Lecture 13

Econ 2001

2015 August 26

Lecture 13 Outline

1. Implicit Function Theorem (General)
2. Envelope Theorem
3. Lebesgue Measure Zero
4. Sard and Transversality Theorems

These are some of the most important tools in economics, and they are conceptually pretty hard.
This is also the slimmest handout.

Announcements:
- *The last exam will be Friday at 10:30am (usual class time), in WWPH 4716.*
- *Tomorrow's lecture will start at 11:30.*

Reminder

Theorem (Inverse Function Theorem)

Suppose $X \subset \mathbf{R}^n$ is open, $f : X \to \mathbf{R}^n$ is C^1 on X, and $\mathbf{x}_0 \in X$. If $\det Df(\mathbf{x}_0) \neq 0$, then there are open neighborhoods U of $\mathbf{x}_0$ and V of $f(\mathbf{x}_0)$ such that

$$\begin{aligned} f : U \to V & \quad \text{is one-to-one and onto} \\ f^{-1} : V \to U & \quad \text{is } C^1 \\ Df^{-1}(f(\mathbf{x}_0)) & = [Df(\mathbf{x}_0)]^{-1} \end{aligned}$$

If, in addition, $f \in C^k$, then $f^{-1} \in C^k$.

Theorem (Simple IFT)

Suppose $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is C^1 and write $f(\mathbf{x}, a)$ where $a \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$. Assume

$$f(\mathbf{x}_0, a_0) = \mathbf{0} \qquad \textit{and} \qquad \det(D_{\mathbf{x}} f(\mathbf{x}_0, a_0)) = \begin{vmatrix} \frac{\partial f^1}{\partial x_1} & \dots & \frac{\partial f^1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f^n}{\partial x_1} & \dots & \frac{\partial f^n}{\partial x_n} \end{vmatrix} \neq 0.$$

Then, there exists a neighborhood of $(\mathbf{x}_0, a_0)$ and a function $g : \mathbb{R} \to \mathbb{R}^n$ defined on the neighborhood of a_0, such that $\mathbf{x} = g(a)$ uniquely solves $f(\mathbf{x}, a) = \mathbf{0}$ on this neighborhood. Furthermore the derivatives of g are given by

$$Dg(a_0) = -[D_{\mathbf{x}} f(\mathbf{x}_0, a_0)]^{-1} D_a f(\mathbf{x}_0, a_0)$$

Implicit Function Theorem

- This generalizes the previous theorem to the case of p exogenous parameters.

Theorem (Implicit Function Theorem)

Let $X \subset \mathbf{R}^n$ and $A \subset \mathbf{R}^p$ be open and $f : X \times A \to \mathbf{R}^n$ is C^1. Assume

$$f(\mathbf{x}_0, \mathbf{a}_0) = \mathbf{0}_n \qquad \text{and} \qquad \det(D_{\mathbf{x}} f(\mathbf{x}_0, \mathbf{a}_0)) = \begin{vmatrix} \frac{\partial f^1}{\partial x_1} & \cdots & \frac{\partial f^1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f^n}{\partial x_1} & \cdots & \frac{\partial f^n}{\partial x_n} \end{vmatrix} \neq 0$$

Then, there are open neighborhoods $U \subset X$ of $\mathbf{x}_0$ and $W \subset A$ of $\mathbf{a}_0$ such that

$$\forall \mathbf{a} \in W \quad \exists! \mathbf{x} \in U \quad \text{such that} \quad f(\mathbf{x}, \mathbf{a}) = \mathbf{0}_n$$

For each $\mathbf{a} \in W$, let $g(\mathbf{a})$ be that unique $\mathbf{x}$. Then $g : W \to X$ is C^1 and

$$\underset{n \times p}{Dg(\mathbf{a}_0)} = -[\underset{n \times n}{D_{\mathbf{x}} f(\mathbf{x}_0, \mathbf{a}_0)}]^{-1} [\underset{n \times p}{D_{\mathbf{a}} f(\mathbf{x}_0, \mathbf{a}_0)}]$$

If, in addition, $f \in C^k$, then $g \in C^k$.

- The determinant condition says that $\mathbf{x}_0$ is a regular point of $f(\cdot, \mathbf{a}_0)$.

