Anscombe & Aumann Expected Utility Betting and Insurance

Econ 2100 Fall 2018

Lecture 11, October 3

Outline

1. Subjective Expected Utility
2. Qualitative Probabilities
3. Allais and Ellsebrg Paradoxes
4. Utility Of Wealth
5. Insurance and Betting

Next Wednesday (October 10)

- **MIDTERM**
  - 75 minutes long,
  - covers everything so far:
    - includes Monday's class (Lecture 12)
    - includes Problem Set 5, due next Wednesday;
  - you can consult the class handouts (in printed form), and any notes you may have written,
  - but there is no access to any other materials (no books, computers, etc).
  - Past midterm exams with Yunyun... but the content of the course has changed over the years.

Anscombe and Aumann Structure

- $\Omega = \{1, 2, \ldots, S\}$ is a finite set of states, with generic element $s \in \Omega$.
- X is a finite set of size n of consequences with a generic element $x \in X$.
- ΔX is the set of all probability distributions over X.
- Elements of Ω reprent subjectively perceived randomness, while elements of ΔX represent objectively perceived randomness.
- Preferences are on $H = (\Delta X)^{\Omega}$, the space of all functions from Ω to ΔX.
  - An act is a function $h : \Omega \rightarrow \Delta X$ that assigns a lottery in ΔX to each $s \in \Omega$.
    - Let $h_s = h(s) \in \Delta X$, and denote $h_s(x) = [h(s)](x) \in [0, 1]$.
    - This is the probability of x conditional on s, given the act h: $(\Pr(x|s, h))$
- The set of constant acts is
$$H_c = \{f \in H : f(s) = f(s') \text{ for all } s \in \Omega\}$$
- If $f, g \in H$, then the function $\alpha f + \beta g : \Omega \rightarrow \Delta X$ is defined by
$$[\alpha f + \beta g](s) = \alpha f(s) + \beta g(s).$$
  - Draw a picture to make sure you see how this works.

Subjective Expected Utility (SEU): Idea

- Starting from preferences, identify a probability distribution $\mu \in \Delta\Omega$ and a utility index $v : X \to \mathbf{R}$ such that a utility representation of these preferences is

$$U(h) = \sum_{s \in \Omega} \mu(s) \left[\sum_{x \in X} v(x) h_s(x) \right]$$

equivalently:

$$f \succsim g \Leftrightarrow \sum_{s \in \Omega} \mu(s) \left[\sum_{x \in X} v(x) f_s(x) \right] \geq \sum_{s \in \Omega} \mu(s) \left[\sum_{x \in X} v(x) g_s(x) \right]$$

Some Accounting Details

- $U(h)$ is the (subjective) expected value of v given act h since

$$\sum_s \mu(s) \underbrace{\left[\sum_x v(x) h_s(x) \right]}_{\mathbf{E}_{h(s)}[v(x)]} = \sum_x v(x) \underbrace{\left[\sum_s \overbrace{\mu(s)}^{\Pr(s)} \overbrace{h_s(x)}^{\Pr(x|s,h)} \right]}_{\Pr(x)}.$$

  - $\sum_s \mu(s) h_s(x)$ is the total or unconditional subjective probability of receiving consequence x under the function h, denoted $\Pr(x)$.
    - Therefore $\Pr(x) = \sum_s \Pr(s)\Pr(x|s,h)$, since $\mu(s)h_s(x) = \Pr(s)\Pr(x|s,h)$.

Last Class: State Dependent Expected Utility

Theorem

The preference relation $\succsim$ on H is complete, transitive, independent and Archimedean if and only if there exists a set of vNM indices $v_1, \ldots, v_S : X \to \mathbf{R}$ such that. the utility representation is

$$U(h) = \sum_{s \in \Omega} \sum_{x \in X} v_s(x) h_s(x)$$

- This theorem does not define a unique probability distribution over Ω (see today's homework).
- To pin down a probability distribution one needs one more assumption:
- The binary relation $\succsim$ on H is state-independent if, for all non-null states $s, t \in \Omega$, for all acts $h, g \in H$, and for all lotteries $\pi, \rho \in \Delta X$,

$$(h_{-s}, \pi) \succsim (h_{-s}, \rho) \quad \Rightarrow \quad (g_{-t}, \pi) \succsim (g_{-t}, \rho).$$

