Structural Properties of Utility Functions
Walrasian Demand

Econ 2100 Fall 2018

Lecture 4, September 10

Outline

1. Structural Properties of Utility Functions
   1. Local Non Satiation
   2. Convexity
   3. Quasi-linearity
2. Walrasian Demand

From Last Class

Definition

The utility function $u : X \to \mathbb{R}$ represents the binary relation $\succsim$ on X if

$$x \succsim y \Leftrightarrow u(x) \geq u(y).$$

Theorem (Debreu)

Suppose $X \subseteq \mathbb{R}^n$. A binary relation $\succsim$ on X is complete, transitive, and continuous if and only if it admits a continuous utility representation $u : X \to \mathbb{R}$.

- We are interested in connections between utility functions and preferences.

Structural Properties of Utility Functions

- The main idea is to understand the relation between properties of preferences and characteristics of the utility function that represents them.

NOTATION:

- We assume $X = \mathbb{R}^n$.
- If $x_i \geq y_i$ for each i, we write $x \geq y$.

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Example: The lexicographic preference on $\mathbb{R}^2$ is locally nonsatiated

- Fix (x_1, x_2) and $\varepsilon > 0$.
- Then $(x_1 + \frac{\varepsilon}{2}, x_2)$ satisfies $\|(x_1 + \frac{\varepsilon}{2}, x_2) - (x_1.x_2)\| < \varepsilon$
- and $(x_1 + \frac{\varepsilon}{2}, x_2) \succ (x_1, x_2)$.

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$$||y - x|| = \sqrt{\sum_{i=1}^{n} \left(\frac{\varepsilon}{n}\right)^2}$$

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$$||y - x|| = \sqrt{\sum_{i=1}^{n} \left(\frac{\varepsilon}{n}\right)^2} = \frac{\varepsilon}{\sqrt{n}} < \varepsilon.$$
- Thus $\succsim$ is locally nonsatiated. □

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Question

- What does convexity imply for the utility function representing $\succsim$?

Convex Preferences: An Example

Let $\succsim$ on $\mathbb{R}^2$ be defined as $x \succsim y$ if and only if $x_1 + x_2 \geq y_1 + y_2$ is convex

Proof: Suppose $x \succsim y$, i.e. $x_1 + x_2 \geq y_1 + y_2$, and fix $\alpha \in (0, 1)$.

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- So,
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proving $\alpha x + (1-\alpha)y \succsim y$.

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proving $\alpha x + (1 - \alpha)y \succsim y$.
- This is not strictly convex, because $(1, 0) \succsim (0, 1)$ and $(1, 0) \neq (0, 1)$ but
$$\frac{1}{2}(1, 0) + \frac{1}{2}(0, 1) = (\frac{1}{2}, \frac{1}{2}) \precsim (0, 1).$$

Convexity and Quasiconcave Utility Functions

- Convexity is equivalent to quasi concavity of the corresponding utility function.

Proposition

If u represents $\succsim$, then:

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Proof.

Question 5b. Problem Set 2, due next Tuesday. □

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  - Let $x, y \in X$ and $\alpha \in [0, 1]$, and suppose $u(x) \geq u(y)$.
  - Then $x \succsim y$ and $y \succsim y$, and so x and y are both in $\succsim(y)$.

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  - Since $\succsim(y)$ is convex (by assumption), then $\alpha x + (1 - \alpha)y \succsim y$.

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  - Since u represents $\succsim$,
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    Thus u is quasiconcave.

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If $\succsim$ is convex, then $C_{\succsim}(A)$ is convex for all convex A.
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  - Convexity of $\succsim$ implies $\alpha x + (1-\alpha)y \succsim y$.
  - By definition of $C_{\succsim}$, $y \succsim z$ for all $z \in A$.

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If $\succsim$ is convex, then $C_{\succsim}(A)$ is convex for all convex A.
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  - Since $x \succsim y$ and $x \neq y$, strict convexity implies $\alpha x + (1-\alpha)y \succ y$, but this contradicts the fact that $y \in C_{\succsim}(A)$.