Implicit Function Theorem: Practical Comments

1. The formula for $Dg(\cdot)$ comes from the Chain rule:
   $$f(\mathbf{x}_0, \mathbf{a}_0) = 0 \Rightarrow Df(g(\mathbf{a}), \mathbf{a})(\mathbf{a}_0) = D_{\mathbf{x}} f(\mathbf{x}_0, \mathbf{a}_0) Dg(\mathbf{a}_0) + D_{\mathbf{a}} f(\mathbf{x}_0, \mathbf{a}_0) = \mathbf{0}$$
   hence
   $$Dg(\mathbf{a}_0) = -[D_{\mathbf{x}} f(\mathbf{x}_0, \mathbf{a}_0)]^{-1} D_{\mathbf{a}} f(\mathbf{x}_0, \mathbf{a}_0)$$
2. Make sure you keep track of the dimensions of the various matrices.
   1. remember you usually need n variables to solve n equations.
   2. therefore, the domain of $f(\cdot)$ has p extra dimensions; typically you have p parameters and the solution function goes from $\mathbb{R}^p$ into $\mathbb{R}^n$.
3. The implicit function theorem proves that a system of equations has a solution if you already know that a solution exists at a point.
   1. If you can solve the system once, then you can solve it locally.
   2. The theorem does not guarantee existence of a solution.
4. IFT provides an explicit formula for the derivatives of the implicit function. One computes the derivatives of the implicit function by "implicitly differentiating" the system of equations.

Implicit Differentiation

- The technique of "implicit differentiation" is fully general.
- In the examples we saw yesterday, $p = n = 1$ so there is one equation and one derivative to find.
- In general, there is an identity in $n + p$ variables and n equations.
  - Differentiating one of the n equations with respect to one of the p parameters, you get 1 linear equation for the derivative.
  - Repeat this p times for each of the p parameters (p linear equations for the derivative).
  - Repeat this n times for each of the n implicit functions ($n \times p$ linear equations for the derivative).
- The system has a solution if the invertibility condition in the theorem holds.

Implicit Function Theorem: A Corollary

Corollary

Suppose $X \subset \mathbf{R}^n$ and $A \subset \mathbf{R}^p$ are open and $f : X \times A \to \mathbf{R}^n$ is C^1. If $\mathbf{0}$ is a regular value of $f(\cdot, \mathbf{a}_0)$, then the correspondence

$$\mathbf{a} \mapsto \{\mathbf{x} \in X : f(\mathbf{x}, \mathbf{a}) = \mathbf{0}_n\}$$

is lower hemicontinuous at $\mathbf{a}_0$.

Proof.

Assume $\mathbf{0}_n$ is a regular value of $f(\cdot, \mathbf{a}_0)$; then, given any $\mathbf{x}_0 \in \{\mathbf{x} \in X : f(\mathbf{x}, \mathbf{a}_0) = \mathbf{0}_n\}$ we can find a local implicit function g;

- in other words, if $\mathbf{a}$ is sufficiently close to $\mathbf{a}_0$, then there exist a function g such that $g(\mathbf{a}) \in \{\mathbf{x} \in X : f(\mathbf{x}, \mathbf{a}) = \mathbf{0}\}$;
- IFT says that g is continuous, hence it proves that the correspondence $\{\mathbf{x} \in X : f(\mathbf{x}, \mathbf{a}) = \mathbf{0}\}$ is lower hemicontinuous at $\mathbf{a}_0$. □

- This will come handy in micro... where $f(\mathbf{x}, \mathbf{a}) = \mathbf{0}$ will be given by the first order conditions of the utility maximization problem, and $\mathbf{a}$ will represent prices and/or income.

Envelope Theorem: Motivation

Given a function $f : X \times A$, *where* $X \subset \mathbf{R}^n$ *and* $A \subset \mathbf{R}$, *assume we have solved*

$$\max_{\mathbf{x} \in X} f(\mathbf{x}, a)$$

We want to know how changes in a affect the maximizer

$$\mathbf{x}^* = \arg\max_{\mathbf{x} \in X} f(\mathbf{x}, a)$$

and the maximized value

$$f(\mathbf{x}^*, a)$$

If the maximizer can be described by some function g, *then*

$$V(a) = f(g(a), a)$$

- Using the Implicit Function Theorem, we can get a sufficient condition for existence of g and g to be differentiable as well as a formula for its derivative; a by-product of IFT also gives information about $V'(a)$.
- The main idea is to apply the implicit function theorem to the first order conditions of the maximization problem.
- These conditions yield an equality that will be the equivalent of "$f(\mathbf{x}, a) = \mathbf{0}$" in the IFT.
    - We can add constraints to this procedure (in the fall).