Subjective Expected Utility Theorem

Theorem (Expected Utility Theorem, Anscombe and Aumann)

A preference relation $\succsim$ on H is complete, transitive, independent, Archimedean, and state-independent if and only if there exists a vNM index $v : X \to \mathbf{R}$ and a probability $\mu \in \Delta\Omega$ such that

$$U(h) = \sum_{s \in \Omega} \mu(s) \left[\sum_{x \in X} v(x) h_s(x) \right]$$

is a utility representation of $\succsim$.
Moreover, this representation is unique up to affine transformations provided at least two acts can be ranked strictly.

- Therefore:

$$h \succsim g \Leftrightarrow \sum_{s \in \Omega} \mu(s) \left[\sum_{x \in X} v(x) h_s(x) \right] \geq \sum_{s \in \Omega} \mu(s) \left[\sum_{x \in X} v(x) g_s(x) \right]$$

- The second part says: if there exist $h, g \in H$ such that $h \succ g$ and $U(h)$ defined above represents $\succsim$, then $U'(h) = \sum_s \mu'(s) \left[\sum_x v'(x) h_s(x) \right]$ also represents $\succsim$ if and only if $\mu' = \mu$ and $v' = av + b$ for some $a > 0$ and $b \in \mathbf{R}$.

Remark

- The probability distribution μ is unique.

Subjective Expected Utility Theorem

Theorem (Expected Utility Theorem, Anscombe and Aumann)

A preference relation $\succsim$ on H is independent, Archimedean, and state-independent if and only if there exists a vNM index $v : X \to \mathbf{R}$ and a probability $\mu \in \Delta\Omega$ such that

$$U(h) = \sum_{s\in\Omega} \mu(s) \left[\sum_{x\in X} v(x) h_s(x) \right] = \sum_{x\in X} v(x) \underbrace{\left[\sum_{s\in\Omega} \overbrace{\mu(s)}^{\Pr(s)} \overbrace{h_s(x)}^{\Pr(x|s)} \right]}_{\Pr(x)}$$

is a utility representation of $\succsim$. This representation is unique up to affine transformations.

- Preferences identify two things:
  - the utility index over consequences $v : X \to \mathbf{R}$ and
  - the probability distribution over states $\mu \in \Delta\Omega$.
- Different preferences may imply different beliefs μ on Ω.
- von Neumann–Morgenstern's Theorem only identifies the utility index v.

For this reason, this is called Subjective Expected Utility.

Anscombe & Aumann Proof

- We will not do it in detail, but here is a breakdown of what would happen.

Proof.

- First, convert each act h to a vector in $[0,1]^S$, by assigning each dimension $s \in \{1, ..., S\}$ a vNM expected utility for h_s
  - this gives $\left[\sum_x v(x) h_s(x)\right]$ in each state s;
  - monotonicity, a consequence of state-independence, is essential for this step.
- Then, construct an independent and Archimedean preference relation on $[0,1]^S$ and the measure μ is the dual of the affine utility on $[0,1]^S$.
- The necessity and uniqueness parts are as usual.

□

- Notice that this argument involves using the mixture space theorem (or vNM theorem) twice.
  - the first time to find $\left[\sum_x v(x) h_s(x)\right]$ in each state s (this is $\mathbf{E}_{h(s)}[v(x)]$);
  - the second to find the measure μ as the dual of the affine utility on $[0,1]^S$.

Savage Subjective Expected Utility

- Anscombe & Aumann assume the existence of an objective randomizing device (preferences are defined over functions from states to objective lotteries).
  - ΔX is a convex space and one can apply the mixture space theorem.

