
- Conversely, if A is compact, A is closed and bounded by a previous result. □

Heine-Borel Theorem

Theorem (Heine-Borel)

If $A \subseteq \mathbf{E}^n$, then A is compact if and only if A is closed and bounded.

Example

A closed interval in $\mathbf{R}^n$ is defined as:

$$[a, b] = \{x \in \mathbf{R}^n : a_i \leq x_i \leq b_i \text{ for each } i = 1, \ldots, n\}$$

$[a, b]$ is compact in $\mathbf{E}^n$ for any $a, b \in \mathbf{R}^n$.

Continuous Functions and Compact Sets

- Continuous Images of Compact Sets Are Compact

Theorem

Let (X, d) and (Y, ρ) be metric spaces. If $f : X \to Y$ is continuous and C is a compact subset of (X, d), then $f(C)$ is compact in (Y, ρ).

Proof.

Problem Set 6. □

- Continuous functions of compact sets are uniformly continuous.

Theorem

Let (X, d) and (Y, ρ) be metric spaces, C a compact subset of X, and $f : C \to Y$ continuous. Then f is uniformly continuous on C.

Proof.

Problem Set 6. □

Extreme Value Theorem (Again!)

Corollary (Extreme Value Theorem)

Let C be a compact set in a metric space (X, d), and suppose $f : C \to \mathbf{R}$ is continuous. Then f is bounded on C and attains its minimum and maximum on C.

Proof.

$f(C)$ is compact by a previous Theorem, hence it is closed and bounded.

- Let $M = \sup f(C)$; w know $M < \infty$.
- Then there exists $y_m \in f(C)$ such that
  $$M - \frac{1}{m} \leq y_m \leq M$$
  - so M is a limit point of $f(C)$.
- Since $f(C)$ is closed, $M \in f(C)$, i.e. there exists $c \in C$ such that $f(c) = M = \sup f(C)$, so f attains its maximum at c.
- The proof for the minimum is similar (do it). □

Theorem of the Maximum: a Prelude

- A major interest in economics is constrained optimzation. In particular, we would like to know how the maximum value and the maximizing vectors in a constrained maximization problem depend on some exogenous parameters.

- Let $f : A \times X \to \mathbf{R}$, where $A \subset \mathbf{R}^m$ and $X \subset \mathbf{R}^n$.
- For each $a \in A$, let $\varphi(a)$ be a non-empty subset of X.
- A maximization problem is defined as

$$\max_{b \in \varphi(a)} f(a, x).$$

- For each $a \in A$, let

$$h(a) = \max_{b \in \varphi(a)} f(a, x) \quad \text{and} \quad \mu(a) = \{b \in \varphi(a) : h(a) = f(a, x)\}$$

  These are, respectively, the maximized value of the objective and the maximal choice, as functions of the parameters a.
- $\varphi(a)$ is typically a set, and $\mu(a)$ is typically not unique ($\mu(a) \subset X$). These are point to set mappings.
- The Theorem of the Maximum gives conditions for h and μ to be continuous in a.
- But what does continuity mean for set valued mappings?

Correspondences

- A set valued mapping takes each point in the domain to a set in the range.

Definition

A correspondence from X to Y, denoted $\varphi : X \to 2^Y$, is a mapping from X to 2^Y; that is, $\varphi(x) \subset Y$ for every $x \in X$.

- This maps every point in X to a subset of Y.
- Obviously, a function from X to Y is a correspondence that is single-valued for each x.
  - Let $f : X \to Y$ be a function. Define $\varphi : X \to 2^Y$ by

$$\varphi(x) = \{f(x)\} \text{ for each } x \in X$$

  - A function is a special case of a correspondence.
- We typically deal with metric spaces (X, d) and (Y, d), where $X \subset \mathbf{R}^n$ and $Y \subset \mathbf{R}^m$, and d is the Euclidean metric.

Correspondences Are Commonplace in Economics

The Budget Set is a Correspondence

A consumer uses her income to buy any combination of the L goods at given prices.