Envelope Theorem: Motivation

- Assume the solution to $\max_{\mathbf{x} \in X} f(\mathbf{x}, a)$ is characterized by the first order conditions, then the maximum is given by the solution to a system equations (first derivative equal to $\mathbf{0}$); thus, we can apply the implicit function theorem to this system.

- In other words, if
$$\mathbf{x}^* = \arg\max_{\mathbf{x} \in X} f(\mathbf{x}, a) \qquad \text{if and only if} \qquad D_{\mathbf{x}} f(\mathbf{x}^*, a) = \mathbf{0}$$
$\mathbf{x}^*$ is implicitly determined as the solution to $D_{\mathbf{x}} f(\mathbf{x}, a) = \mathbf{0}$ (a system of n equations).
- Now define
$$V(a) = f(\mathbf{x}^*, a) = f(g(a), a)$$
where $\mathbf{x}^* = g(a)$.
- Using the chain rule:
$$V'(a) = D_{\mathbf{x}} f(g(a), a) Dg(a) + D_a f(g(a), a)$$
and since $D_{\mathbf{x}} f(\mathbf{x}^*, a) = \mathbf{0}$ at any optimum:
$$V'(a) = D_a f(\mathbf{x}^*, a)$$
- Envelope theorem: the value function $V(a)$ is tangent to a family of functions $f(\mathbf{x}, a)$ when $\mathbf{x} = g(a)$. On the other hand, $V(a) \geq f(\mathbf{x}, a)$ for all a, so the V curve looks like an "upper envelope" to the f curves.
- Assumpion needed: the derivative of $D_{\mathbf{x}} f(\mathbf{x}^*, a)$ is non-singular. (why?)

Envelope Theorem and Comparative Statics

Let $u : X \times A \to \mathbb{R}$, with $\mathbf{x} \in X \subset \mathbb{R}^n$ and $\mathbf{a} \in A \subset \mathbb{R}^p$ (as usual, think of $\mathbf{x}$ as endogenous while $\mathbf{a}$ is exogenous). The optimization problem is $\max_{\mathbf{x} \in X} u(\mathbf{x}, \mathbf{a})$ and suppose the first order conditions define the maximizer: $D_{\mathbf{x}} u(\mathbf{x}^*, \mathbf{a}) = \mathbf{0}$

- Given some $\mathbf{a}_0 \in A$, let $\mathbf{x}_0^*$ be the corresponding solution, and assume $\det(D_{\mathbf{xx}} u(\mathbf{x}_0^*, \mathbf{a}_0)) \neq 0$.

Invoke the Implicit Function Theorem

There exist a functon describing the relationship between $\mathbf{x}^*$ and $\mathbf{a}$ close to $\mathbf{a}_0$; furthermore, the maximizer's derivative with respect to $\mathbf{a}$ is given by the theorem.

How does it work?

- The function $f(\mathbf{x}, \mathbf{a}) = \mathbf{0}$ in IFT here is $D_x u(\mathbf{x}^*, \mathbf{a}) = \mathbf{0}$.
- The function $g(\mathbf{a})$ in IFT here is $\mathbf{x}^* : A \to X$:

$$\mathbf{x}^*(\mathbf{a}) = \arg\max_{\mathbf{x} \in X} u(\mathbf{x}, \mathbf{a})$$

  gives the maximizer depending on the parameters.

Envelope Theorem

Suppose $u : X \times A \to \mathbb{R}$, with $\mathbf{x} \in X \subset \mathbb{R}^n$ and $\mathbf{a} \in A \subset \mathbb{R}^p$, is C^2. Consider the maximization problem

$$\mathbf{x}^*(\mathbf{a}) = \arg\max_{\mathbf{x}\in X} u(\mathbf{x}, \mathbf{a});$$

Assume: $D_{\mathbf{x}} u(\mathbf{x}, \mathbf{a}) = \mathbf{0}$ defines $\mathbf{x}^*(\cdot)$, $\mathbf{x}_0^*, \mathbf{a}_0$ is a solution, and $\det D_{\mathbf{xx}} u(\mathbf{x}_0^*, \mathbf{a}_0) \neq 0$.

