General Equilibrium Notation
Pareto Optimality
Core

Econ 2100 Fall 2018

Lecture 13, October 16

Outline

1. General equilibrium notation
2. The economy
3. Feasibility
4. Pareto optimality
5. The core

General Equilibrium

General Equilibrium Theory

Models interactions between anonymous agents who take prices as given (everyone is "small" relative to the market).
Resources are exogenous, but all markets are considered at once, hence general (as opposed to partial) equilibirum.

1. What would a social planner do?
   1. The main concept is Pareto efficiency: outcomes cannot be redistributed to improve welfare.
2. How do competitive markets allocate resources?
   1. The main concept is competitive equilibrium: all agents makes optimal choices, and prices are such that demand and supply are equal.
   2. The main assumption is that agents are price-taking optimizers.
3. Important questions:
   1. when is a competitive equilibirum Pareto efficient? First Welfare Theorem;
   2. when is a Pareto efficient outcome an equilibirum? Second Welfare Theorem;
   3. when does a competitive equilibirum exhist? Arrow-Debreu-McKenzie Theorem;
   4. when is a competitive equilibrium unique?
   5. is a competitive equilibrium robust to small perturbations of the parameters?
   6. how do we get to a competitive equilibrium and what happens away from it?
   7. what happens if we add time or uncertainty?

The Economy

- An economy consists of I individuals and J firms who consume and produce L perfectly divisible commodities.
- Each firm $j = 1, ..., J$ is described by a production set $Y_j \subset \mathbb{R}^L$.
- Each consumer $i = 1, ..., I$ is described by four objects:
  - a consumption set $X_i \subset \mathbb{R}^L$;
  - preferences $\succsim_i$;
  - an individual endowment given by a vector of commodities $\omega_i \in X_i$; and
  - a non-negative share θ_{ij} of the profits of each firm j, with $\theta_{ij} \in [0, 1]$ for all j.

Definition

An economy with private ownership is a tuple

$$\left\{ \{X_i, \succsim_i, \omega_i, (\theta_{i1}, ..., \theta_{iJ})\}_{i=1}^{I}, \{Y_j\}_{j=1}^{J} \right\}$$

- The resources available to the economy are given by the aggregate endowment:

$$\omega = \sum_{i=1}^{I} \omega_i$$

- If we do not need to track who owns what, we can define an economy as

$$\left\{ \{X_i, \succsim_i\}_{i=1}^{I}, \{Y_j\}_{j=1}^{J}, \omega \right\}$$

  where $\omega \in \mathbb{R}^L$ is the aggregate endowment.

Allocations

- First, we define a possible outcome for the economy.

Definition

An allocation specifies a consumption vector $x_i \in X_i$ for each consumer $i = 1, ..., I$, and a production vector $y_j \in Y_j$ for each firm $j = 1, ..., J$; an allocation is

$$(x, y) \in \mathbb{R}^{L(I+J)}$$

- This lists what everyone consumes and what every firm produces.
- We can rewrite an allocation as

$$(x_1, ..., x_I, y_1, ..., y_J) \in X_1 \times ... \times X_I \times Y_1 \times ... \times Y_J$$

- Notice that by definition consumption must be possible ($x_i \in X_i$), and production must be possible ($y_j \in Y_j$).
- Is any allocation a possible outcome for the economy?
  - No, as the definition of allocation does not take into account the initial resources available;
  - these initial resources constrain what can be achieved.

Feasibility

Definition

An allocation (x, y) is feasible if

$$\sum_{i=1}^{I} x_i \leq \sum_{i=1}^{I} \omega_i + \sum_{j=1}^{J} y_j$$

- Consumption and production are compatible with the aggregate endowment.
- Feasibility has nothing to do with choices by consumers or firms; it just states that a particular allocation could be consumed and produced.

Definition

The set of feasible allocations is denoted by

$$\{(x, y) \in X_1 \times ... \times X_I \times Y_1 \times ... \times Y_J : \sum_{i=1}^{I} x_i \leq \omega + \sum_{j=1}^{J} y_j\} \subset \mathbb{R}^{L(I+J)}$$

Edgeworth Box Economy

- There is no production. There are two goods, denoted 1 and 2, and two consumers, denoted A and B.
- An allocation is given by four numbers
$$x_A = (x_{1\,A}, x_{2\,A}) \text{ and } x_B = (x_{1\,B}, x_{2\,B})$$
- A feasible allocation is described as
$$\left\{(x_A, x_B) \in \mathbb{R}^4 : x_A \in \mathbb{R}^2_+,\ x_B \in \mathbb{R}^2_+, \text{ and } x_A + x_B \leq \omega_A + \omega_B\right\}$$
so a feasible allocation must satisfy
$$x_{1\,A} + x_{1\,B} \leq \omega_{1\,A} + \omega_{1\,B} \text{ and } x_{2\,A} + x_{2\,B} \leq \omega_{2\,A} + \omega_{2\,B}$$
- The inequality is important as we do not want to force all that is available to be consumed. So, to do this rigorously, we think of the economy having a firm that has only one technology:
$$Y_1 = -\mathbb{R}^2_+$$
it can destroy some amounts of the goods.
- We can represent all feasible allocation by means of an Edgeworth Box. Draw it (with all details).

