The Core
Pareto Optimality and Social Welfare Maximizationty

Econ 2100 Fall 2018

Lecture 14, October 17

Outline

1. The Core
2. Core, indiviual rationality, and Pareto efficiency
3. Social welfare function
4. Pareto optimality and social welfare maximization

From Last Class

- Firm $j = 1, ..., J$ are described by a production set, $Y_j \subset \mathbb{R}^L$.
- Consumers $i = 1, ..., I$ are described by a consumption set $X_i \subset \mathbb{R}^L_+$, preferences $\succsim_i$, endowments $\omega_i \in X_i$, and shares $\theta_i \in [0,1]^J$ of the profits of each firm (with $1 = \sum_i \theta_{ij}$).
  - For a while we will not care about firm ownership.
- An allocation: $(x, y) \in \mathbb{R}^{L(I+J)}$ where $x_i \in X_i$ for each $i = 1, ..., I$, and $y_j \in Y_j$ for each $j = 1, ..., J$.
- An allocation (x, y) is feasible if $\sum_{i=1}^{I} x_i \leq \sum_{i=1}^{I} \omega_i + \sum_{j=1}^{J} y_j$
- An allocation (x, y) is individually rational if $x_i \succsim_i \omega_i$ for all i.

Definition

A feasible allocation (x, y) is Pareto optimal (efficient) if there is no other *feasible* allocation (x', y') such that

$$x'_i \succsim_i x_i \text{ for all } i = 1, ..., I \quad \text{and} \quad x'_i \succ_i x_i \text{ for some } i$$

Coalitions and Blocking

- We can think of a different notion of 'making everyone happy' which takes into account individual rationality as well as some notion of individuals making their own choices.
- This is easier to define for an exchange economy: let $J = 1$, and $Y_J = -\mathbb{R}^L_+$.

Definition

A coalition is a subset of $\{1, ..., I\}$.

Definition

A coalition $S \subset \{1, ..., I\}$ blocks the allocation x if for each $i \in S$ there exist $x'_i \in X_i$ such that

$$x'_i \succ_i x_i \text{ for all } i \in S \qquad \text{and} \qquad \sum_{i \in S} x'_i \leq \sum_{i \in S} \omega_i$$

- A blocking coalition can make all its members better off.
  - One can think of a weaker definition where a coalition benefits at least one of its members strictly without hurting the others.

The Core

- Next, we define the idea that no group of consumers can gain by 'seceding' from the economy.

Definition

A feasible allocation x is in the core of an economy if there is no coalition that blocks it.

- The idea is that no sub-group of consumers can improve their situation by separating from the economy.
- One could define these concepts with firms, but things get complicated in defining what is feasible for a particular coalition as one needs to allocate production.

Pareto Optimality, Individual Rationality, and the Core

Easy to Prove Results

- Any allocation in the core of an economy is also Pareto optimal.
  - Obvious since the 'whole' (sometimes called 'grand coalition' $S = \{1, ..., I\}$) is not a blocking coalition.
- Not all Pareto optimal allocations are in the core.
  - Slightly less obvious: $x_i = \omega$ (consumer i gets everything) is Pareto optimal but not in the core.
    - Any assumptions requred here?
- In an Edgeworth box, the core is the set of all individually rational Pareto optimal allocations.
  - This is an (easy) homework problem.
  - With more consumers this result does not hold.
    - As the number of consumers grows, there are more possible coalitions, and more allocations will be blocked. So the core is typically much smaller than the set of Pareto optimal allocations that are individually rational.

Characterization of Pareto Optimal Allocations

- Definitions are well and good, but how do we know Pareto optimal allocations exist?
- Furthermore, how do we find Pareto optimal allocations?
- To answer these questions, we restrict attention to the case of continuous, complete, and transitive preference relations, so that one can work with individuals' utility functions.
- Existence turns out to be an easy problem.
- Characterization goes through a particular maximization problem: an allocation is Pareto optimal if and only if it maximizes a particular "welfare function" that econmpasses everyone's utility.

Definitions With Utility Functions

- Suppose individuals' preferences are represented by a utility function.
- Consumer i's utility function is denoted $u_i(x_i)$.
- We can rewrite Pareto efficiency and individual rationality as follows.