- Savage's theory instead considers all uncertainty as subjective: no probabilities are assumed a priori; all probabilities are identified by preferences over basic objects.
  - This comes at the cost of higher mathematical complexity (see Kreps' Theory of Choice).
- Ω is the state space, and X is a finite set of (sure) consequences.
- Savage acts are elements of X^Ω, the space of all functions from Ω to X.
- The decision maker has preferences $\succsim$ over these acts.
- The expected utility representation is something like

$$U(f) = \sum_{x \in X} v(x)\mu(f^{-1}(x)) = \int_\Omega v \circ f d\mu = \mathbf{E}_\mu[v \circ f],$$

  where f is any act, $v : X \to \mathbf{R}$ is a vNM utility index on consequences, and μ is a probability measure on Ω.
- Utility and probability are fully endogenous.

Probability Distributions

- Economists think of preference as primitive, and derive subjective probability as a piece of a subject's utility function.
- Some statisticians think about probabilities the same way economists think about utility functions.

Definition

A (finitely additive) probability measure on Ω is a function $\mu : 2^{\Omega} \to [0, 1]$ such that:

1. $\mu(\Omega) = 1$;
2. If $A \cap B = \emptyset$, then $\mu(A \cup B) = \mu(A) + \mu(B)$.

- They usually assume that μ is also countably additive, which is very useful technically.

Qualitative Probabilities

Definition

A binary relation $\succeq^*$ on 2^Ω is a qualitative probability if:

1. $\succeq^*$ is complete and transitive;
2. $A \succeq^* \emptyset$, for all $A \subseteq \Omega$;
3. $\Omega \succ^* \emptyset$;[a]
4. $A \succeq^* B$ if and only if $A \cup C \succeq^* B \cup C$, for all $A, B, C \subseteq \Omega$ such that $A \cap C = B \cap C = \emptyset$.

[a] By $A \succ^* B$, we mean $A \succeq^* B$ and not $B \succeq^* A$, i.e. $\succ^*$ is the asymmetric component of $\succeq^*$.

- Think of this as a binary relation $\succeq^*$ expressing the idea of "more likely than".
- This is a "personal" characteristic; it varies from individual to individual and can be used to infer the existence of "personal probabilities".

Qualitative Probabilities and Probability Measures

- The relationship between qualitative probability relations and probability measures is similar to the one between preference orders and utility functions.

Definition

The probability measure μ represents the binary relation $\succeq^*$ on 2^Ω if:

$$\mu(A) \geq \mu(B) \text{ if and only if } A \succeq^* B.$$

Result

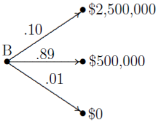
- All binary relations that are represented by a probability measure are qualitative probabilities.
- The converse is false: not all qualitative probabilities can be represented by a probability measure.
- A stronger version of additivity is required to guarantee representation by a probability measure (see Fishburn 1986 in *Statistical Science*).

- Our theorems about preferences infer the existence of probabilities. Their theorem about probabilities infer the existence of a preference relation.

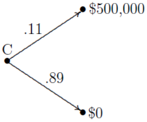
Allais Paradox

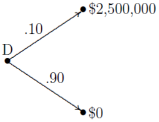
The decision maker must choose between the following objective lotteries.

A : \$500,000 for sure

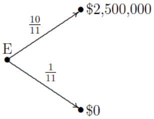


and between





- One reasonable prefrence may rank $A \succ B$ and $D \succ C$.
- Any such preferences violates independence:
- Consider the following lotteries E and F:



F : \$0 for sure

$$0.11A + .89A = A \succ B = 0.11E + .89A$$

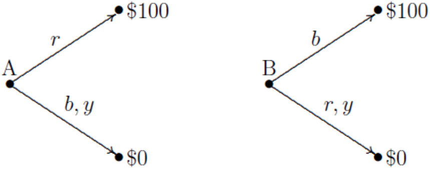
$$0.11A + .89F = C \prec D = 0.11E + 0.89F$$

- But independence says for all $f, g, h \in H$ and $\alpha \in (0,1)$,

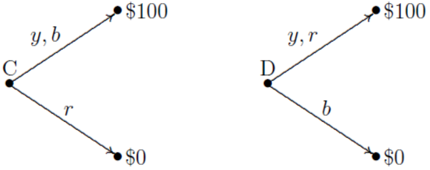
$$f \succsim g \Leftrightarrow \alpha f + (1-\alpha)h \succsim \alpha g + (1-\alpha)h.$$