- By the implicit function theorem (with $D_{\mathbf{x}} u(\mathbf{x}, \mathbf{a})$ as $f(\mathbf{x}, \mathbf{a})$), close to $\mathbf{x}_0^*, \mathbf{a}_0$:
  - $\mathbf{x}^*(\mathbf{a})$ is continuously differentiable (like $g(\mathbf{a})$ in IFT).
  - IFT gives $\mathbf{x}^*(\mathbf{a})$'s derivative $(Dg(\mathbf{a}_0) = [D_{\mathbf{x}} f(\mathbf{x}_0, \mathbf{a}_0)]^{-1} D_{\mathbf{a}} f(\mathbf{x}_0, \mathbf{a}_0))$.
    - Thus:

$$\begin{aligned} \underset{n\times p}{D_{\mathbf{a}}\mathbf{x}^*(\mathbf{a}_0)} &= -[D_{\mathbf{x}}\,[\underset{n\times n}{D_{\mathbf{x}} u(\mathbf{x}^*(\mathbf{a}_0), \mathbf{a}_0)}]]^{-1}[D_{\mathbf{a}}\,[\underset{n\times p}{D_{\mathbf{x}} u(\mathbf{x}^*(\mathbf{a}_0), \mathbf{a}_0)}]] \\ &= -\,[D_{\mathbf{xx}} u(\mathbf{x}^*(\mathbf{a}_0), \mathbf{a}_0)]^{-1}\, D_{\mathbf{ax}} u(\mathbf{x}^*(\mathbf{a}_0), \mathbf{a}_0) \end{aligned}$$

    - If $\mathbf{x}$ and $\mathbf{a}$ are scalars, $D_a \mathbf{x}^*(\mathbf{a})$ becomes $\frac{\partial x^*}{\partial a} = -\frac{\frac{\partial^2 u}{\partial a \partial x}}{\frac{\partial^2 u}{\partial x^2}}$.
- By the Chain Rule: the derivative of $u(\mathbf{x}^*(\mathbf{a}), \mathbf{a})$ with respect to $\mathbf{a}$ is:

$$\begin{aligned} D_{\mathbf{a}} u(\mathbf{x}^*(\mathbf{a}_0), \mathbf{a}_0) &= D_{\mathbf{a}} u(\mathbf{x}, \mathbf{a})|_{\mathbf{x}=\mathbf{x}^*(\mathbf{a}_0), \mathbf{a}=\mathbf{a}_0} + \overbrace{D_{\mathbf{x}} u(\mathbf{x}, \mathbf{a})|_{\mathbf{x}=\mathbf{x}^*(\mathbf{a}_0), \mathbf{a}=\mathbf{a}_0}}^{=0} D_{\mathbf{a}}\mathbf{x}^*(\mathbf{a}) \\ &= D_{\mathbf{a}} u(\mathbf{x}, \mathbf{a})|_{\mathbf{x}=\mathbf{x}^*(\mathbf{a}), \mathbf{a}=\mathbf{a}_0} \end{aligned}$$

  - Close to a solution, the "second order effect" of how the maximizer x^* responds to $\mathbf{a}$ is irrelevant because the first order conditions must hold.

Envelope Theorem for Unconstrained Optimization

Let $u : X \times A \to \mathbb{R}$*, with* $\mathbf{x} \in X \subset \mathbb{R}^n$ *and* $\mathbf{a} \in A \subset \mathbb{R}^p$*, be* C^2*. Define*

$$\mathbf{x}^*(\mathbf{a}) \equiv \arg\max_{\mathbf{x}\in X} u(\mathbf{x}, \mathbf{a}) \qquad \textit{and} \qquad V(\mathbf{a}) \equiv u(\mathbf{x}^*(\mathbf{a}), \mathbf{a})$$

Assume solutions are characterized by the first order conditions alone, so that $D_{\mathbf{x}} u(\mathbf{x}, \mathbf{a}) = \mathbf{0}$ *defines* $\mathbf{x}^*(\cdot)$*, and let* $\mathbf{x}^*, \mathbf{a}_0$ *be a solution with* $\det D_{\mathbf{xx}} u(\mathbf{x}^*, \mathbf{a}_0) \neq 0$*.*