- Prices are L non-negative numbers measuring how much each unit of each good costs. They are a vector in $\mathbf{R}^L_+$.
- Possible consumption choices are described by L non negative quantities, one for each good. This is also a vector in $\mathbf{R}^L_+$.
- Given prices and income, the consumer can only choose combinations of the goods that she can afford: $\sum_{l=1}^{L} p_l x_l \leq I$.
- For given prices and income, the affordable consumption bundles for a set
$$\left\{ \mathbf{x} \in \mathbf{R}^L_+ : \mathbf{p} \cdot \mathbf{x} = \sum_{l=1}^{L} p_l x_l \leq I \right\}$$
- The correspondence $B : \mathbf{R}^L_+ \times \mathbf{R}_{++} \to \mathbf{R}^L_+$ defined as follows:
$$B(\mathbf{p}, I) = \left\{ \mathbf{x} \in \mathbf{R}^L_+ : \mathbf{p} \cdot \mathbf{x} = \sum_{l=1}^{L} p_l x_l \leq I \right\} \subset \mathbf{R}^L_+$$
describes all affordable consumption bundles (elements of $\mathbf{R}^L_+$) for different price vectors and income levels.

Correspondences Are Useful In Economics

A Consumer's Demand is a Correspondence

The consumer ranks consumption bundles using a utility function $u : \mathbf{R}^L_+ \to \mathbf{R}$.

- Define $x^* : \mathbf{R}^L_{++} \times \mathbf{R}_{++} \to 2^{\mathbf{R}^L_+}$ by
$$x^*(\mathbf{p}, I) = \arg \max_{\mathbf{x} \in B(\mathbf{p}, I)} u(\mathbf{x})$$
- This is a consumer's demand and can be multi-valued (and thus a correspondence).
- A typical question an economist likes to answer is: what happens to demand when prices and/or income change? That is, how does x^* change as p and I change?
- You can answer this questions if you know what the correspondence $x^*(\mathbf{p}, I)$ looks like.
- Why assume that income is strictly positive? What if income is zero?
- Why assume that all prices are strictly positive? What if some prices are zero?

Definitions for Correspondences

- Since correspondences are sets, one can talk about closedness and/or compactness of these sets.

Definition

A correspondence $\varphi : X \to 2^Y$ is closed-valued if $\varphi(x)$ is a closed subset of Y for all x.

Definition

A correspondence $\varphi : X \to 2^Y$ is compact-valued if $\varphi(x)$ is compact for all x.

- The graph of a correspondence is defined similarly to the graph of a function.

Definition

The graph of a correspondence $\varphi : X \to 2^Y$ is the set

$$G_\varphi = \{(x, y) \in X \times Y : y \in \varphi(x)\}$$

Continuity for Correspondences

We will see three different notions of continuity for correspondences, each motivated by continuity for functions.

- A function $f : \mathbf{R} \to \mathbf{R}$ may be discontinuous at a point x_0 because it "jumps down:"

$$\exists x_n \to x_0 \text{ such that } f(x_0) < \liminf f(x_n)$$

- or it "jumps up"

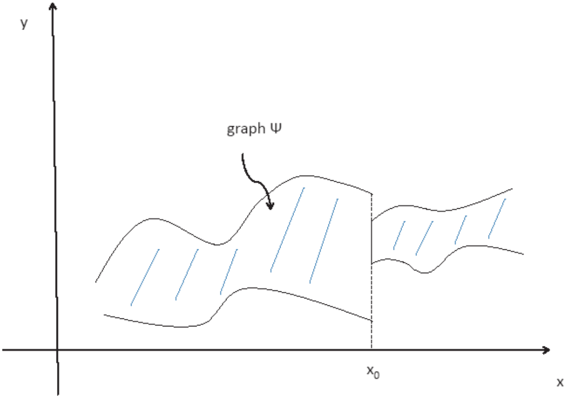
$$\exists x_n \to x_0 \text{ such that } f(x_0) > \limsup f(x_n)$$

- In either case, it does not matter whether the sequence x_n approaches x_0 from the left or the right, the function is not continuous at x_0.
- For sets, however, this matters and makes for different possible views of continuity. Hence all the different definitions.