Ellseberg Paradox

An urn contains 90 balls. 30 balls are red (r) and 60 are either blue (b) or yellow (y). One ball will be drawn at random. Consider the following bets.



and



- Many prefer A to B and C over D. Is this consistent with expected utility?
  - By expected utility, there exists a (subjective) probability distribution μ over $\{r, y, b\}$.
  - From $A \succ B$ and $C \succ D$ we conclude:

$$\begin{aligned} \mu(r)U(\$100) + \underbrace{(1-\mu(r))}_{\mu(b)+\mu(y)} U(\$0) &> \mu(b)U(\$100) + \underbrace{(1-\mu(b))}_{\mu(r)+\mu(y)} U(\$0) \\ \Rightarrow \mu(r) &> \mu(b) \end{aligned}$$

and

$$\begin{aligned} (\mu(b)+\mu(y))U(\$100) + \mu(r)U(\$0) &> (\mu(r)+\mu(y))U(\$100) + \mu(b)U(\$0) \\ \Rightarrow \mu(r) &< \mu(b) \end{aligned}$$

- This contradicts the hypothesis of consistent subjective beliefs.

(Knightian) Uncertainty vs. Risk

- Risk

B R

$B = 30$ $R = 70$

B+R=100

$P(B) = \frac{3}{10}$ $P(R) = \frac{7}{10}$

- Knightian Uncertainty (Ambiguity)

B R

$20 \leq B \leq 50$ $50 \leq R \leq 80$

B+R=100

$P(B) = ?$ $P(R) = ?$

Some theories that can account for Knightian Uncertainty.

- Gilboa & Schmeidler (weaken independence): Maxmin Expected Utility

$$x \succsim y \Leftrightarrow \min_{\mu \in C} \sum_s \mu(s) u(x_s) \geq \min_{\mu \in C} \sum_s \mu(s) u(y_s)$$

- Truman Bewley (drop completeness): Expected Utility with Sets

$$x \succsim y \Leftrightarrow \sum_s \mu(s) u(x_s) \geq \sum_s \mu(s) u(y_s) \qquad \text{for all } \mu \in C$$

Probability Distribution On Wealth

- Many applications of expected utility consider preferences on probability distributions of wealth (a continuous variable).
- A probability distribution is characterized by its cumulative distribution function.

Definition

A cumulative distribution function (cdf) $F : \mathbf{R} \to [0,1]$ satisfies:

- $x \geq y$ implies $F(x) \geq F(y)$ (nondecreasing);
- $\lim_{y \downarrow x} F(y) = F(x)$ (right continuous);[a]
- $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to \infty} F(x) = 1$.

[a] Recall that, if it exists, $\lim_{y \downarrow x} f(y) = \lim_{n \to \infty} f(x + \frac{1}{n})$.

Notation

- μ_F denotes the mean (expected value) of F, i.e. $\mu_F = \int x \, dF(x)$.
- δ_x is the degenerate distribution function at x; i.e. δ_x yields x with certainty:

$$\delta_x(z) = \begin{cases} 0 & \text{if } z < x \\ 1 & \text{if } z \geq x \end{cases}.$$

Expected Utility Of Wealth

- As usual, the space of all distribution functions is convex and one can define preferences on it.
- The utility index $v : \mathbf{R} \to \mathbf{R}$ is defined over wealth (can be negative).
- The expected utility is the integral of v with respect to F

$$\int v(x) dF(x) = \int v dF$$

  - If F is differentiable, the expectation is computed using the density $f = F'$:
$$\int v\, dF = \int v(x) f(x)\, dx.$$

von Neumann and Morgenstern Expected Utility

Under some axioms, there exists a utility function U on distributions defined as $U(F) = \int v\, dF$, for some continuous index $v : \mathbf{R} \to \mathbf{R}$ over wealth, such that

$$F \succsim G \iff \int v\, dF \geq \int v\, dG$$

- Axioms not important from now on (need a stronger continuity assumption).
- We always think of v as a weakly increasing function (more wealth cannot be bad).