- *Then: close to* $\mathbf{x}^*, \mathbf{a}_0$ *the derivative of* $V(\mathbf{a})$ *with respect to* $\mathbf{a}$ *is:*

$$D_{\mathbf{a}} u(\mathbf{x}^*(\mathbf{a}_0), \mathbf{a}_0) = D_a u(\mathbf{x}, \mathbf{a})\big|_{\mathbf{x}=\mathbf{x}^*(\mathbf{a}),\mathbf{a}=\mathbf{a}_0}$$

  - *Close to a solution, only the "first order effect" of how* $\mathbf{a}$ *changes the objective function evaluated at the fixed maximizer* $x^*(\mathbf{a})$ *matters.*

- The envelope theorem tells how to compute the derivative of the value function, even before we can solve explicitly the problem.
- We can then use this derivative to discover general properties of the solution.
- Notice that

$$V(\mathbf{a}) \equiv u(\mathbf{x}^*(\mathbf{a}), \mathbf{a}) \geq u(\mathbf{x}^*(\mathbf{a}_0), \mathbf{a}) \equiv G(\mathbf{a})$$

  with equality at $\mathbf{a} = \mathbf{a}_0$.

  - This justifies the "envelope" in the name as $V(\cdot)$ is the upper envelope of $G(\cdot)$.

Envelope Theorem: Example

Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined as

$$f(x; a) = -x^2 + 2ax + 4a^2$$

and think of maximizing this function with respect to x.

- For a given value of a, the critical points of f are given by

$$\frac{\partial f}{\partial x} = -2x + 2a = 0 \qquad \Leftrightarrow \qquad x = a$$

- The solution yields is a local (and global) maximum (how do I know? draw $f(x; a)$).
- Thus, we know that $x^*(a) = a$ and the value function at the optimum is

$$V(a) = f(x^*(a); a) = -a^2 + 2a^2 + 4a^2 = 5a^2$$

- Hence, the derivative of the value function is given by

$$\frac{\partial V}{\partial a} = \frac{\partial f(x^*(a); a)}{\partial a} = \frac{\partial (5a^2)}{\partial a} = 10a$$

- We could derived this directly using the envelope theorem:

$$\frac{\partial V}{\partial a} = \left.\frac{\partial f}{\partial a}\right|_{x=x^*(a)} = 2x + 8a|_{x=x^*(a)} = 2a + 8a = 10a$$

at $x^*(a) = a$ since $\frac{\partial f}{\partial a} = 2x + 8a$.

Envelope Theorem: Another Example

The problem is $\max_q \pi(p,q) = pq - \frac{1}{2}q^2$ with $q, p \in \mathbb{R}$, so define

$$q^*(p) = \arg\max_q \pi(p,q) \qquad \text{and} \qquad \Pi(p) = \pi(q^*(p); p)$$

$\Pi(p)$ are the profits the firm makes at a given price after optimally choosing how much to produce at that price.

- find $\frac{d\Pi}{dp}$ without solving explicitly for the argmax $q^*(p)$:
  - use the Envelope theorem to find
    $$\frac{d\Pi}{dp} = \left.\frac{\partial \pi(q,p)}{\partial p}\right|_{q=q^*(p)} = q\big|_{q=q^*(p)} = q^*(p)$$
    - Notice that we can immediately conclude that $\frac{d\Pi}{dp} \geq 0$ (why?).
  - If you solve for $q^*(p)$ explicitly (do it), you will find that $q^*(p) = p$ to confirm the result.
- By definition,
  $$\Pi(p) = \pi(q^*(p); p) \geq \pi(q^*(\hat{p}); p) = G(p)$$
  for some $\hat{p}$, and equality holds when $p = \hat{p}$.
  - If $\hat{p} = 2$, for example, $G(p) = p2 - \frac{1}{2}(2)^2 = 2p - 2$ which is always below $\frac{1}{2}p^2$, and it is equal to it at $p = 2$. Graph this, and a few other values of $\hat{p}$.

Lebesgue Measure Zero

- We want to talk about sets being small in $\mathbf{R}^n$.
- The idea is that a set is small if one can squeeze it inside an arbitrarily small rectangle.

Definition

A rectangle is defined as

$$I_k = \times_{j=1}^{n} (a_j^k, b_j^k)$$

for some $a_j^k < b_j^k \in \mathbf{R}$.