Continuity for Correspondences

How can a set "jump" at the limit x_0? Maybe the set suddenly gets smaller.

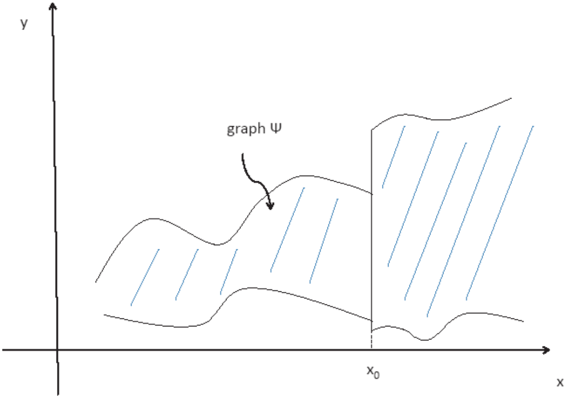
- There is a sequence $x_n \longrightarrow x_0$, with image $\varphi(x_0)$, but as $n \longrightarrow \infty$ the points $y_n \in \varphi(x_n)$ are far from every point of $\varphi(x_0)$.



Continuity for Correspondences

A second way a set can “jump” at the limit is to suddenly get larger.

- There is a point y in $\varphi(x_0)$ and a sequence $x_n \to x_0$ such that y is far from every point of $\varphi(x_n)$ as $n \to \infty$.



Upper Hemicontinuity

Definition

Let $X \subset \mathbf{R}^n$, $Y \subset \mathbf{R}^m$, and $\varphi : X \to 2^Y$.

- φ is upper hemicontinuous (uhc) at $x_0 \in X$ if,

  for every open set $V \supset \varphi(x_0)$ there is an open set U with $x_0 \in U$ such that $\varphi(x) \subset V$ for every $x \in U \cap X$

- φ is upper hemicontinuous if it is upper hemicontinuous at every $x \in X$.

Draw a Picture

- In other words, for every $x_0 \in X$ and every $\varepsilon > 0$, there is a $\delta > 0$ such that $\varphi(x) \subset B_\varepsilon(\varphi(x_0))$ when $\|x - x_0\| < \delta$.

Lower Hemicontinuity

Definition

Let $X \subset \mathbf{R}^n$, $Y \subset \mathbf{R}^m$, and $\varphi : X \to 2^Y$.

- φ is lower hemicontinuous (lhc) at $x_0 \in X$ if, for every open set V such that $\varphi(x_0) \cap V \neq \emptyset$ there is an open set U with $x_0 \in U$ such that $\varphi(x) \cap V \neq \emptyset$ for every $x \in U \cap X$
- φ is lower hemicontinuous if it is lower hemicontinuous at every $x \in X$.

Draw a Picture

Continuity for Correspondences

A correspondence is continuous if it is both upper and lower hemicontinuous.

Definition

Let $X \subset \mathbf{R}^n$, $Y \subset \mathbf{R}^m$, and $\varphi : X \to 2^Y$.

- φ is continuous at $x_0 \in X$ if it is both upper hemicontinuous and lower hemicontinuous at x_0.
- φ is continuous if it is continuous at every $x \in X$.

Hemicontinuity and Functions

If the correspondence is a function, upper hemi-continuity and continuity coincide.

Theorem

Let $X \subset \mathbf{R}^n$, $Y \subset \mathbf{R}^m$ and $f : X \to Y$. Let $\varphi : X \to 2^Y$ be given by $\varphi(x) = \{f(x)\}$ for all $x \in X$. Then $\varphi(x)$ is uhc if and only if f is continuous.

Proof.