Definitions with utility functions

- A feasible allocation (x, y) is Pareto optimal if there is no other feasible allocation (x', y') such that

$$u_i(x_i') \geq u_i(x_i) \text{ for all } i \qquad \text{and} \qquad u_i(x_i') > u_i(x_i) \text{ for some } i.$$

- A feasible allocation (x, y) is individually rational if

$$u_i(x_i) \geq u_i(\omega_i) \qquad \text{for all } i.$$

Do Pareto Optimal Allocations Exist?

- Pareto optimal allocations exist if the set of feasible allocations is well behaved.

Theorem

Any economy such that the set of feasible allocations is non-empty, closed, and compact, and each $\succsim_i$ is complete, transitive, and continuous, has a Pareto efficient allocation.

Proof.

- Let $\mathbb{A}$ be the set of feasible allocations; let $U : \mathbb{A} \to \mathbb{R}$ be defined by

$$U(x, y) = \sum_{i=1}^{I} u_i(x_i)$$

  - This is well defined because, by Debreu's theorem, each $\succsim_i$ can be represented by a continuous function u_i.
  - U is the sum of continuous functions, thus it is also continuous.
- Since $\mathbb{A}$ is compact and nonempty, the Extreme Value Theorem implies that there is a feasible allocation that maximizes U.
- This allocation must be Pareto optimal because if another feasible allocation Pareto dominates it, that allocation must give a larger value of U, implying that someone reaches a higher utility value for that allocation. □

Pareto Efficiency and Utility Possibility Set

Definitions

The utility possibility set is

$$\mathbb{U} = \left\{ (v_1, ..., v_I) \in \mathbb{R}^I : \begin{array}{c} \text{there exists a feasible } (x, y) \\ \text{such that } v_i \leq u(x_i) \text{ for } i = 1, ..., I \end{array} \right\}$$

The utility possibility frontier is

$$\mathbb{UF} = \{(\bar{v}_1, ..., \bar{v}_I) \in \mathbb{U} : \text{there is no } v \in \mathbb{U} \text{ such that } v > \bar{v}\}$$

- Draw a picture for an Edgeworth box economy.
- The utility possibility frontier is the boundary of $\mathbb{U}$, and a Pareto optimal allocation must belong to the frontier.
- In the picture, a point on the frontier can be characterized as the solution to the following optimization problem

$$\max_{(x,y) \text{ is a feasible allocation}} u_i(x_i) \text{ such that } u_j(x_j) \geq v_j \text{ for } j \neq i$$

  If an allocation is Pareto efficient then it maximizes the utility of one consumer subject to the constraint that all others get some fixed (feasible) amount.

Social Welfare and Planner's Problem

Definitions

A (linear) social welfare function is a weighted sum of the individuals' utilities:

$$W = \sum_{i=1}^{I} \lambda_i u_i = \lambda \cdot v \qquad \text{with } \lambda_i \geq 0$$

The social welfare maximization problem is

$$\max_{v \in \mathbb{U}} \sum_i \lambda_i v_i$$

- This maximization problem is sometimes called the "planner's problem".
- Later in the course, we will encounter another planner's problem: maximize the utility of one consumer, subject to all other consumers getting a predifined utility value.

Pareto Efficiency and Social Welfare

- One can use the planner's problem to find Pareto optimal allocations.

Theorem

If the allocation $(\hat{x}, \hat{y})$ is feasible for the economy $\mathcal{E} = \left\{ \{u_i, \omega_i\}_{i=1}^{I}, \{Y_j\}_{j=1}^{J} \right\}$ and solves the problem

$$\max_{(x,y) \text{ is a feasible allocation}} \sum_{i=1}^{I} \lambda_i u_i(x_i) \qquad \text{where } \lambda_i > 0 \text{ for all } i$$

then $(\hat{x}, \hat{y})$ is Pareto optimal.

Proof.

By contradiction. Suppose $(\hat{x}, \hat{y})$ is not Pareto optimal. Then, it is Pareto dominated by some feasible allocation (x, y).