Exchange Economy

- In an exchange economy there is no production; all consumers can do is trade with each other.
  - This is like an Edgeworth Box economy, but there are I consumers and L goods.
- In an exchange economy there is a single firm that has the ability to dispose freely of any amount of any of the goods.
- Formally, $J = 1$, and $Y_j = -\mathbb{R}^L_+$.

Example

In an exchange economy the set of feasible allocations is

$$\{x \in X_1 \times ... \times X_I : \sum_{i=1}^{I} x_i \leq \omega\} \subset \mathbb{R}^{LI}$$

- This is a simple environment in which many results are easier to prove, and that most of the time has the same intuition as an economy with production.

Edgeworth Box

- An Edgeworth Box is an exchange economy with two consumers and two goods.
- Formally $L = 2$, $I = 2$, $X_1 = X_2 = \mathbb{R}^2_+$, $J = 1$, and $Y_j = -\mathbb{R}^2_+$.

Robinson Crusoe Economy

- There are two goods, food F and labor L, one firm, and one consumer (both are Robinson).
- Robinson's endowment is $\omega = (1, 0)$ (one unit of labor and no food).
- Robinson's technology transforms labor into food:

$$Y = \left\{ (y_L, y_F) \in \mathbb{R}^2 : y_L \leq 0, \text{ and } f_F \leq f(-y_L) \right\}$$

  - The function $f(\cdot)$ is a standard production function.
  - This technology has the propery that if $y \in Y$ and $y' \leq y$ then y' is also an element of Y. This is free disposal property.
  - Draw a picture of the production function and the production possibility set.

- A feasible allocation is described as

$$\left\{ (x, y) \in \mathbb{R}^4 : x \in \mathbb{R}^2_+, \, y \in Y, \text{ and } x \leq \omega + y \right\}$$

  - Notice that because of free disposal we may think of the inequality as an equality without loss.
  - Draw a picture of the feasible set.

Feasible Allocations: Conditions

Proposition

If the following conditions are satisfied, the set of feasible allocations is closed and bounded.[a]

1. X_i *is closed and bounded below for each* $i = 1, \ldots, I$.[b]
2. Y_j *is closed for each* $j = 1, \ldots, J$.
3. $Y = \sum_j Y_j$ *is convex and satisfies*
    1. $\mathbf{0}_L \in Y$ *(inaction),*
    2. $Y \cap \mathbf{R}_+^L \subseteq \{\mathbf{0}_L\}$ *(no free-lunch), and*
    3. *if* $y \in Y$ *and* $y \neq \mathbf{0}_L$ *then* $-y \notin Y$ *(irreversibility).*

Moreover, if $-\mathbb{R}_+^L \subset Y_j$ *(free-disposal) and there are* $x_i \in X_i$ *for each* i *such that* $\sum_i x_i \leq \omega$*, then the set of feasible allocations is also non empty.*

[a] Closed and bounded means there exists an $r > 0$ such that for each $l = 1, \ldots, L$, $i = 1, \ldots, I$, and $j = 1, \ldots, J$, we have $|x_{l,i}| < r$ and $|y_{l,j}| < r$ for each $(x, y) \in A$.

[b] Bounded below means there exists an $r > 0$ such that $x_{l,i} > -r$ for each $l = 1, \ldots, L$.

- When these conditions are not satisfied, the latter in particular, we may have nothing to talk about. So we take them for granted.

Pareto Optimality (Efficiency)

- How should we think about efficiency? A minimal requirement is that there is no waste of resources.

Definition

An allocation $(\bar{x}, \bar{y})$ Pareto dominates the allocation (x, y) if

$$\bar{x}_i \succsim_i x_i \text{ for all } i = 1, ..., I \quad \text{and} \quad \bar{x}_i \succ_i x_i \text{ for some } i$$

Definition

A feasible allocation (x, y) is Pareto optimal if there is no other *feasible* allocation (x', y') such that

$$x'_i \succsim_i x_i \text{ for all } i = 1, ..., I \quad \text{and} \quad x'_i \succ_i x_i \text{ for some } i$$

- A feasible allocation is Pareto optimal if no feasible allocation Pareto dominates it. Feasibility is necessary.
- Nothing is left on the table: an allocation is efficient if we cannot find another one that harms no one and benefits someone.
- One only looks at consumers: firms matter via what they produce.

Pareto Optimality: Examples

- Draw a Pareto optimal allocation for an Edgeworth box economy.

- Draw a Pareto optimal allocation for a Robinson Crusoe economy.

Individual Rationality

- Suppose we let the individuals exchange goods with each other.

Observation

- A consumer should not want allocations that do not improve over her initial conditions.
- Therefore, she may not want outcomes that are worse than consuming her initial endowment since she can always refuse to trade.

Definition

A feasible allocation (x, y) is individually rational if $x_i \succsim_i \omega_i \quad$ for all i.

- An individually rational allocation represents a trade that improves everyone position relative to their initial endowment.
- Pareto optimal allocations are not necessarily individually rational.
- Draw a picture of Pareto optimal allocation that are not individually rational.

Next Class

- Core
- Social Welfare Functions