Definition

The volume of a rectangle is defined as

$$\mathrm{Vol}(I_k) \prod_{j=1}^{n} \left| b_j^k - a_j^k \right|$$

Lebesgue Measure Zero

Definition

Suppose $A \subset \mathbf{R}^n$. A has Lebesgue measure zero if for every $\varepsilon > 0$ there is a countable collection of rectangles $I_1, I_2, \ldots$ such that

$$\sum_{k=1}^{\infty} \mathrm{Vol}(I_k) < \varepsilon \text{ and } A \subset \bigcup_{k=1}^{\infty} I_k$$

- Sometimes these are called Null Sets.
- This defines Lebesgue measure zero without defining Lebesgue measure.
  - You will talk about measurability at the end of the Fall math class.
- Without specifying a probability measure explicitly, this expresses the idea that if $x \in \mathbf{R}^n$ is chosen at random, then the probability that $x \in A$ is zero.
- Lebesgue measure zero is a natural formulation of the notion that A is a small set in $\mathbf{R}^n$.

Lebesgue Measure Zero: Examples

Lebesgue Measure Zero Sets

- "Lower-dimensional" sets have Lebesgue measure zero.
  - For example, the horizontal axes in $\mathbf{R}^2$:
    $$A = \{x \in \mathbf{R}^2 : x_2 = 0\}$$
    has measure zero.
  - A circle or a straight line also have Lebesgue measure zero in $\mathbf{R}^2$.
- Any finite set has Lebesgue measure zero in $\mathbf{R}^n$.
- $\mathbf{Q}$ and (every countable set) has Lebesgue measure zero in $\mathbf{R}$.
  - Notice that this holds even though the rationals are dense in the reals.

Proposition

If A_n has Lebesgue measure zero $\forall n$ then $\cup_{n \in \mathbf{N}} A_n$ has Lebesgue measure zero.

Lebesgue Measure Zero: Examples

Open Sets Are Not Lebesgue Measure Zero

- No open set in $\mathbf{R}^n$ has Lebesgue measure zero.
- If $O \subset \mathbf{R}^n$ is open, then there exists a rectangle R such that $\bar{R} \subset O$ and such that
$$\mathrm{Vol}(R) = r > 0$$
- If $\{I_j\}$ is any collection of rectangles such that
$$O \subset \cup_{j=1}^{\infty} I_j,$$
- then
$$\bar{R} \subset O \subset \cup_{j=1}^{\infty} I_j,$$
- so
$$\sum_{j=1}^{\infty} \mathrm{Vol}(I_j) \geq \mathrm{Vol}(R) = r > 0$$

Genericity and Sard's Theorem

- Lebesgue measure zero captures the idea that certain sets are rare. They are not generic.
- This can be used to ask how rare are critical points of a function.
- A function may have many critical points.
  - For example, if a function is constant on an interval, then every element of the interval is a critical point.
  - But even in that case a function does not have many critical values.
- Critical values are not generic.

Theorem (Sard's Theorem)

Let $X \subset \mathbf{R}^n$ be open, and $f : X \to \mathbf{R}^m$ be C^r with $r \geq 1 + \max\{0, n-m\}$. Then the set of all critical values of f has Lebesgue measure zero.

Genericity and Sard's Theorem

Theorem (Sard's Theorem)

Let $X \subset \mathbf{R}^n$ be open, and $f : X \to \mathbf{R}^m$ be C^r with $r \geq 1 + \max\{0, n - m\}$. Then the set of all critical values of f has Lebesgue measure zero.

- Sard's Theorem has many interesting implications.

Consequence of Sard's Theorem

- Given a randomly chosen function f, it is very unlikely that 0 will be a critical value of f.
- If by some fluke 0 is a critical value of f, then a slight perturbation of f will make 0 a regular value.

- Next, we formalize this idea.

Transversality

- Let $g : \mathbf{R}^n \to \mathbf{R}^n$ be C^1. Consider the family of n equations in n variables:
$$g(\mathbf{x}) = 0$$
- Suppose for some x such that $g(\mathbf{x}) = \mathbf{0}$,
$$\mathrm{rank}(Dg(\mathbf{x})) < n.$$
  - That is, some $\mathbf{x} \in g^{-1}(\mathbf{0})$ is a critical point of g, thus $\mathbf{0}$ is a critical value of g.
- By Sard's Theorem, "almost every" $\mathbf{a} \neq 0$ is a regular value of g.
  - So for $\mathbf{a}$ outside a set of Lebesgue measure *zero*, $Dg(\mathbf{x})$ has full rank for every $\mathbf{x}$ that solves $g(\mathbf{x}) = \mathbf{a}$.
  - Therefore, for any such $\mathbf{a}$ and any $\mathbf{x} \in g^{-1}(\mathbf{a})$, we can use the Inverse Function Theorem to show that a local inverse $x(\mathbf{a})$ exists, and give a formula for $Dx(\mathbf{a})$.