□

Quasi-linear Utility

Definition

The function $u : \mathbb{R}^n \to \mathbb{R}$ is quasi-linear if there exists a function $v : \mathbb{R}^{n-1} \to \mathbb{R}$ such that $u(x, m) = v(x) + m$.

We usually think of the n-th good as money (the numeraire).

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Proof.

Question 5c. Problem Set 2, due next Tuesday. □

Quasi-linear Preferences and Utility

Proposition

Suppose that the preference relation $\succsim$ on $\mathbf{R}^n$ admits two quasi-linear representations: $v(x) + m$, and $v'(x) + m$, where $v, v' : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$. Then there exists $c \in \mathbf{R}$ such that $v'(x) = v(x) - c$ for all $x \in \mathbf{R}^{n-1}$.

Proof.

Exercise □

Homothetic Preferences and Utility

- Homothetic preferences are also useful in many applications, in particular for aggregation problems and macroeconomics.

Definition

The preference relation $\succsim$ on X is homothetic if for all $x, y \in X$,

$$x \sim y \Rightarrow \alpha x \sim \alpha y \text{ for each } \alpha > 0$$

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The continuous preference relation $\succsim$ on $\mathbf{R}^n$ is homothetic if and only if it is represented by a utility function that is homogeneous of degree 1.

- A function is homogeneous of degree r if $f(\alpha x) = \alpha^r f(x)$ for any x and $\alpha > 0$.

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Demand Theory

Main Questions

- Suppose the consumer uses her income to purchase goods (commodities) at the exogenously given prices:
  - What are the optimal consumption choices?
  - How do they depend on prices and income?

- Typically, we answer this questions solving a constrained optimization problem using calculus.
- That means the utility function must be not only continuous, but also differentiable.
  - Differentiability, however, is not a property we can derive from preferences.
- Sometimes, calculus is not necessary, and we can talk about optimal choices even when preferences are not necessarily represented by a utility function.

Budget Set

- First, we define what a consumer can buy.

Definition

The Budget Set $B(\mathbf{p}, w) \subset \mathbf{R}^n$ at prices $\mathbf{p}$ and income w is the set of all affordable consumption bundles and is defined by

$$B(\mathbf{p}, w) = \{\mathbf{x} \in \mathbf{R}^n_+ : \mathbf{p} \cdot \mathbf{x} \leq w\}.$$

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Exercise

Suppose $w = \$100$. There are two commodities, electricity and food. Each unit of food costs \$1. The first $20Kwh$ electricity cost \$1 per Kwh, but the price of each incremetal unit of electricity is \$1.50 per Kwh. Write the consumer's budget set formally and draw it.

Walrasian Demand

Main Idea

- The optimal consumption bundles are those that are weakly preferred to all other affordable bundles.

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Walrasian Demand With Utility

- Although we do not need the utility function to exist to define Walrasian demand, if a utility function exists there is an equivalent definition.

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- We can derive some properties of Walrasian demand directly from assumptions on preferences and/or utility.

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- Since the constraints are the same, the optimal choices must also be the same. □

The Consumer Spends All Her Income

This is sometimes known as Walras' Law for individuals

Proposition (Full Expenditure)

If $\succsim$ is locally nonsatiated , then

$$\mathbf{p} \cdot \mathbf{x} = w \qquad \text{for any } \mathbf{x} \in x^*(\mathbf{p}, w)$$

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This is sometimes known as Walras' Law for individuals

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(why does such a $\mathbf{y}$ always exist?)
- By local non satiation, this implies $\mathbf{y} \succ \mathbf{x}$ contradicting $\mathbf{x} \in x^*(\mathbf{p}, w)$. □

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Proposition

If u is quasiconcave, then $x^*(\mathbf{p}, w)$ is convex.
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Proof.

Suppose $\mathbf{x}, \mathbf{y} \in x^*(\mathbf{p}, w)$ and pick $\alpha \in [0, 1]$.