Simple Probability

- A simple probability distribution π on $X \subset \mathbf{R}$ is specified by
  - a finite subset of X called the support and denoted $supp(\pi)$, and
  - for each $x \in X$, $\pi(x) > 0$ with $\sum_{x \in supp(\pi)} \pi(x) = 1$
- If we restrict attention to simple probability distributions, then even if X is infinite, only elements with strictly positive probability count.
- The utility index $v : X \to \mathbf{R}$ is defined over wealth (can be negative).
- The expected utility is the expected value of v with respect to π

$$\sum_{x \in supp(\pi)} \pi(x)v(x)$$

- One can write more money is better as: for each $x,y \in X$ such that $x > y$ then $\delta_x \succ \delta_y$.
- We can use this setting to think about many applied choice under uncertainty problems like betting and insurance.

Betting

A Gamble

- Suppose an individual is offered the following bet:

win ax with probability p	lose x with probability $1-p$

- The expected value of this bet is

$$pax + (1-p)(-x) = [pa + (1-p)(-1)]\,x$$

Definition

A bet is actuarially fair if it has expected value equal to zero (i.e. $a = \frac{1-p}{p}$); it is better than fair if the expected value is positive and worse than fair if it is negative.

How does she evaluate this bet? Use the expected utility model to find out

- If vNM index is $v(\cdot)$ and initial wealth is w, expected utility is:

$$\overset{\text{probability of winning}}{p}\;\underset{\text{utility of wealh if win}}{v(w+ax)} + \overset{\text{probability of losing}}{(1-p)}\;\underset{\text{utility of wealth if lose}}{v(w-x)}$$

- How much does she want of this bet? Answer by finding the optimal x.

Betting and Expected Utility

win ax with probability p	lose x with probability $1-p$

- The consumer solves

$$\max_x pv(w+ax) + (1-p)v(w-x)$$

- The FOC is

$$pav'(w+ax) = (1-p)v'(w-x)$$

  - rearranging

$$\frac{pa}{(1-p)} = \frac{v'(w-x)}{v'(w+ax)}$$

- If the bet is fair, the left hand side is 1. Therefore, at an optimum, the right hand side must also be 1.
- If the vNM utility function is strictly increasing and strictly concave ($v' > 0$ and $v'' < 0$), the only way a fair bet can satisfy this FOC is to solve

$$w + ax = w - x$$

  which implies $x = 0$.
- She will take no part of a fair bet.
- What happens with a better than fair bet?

Insurance

An Insurance Problem

- An individual faces a potential "accident":

the loss is L with probability π	nothing happens with probability $1-\pi$

Definition

An insurance contract establishes an initial premium P and then reimburses an amount Z if and only if the loss occurs.

Definition

Insurance is actuarially fair when its expected cost is zero; it is less than fair when its expected cost is positive.

- The expected cost (to the individual) of an insurance contract is

$$P - \left[\underbrace{\pi(-Z)}_{\text{loss}} + \underbrace{(1-\pi)\ (0)}_{\text{no loss}} \right] = P - \pi Z$$

- Fair insurance means

$$P = \pi Z$$

Insurance and Expected Utility

An Insurance Problem

- An individual with current wealth W and utility function $v(\cdot)$ faces a potential accident:

lose L with probability π	or	lose zero with probability $1-\pi$

- If she buys insurance, her expected utility is

$$\pi v(\underbrace{W - L - P + Z}_{\text{wealth if loss}}) + (1-\pi)\, v(\underbrace{W - P}_{\text{wealth if no loss}})$$

  - For example, if the loss is fully reimbursed ($Z = L$), this becomes
$$\pi v\,(W-P) + (1-\pi)\, v\,(W-P) = v\,(W-P)$$

- Will she buy any insurance? Yes if

$$\underbrace{\pi v\,(W-L-P+Z) + (1-\pi)\, v\,(W-P)}_{\text{expected utility with insurance}} \geq \underbrace{\pi v\,(W-L) + (1-\pi)\, v\,(W)}_{\text{expected utility without insurance}}$$

- How much coverage will she want if she buys any coverage?
  - Find the optimal Z.
- The answer depends on the premium set by the insurance company P (which could depend on Z) as well as the curvature of the utility function v.

Next Class

- Risk aversion