Transversality

- Suppose $f : \mathbf{R}^n \times \mathbf{R}^p \to \mathbf{R}^m$. We have a parameterized family of equations
$$f(\mathbf{x}, \mathbf{a}) = \mathbf{0}$$
where, as before, we interpret $\mathbf{a} \in \mathbf{R}^p$ to be a vector of parameters that indexes the function $f(\cdot, \mathbf{a})$.
- For a given $\mathbf{a}$, we are interested in the set of solutions
$$\{\mathbf{x} \in X : f(\mathbf{x}, \mathbf{a}) = \mathbf{0}\}$$
and the way that this correspondence depends on $\mathbf{a}$.

Transversality Theorem

Theorem (Transversality Theorem)

Let $X \subset \mathbf{R}^n$ and $A \subset \mathbf{R}^p$ be open, and $f : X \times A \to \mathbf{R}^m$ be C^r with $r \geq 1 + \max\{0, n - m\}$. Suppose that $\mathbf{0}$ is a regular value of f. Then

- *there is a set $A_0 \subset A$ such that $A \setminus A_0$ has Lebesgue measure zero, and*
- *for all $\mathbf{a} \in A_0$, $\mathbf{0}$ is a regular value of $f_a = f(\cdot, \mathbf{a})$.*

Remark

- Notice the difference between "$\mathbf{0}$ is a regular value of f" which is an assumption, and "$\mathbf{0}$ is a regular value of f_a for a fixed $\mathbf{a} \in A_0$" which is a conclusion.
  - $\mathbf{0}$ is a regular value of f if and only if $Df(\mathbf{x}, \mathbf{a})$ has full rank for every $(\mathbf{x}, \mathbf{a})$ such that $f(\mathbf{x}, \mathbf{a}) = 0$.
  - Instead, for fixed $\mathbf{a}_0 \in A_0$, $\mathbf{0}$ is a regular value of $f_{a_0} = f(\cdot, \mathbf{a}_0)$ if and only if $D_x f(\mathbf{x}, \mathbf{a}_0)$ has full rank for every $\mathbf{x}$ such that $f_{a_0}(\mathbf{x}) = f(\mathbf{x}, \mathbf{a}_0) = 0$.

- We can use the implicit function theorem everywhere, except for a set of points that have Lebesgue measure zero.

Transversality and Implicit Function Theorems

Implications of the Transversality Theorem

- Suppose $n = m$ so that there are as many equations (m) as endogenous variables (n). Suppose f is C^1 (note that $1 = 1 + \max\{0, n - n\}$).
- If $\mathbf{0}$ is a regular value of f
  - so $Df(\mathbf{x}, \mathbf{a})$ has rank $n = m$ for every (x, a) such that $f(\mathbf{x}, \mathbf{a}) = 0$
- by the Transversality Theorem
  - there is a set $A_0 \subset A$ such that $A \setminus A_0$ has Lebesgue measure zero and
  - for every $a_0 \in A_0$, $D_x f(x, a_0)$ has rank $n = m$ for all x such that $f(x, a_0) = 0$.
- Fix $a_0 \in A_0$ and x_0 such that $f(x_0, a_0) = 0$.
- By the Implicit Function Theorem, there exist open sets A^* containing a_0 and X^* containing x_0, and a C^1 function $x : A^* \to X^*$ such that
  - $x(a_0) = x_0$
  - $f(x(a), a) = 0$ for every $a \in A^*$
  - if $(x, a) \in X^* \times A^*$ then
    $$f(x, a) = 0 \Leftrightarrow x = x(a)$$
    that is, x_0 is locally unique, and $x(a)$ is locally unique for each $a \in A^*$
  - Moreover: $Dx(a_0) = -[D_x f(x_0, a_0)]^{-1} D_a f(x_0, a_0)$

Tomorrow

We talk about the shape of functions, and about properties that can be preserved across functions.

1. Convexity
2. Concave and Convex Functions
3. Cardinal and Ordinal Properties