- First convexity: need to show $\alpha\mathbf{x} + (1-\alpha)\mathbf{y} \in x^*(\mathbf{p}, w)$.
  - $\mathbf{x} \succsim \mathbf{y}$ by definition of $x^*(\mathbf{p}, w)$.
  - u is quasiconcave, thus $\succsim$ is convex and $\alpha\mathbf{x} + (1-\alpha)\mathbf{y} \succsim \mathbf{y}$.
  - $\mathbf{y} \succsim \mathbf{z}$ for any $\mathbf{z} \in B(\mathbf{p}, w)$ by definition of $x^*(\mathbf{p}, w)$.
  - Transitivity implies $\alpha\mathbf{x} + (1-\alpha)\mathbf{y} \succsim \mathbf{z}$ for any $\mathbf{z} \in B(\mathbf{p}, w)$; thus $\alpha\mathbf{x} + (1-\alpha)\mathbf{y} \in x^*(\mathbf{p}, w)$.
- Now uniqueness.
  - $\mathbf{x}, \mathbf{y} \in x^*(\mathbf{p}, w)$ and $\mathbf{x} \neq \mathbf{y}$ imply $\alpha\mathbf{x} + (1-\alpha)\mathbf{y} \succ \mathbf{y}$ for any $\alpha \in (0, 1)$ because u is strictly quasiconcave ($\succsim$ is strictly convex).
  - Since $B(\mathbf{p}, w)$ is convex, $\alpha\mathbf{x} + (1-\alpha)\mathbf{y} \in B(\mathbf{p}, w)$, contradicting $\mathbf{y} \in x^*(\mathbf{p}, w)$.

□

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If u is continuous, then $x^*(\mathbf{p}, w)$ is nonempty and compact.

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Proof.

Define A by

$$A = B(\mathbf{p}, w) = \{\mathbf{x} \in \mathbb{R}^n_+ : \mathbf{p} \cdot \mathbf{x} \leq w\}$$

- This is a closed and bounded (i.e. compact, set) and
$$x^*(\mathbf{p}, w) = C_{\succsim}(A) = C_{\succsim}(B(\mathbf{p}, w))$$
where $\succsim$ are the preferences represented by u.
- Then $x^*(\mathbf{p}, w)$ is the set of maximizers of a continuous function over a compact set.

$\square$

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Question 6, Problem Set 2; due next Tuesday.

For each of the following utility functions, find the Walrasian demand correspondence. (Hint: pictures may help)

1. $u(\mathbf{x}) = \prod_{i=1}^{n} x_i^{\alpha_i}$ with $\alpha_i > 0$ (Cobb-Douglas).
2. $u(\mathbf{x}) = \min\{\alpha_1 x_1, \alpha_2 x_2, ..., \alpha_n x_n\}$ with $\alpha_i > 0$ (generalized Leontief).
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- Constant elasticity of substitution (CES) preferences are the most commonly used homothetic preferences. Many preferences are a special case of CES.

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How to solve $\max f(x)$ **subject to** $g_i(x) \leq 0$ **with** $i = 1, .., m$

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1. Write the Langrange function $L : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$ as

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The Recipe In Action: Cobb-Dougals Utility

Compute Walrasian demand when the utility function is $u(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$

Here $x^*(\mathbf{p}, w)$ is the solution to

$$\max_{x_1, x_2 \in \{p_1 x_1 + p_2 x_2 \leq w,\ x_1 \geq 0,\ x_2 \geq 0\}} x_1^{\alpha} x_2^{1-\alpha}$$

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4. Find a solution to the above (easy for me to say).

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The Recipe In Action: Cobb-Dougals Utility

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- Substituting back then yields

$$x_1^*(p, w) = \frac{\alpha w}{p_1}, \ x_2^*(p, w) = \frac{(1-\alpha)\, w}{p_2}, \ \text{and } \lambda_w = \left(\frac{\alpha}{p_1}\right)^{\alpha} \left(\frac{1-\alpha}{p_2}\right)^{1-\alpha}$$

Next Week

- More Properties of Walrasian Demand.
- Indirect Utility.
- Comparative Statics.
- Expenditure Minimization.