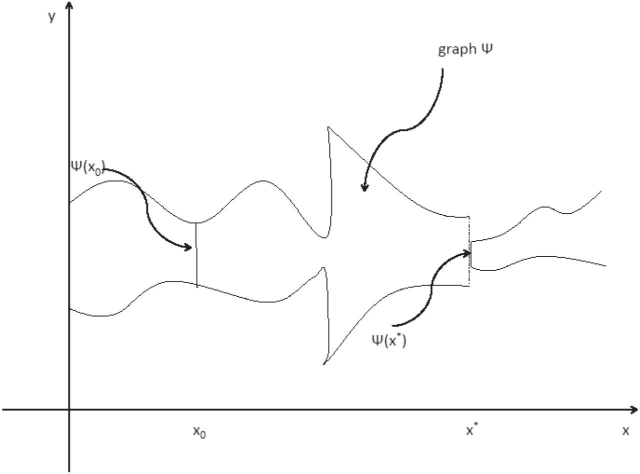
Suppose φ is uhc. Fix V open in Y.

- Then
$$\begin{aligned} f^{-1}(V) &= \{x \in X : f(x) \in V\} \\ &= \{x \in X : \varphi(x) \subset V\} \end{aligned}$$
- Thus, f is continuous if and only if $f^{-1}(V)$ is open in X for each open V in Y.
- Hence, if and only if $\{x \in X : \varphi(x) \subset V\}$ is open in X for each open V in Y.
- Therefore, if and only if φ is uhc (think through why this holds). □

- There is a similar statement for lower hemicontinuity. Prove it.

Hemicontinuity Notions Are Not Nested

In the picture, the correspondece violates uhc at x^* (but not lhc) and it violates lhc between x^* and x_0 (but not uhc).



uhc at $x_0 \in X$: $\forall$ open set $V \supseteq \varphi(x_0)$, $\exists$ an open set U with $x_0 \in U$ s.t. $\varphi(x) \subseteq V$, $\forall x \in U \cap X$

lhc at $x_0 \in X$: $\forall$ open set V s.t. $\varphi(x_0) \cap V \neq \emptyset$, $\exists$ an open set U with $x_0 \in U$ s.t. $\varphi(x) \cap V \neq \emptyset$, $\forall x \in U \cap X$

Continuity and the Graph

Recall that a function $f : \mathbf{R}^n \to \mathbf{R}$ is continuous if and only if whenever $x_n \to x$, $f(x_n) \to f(x)$. We can translate this into a statement about its graph.

- The graph of function $f : X \to Y$ is the set $G_f = \{(x, y) \in X \times Y : y = f(x)\}$.
- Suppose $\{(x_n, y_n)\} \subset G_f$ and $(x_n, y_n) \to (x, y)$. Since f is a function,
$$(x_n, y_n) \in G_f \Leftrightarrow y_n = f(x_n).$$
- So
$$\begin{aligned} f \text{ is continuous} \quad &\Rightarrow \quad y = \lim y_n = \lim f(x_n) = f(x) \\ &\Rightarrow \quad (x, y) \in G_f \end{aligned}$$
- So if f is continuous then each convergent sequence $\{(x_n, y_n)\}$ in G_f converges to a point (x, y) also in G_f.
- Thus, if f is continuous then G_f is a closed set.

Continuity for Correspondences: Closed Graph

Remember, the graph of a correspondence $\varphi : X \to 2^Y$ is the set

$$G_\varphi = \{(x, y) \in X \times Y : y \in \varphi(x)\}$$

Definition

Let $X \subseteq \mathbf{R}^n$, $Y \subseteq \mathbf{R}^m$. A correspondence $\varphi : X \to 2^Y$ has closed graph if its graph is a closed subset of $X \times Y$.

REMARK

The condition that G_φ is closed can be stated using sequences.

- For any sequences $\{x_n\} \subseteq X$ and $\{y_n\} \subseteq Y$ such that

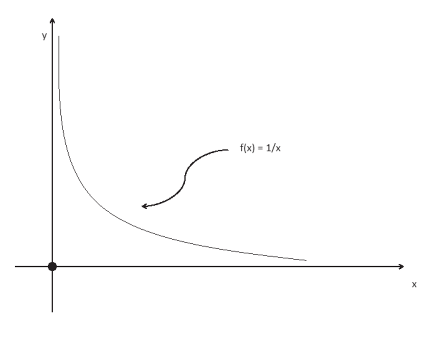
$$\left.\begin{array}{c} x_n \to x \in X \\ y_n \to y \in Y \\ \text{and} \\ y_n \in \varphi(x_n) \text{ for each } n \end{array}\right. \quad \Rightarrow y \in \varphi(x)$$

Closed Graph: Example

Closed Graph but Not Upper Hemicontinuous

Consider the correspondence $\varphi(x) = \begin{cases} \{\frac{1}{x}\} & \text{if } x \in (0,1] \\ \{0\} & \text{if } x = 0 \end{cases}$

- Let $V = (-0.1, 0.1)$. Then $\varphi(0) = \{0\} \subset V$, but no matter how close x is to 0, $\varphi(x) = \{\frac{1}{x}\} \not\subseteq V$
- So φ is not uhc at 0. However, note that φ has closed graph.

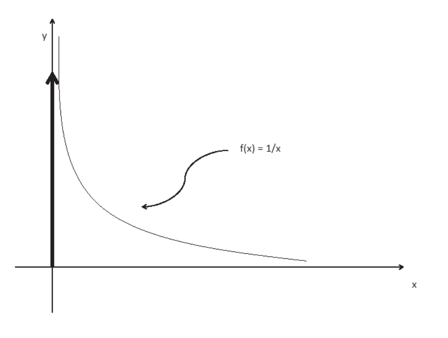


Closed Graph: Examples

Closed Graph and Upper Hemicontinuous

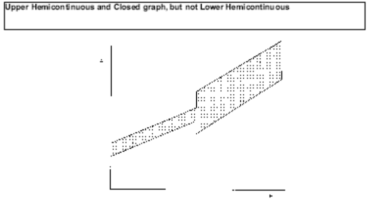
Consider the correspondence $\varphi(x) = \begin{cases} \left\{\frac{1}{x}\right\} & \text{if } x \in (0,1] \\ \mathbf{R}_+ & \text{if } x = 0 \end{cases}$

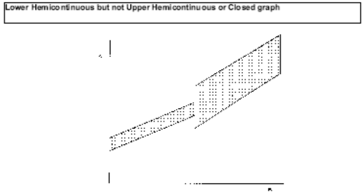
- $\varphi(0) = [0, \infty)$, and $\varphi(x) \subset \varphi(0)$ for every $x \in [0,1]$.
- So if $V \supset \varphi(0)$ then $V \supset \varphi(x)$ for all x.
- Thus, φ is uhc, and has closed graph.



Hemicontinuity and Closed Graph

A closed graph correspondence need not be uhc (see above), and an uhc correspondence need not have closed graph, or even have closed values. A closed graph correspondence need not be lhc.





Continuity in Special Cases

- The three continuity concepts are connected in special cases.
  - A closed-valued upper hemicontinuous correspondence must have closed graph.
  - For a closed-valued correspondence with a compact range, upper hemicontinuity is equivalent to closed graph.

Theorem

Suppose $X \subset \mathbf{R}^n$, $Y \subset \mathbf{R}^m$, *and* $\varphi : X \to 2^Y$.

(i) *If* φ *is closed-valued and uhc, then* φ *has closed graph.*

(ii) *If* φ *has closed graph and*

$$\left.\begin{array}{c}\text{there are}\\ \text{an open set } W \text{ with } x_0 \in W\\ \text{and}\\ \text{a compact set } Z \text{ such that } x \in W \cap X \Rightarrow \varphi(x) \subset Z\end{array}\right. \Rightarrow \varphi \text{ is uhc at } x_0$$

(iii) *If* Y *is compact, then* φ *has closed graph* $\iff$ φ *is closed-valued and uhc.*

- (iii) Follows from (i) and (ii) so there is no need to prove it.