- Since the λ_i are all strictly positive, we must have

$$\sum_{i=1}^{I} \lambda_i u_i(x_i) > \sum_{i=1}^{I} \lambda_i u_i(\hat{x}_i)$$

- But this means $(\hat{x}, \hat{y})$ does not maximize $\sum_{i=1}^{I} \lambda_i u_i(x_i)$ which is a contradiction. □

Pareto Efficiency and Social Welfare

Theorem

If the allocation $(\hat{x}, \hat{y})$ *is feasible for the economy* $\mathcal{E} = \left\{ \{u_i, \omega_i\}_{i=1}^{I}, \{Y_j\}_{j=1}^{J} \right\}$ *and solves the problem*

$$\max_{(x,y) \text{ is a feasible allocation}} \sum_{i=1}^{I} \lambda_i u_i(x_i) \qquad \text{where } \lambda_i > 0 \text{ for all } i$$

then $(\hat{x}, \hat{y})$ *is Pareto optimal.*

- Notice that if a feasible allocation maximizes $\sum_{i=1}^{I} \lambda_i u_i(x_i)$ it also maximizes $K \sum_{i=1}^{I} \lambda_i u_i(x_i)$ where K is a strictly positive number.
- So, without loss of generality we can consider the social welfare function

$$\sum_{i=1}^{I} \frac{\lambda_i}{\sum_{i=1}^{I} \lambda_i} u_i(x_i) = \sum_{i=1}^{I} \hat{\lambda}_i u_i(x_i) \qquad \text{where } \hat{\lambda}_i \geq 0 \text{ for all } i \text{ and } \sum_{i=1}^{I} \hat{\lambda}_i = 1$$

- In other words, a Pareto optimal allocation maximizes a weighted sum of individual utilities.

Planner's Problem: Examples

- Draw the utility possibility set and solve the planner's problem in an Edgeworth box economy.
  - Note that the objective function is linear.

- Draw the utility possibility set and solve the planner's problem in a Robinson Crusoe economy.
  - The objective function is...

Pareto Efficiency and Constrained Planner Problem

Theorem

If the allocation $(\hat{x}, \hat{y})$ *is feasible for the economy* $\mathcal{E} = \left\{ \{u_i, \omega_i\}_{i=1}^{I}, \{Y_j\}_{j=1}^{J} \right\}$ *and solves the problem*

$$\max_{(x,y) \text{ is a feasible allocation}} \sum_{i=1}^{I} \lambda_i u_i(x_i) \qquad \textit{where } \lambda_i > 0 \textit{ for all } i$$

then $(\hat{x}, \hat{y})$ *is Pareto optimal.*

- In order for this to be a full characterization result, one would like to prove a converse: if an allocation is Pareto optimal, then it must solve the planner's problem.

- This works if given a Pareto optimal allocation one can always find some weights that make it a solution to the planner's problem.
- That kind of result needs more assumptions, and even with those, it cannot guarantee that everyone gets a non-zero weight in the social welfare function.

Social Welfare Maximization and Pareto Efficiency

- Consider the linear social welfare function $W(x, y) = \sum_{i=1}^{I} \lambda_i u_i(x_i)$ where $\lambda_i \geq 0$ for all i and (x, y) is an allocation.

- One can think of $W(x, y)$ as the composition of two functions:
  - $U : \mathbb{A} \to \mathbb{R}^I$, defined as $U(x, y) = (u_1(x_1), ..., u_I(x_I))$, where $\mathbb{A}$ is the set of allocations, and
  - $f : \mathbb{R}^I \to \mathbb{R}$, defined as $f(v) = \sum_{i=1}^{I} \lambda_i v_i$.
    - Using the dot product notation, $f(v) = \lambda \cdot v$.
  - Then, $W(x, y) = f(U(x, y))$.
- The image of the set of feasible allocations under the mapping U is given by:

$$\mathbb{V} = \left\{ U(x, y) \in \mathbb{R}^I : (x, y) \text{ is a feasible allocation} \right\}$$

Remark

- $\mathbb{U}$ and $\mathbb{V}$ are not the same set:
  The utility possibility set $\mathbb{U}$ is equal to $\mathbb{V}$ plus all the points dominated by points in $\mathbb{V}$
- $\mathbb{U}$ is a larger set.

Social Welfare Maximization and Pareto Efficiency

Proposition

The allocation $(\hat{x}, \hat{y})$ *solves the problem*

$$\max_{(x,y) \text{ is a feasible allocation}} W(x, y)$$

if and only if the vector $\hat{v} = U(\hat{x}, \hat{y})$ *solves the problem* $\max_{v \in \mathbb{V}} \lambda \cdot v$.

Proof.

Question 4, Problem set 7. □

- Given this poposition, any Pareto optimal allocation maximizes $\lambda \cdot v$ over the set $\mathbb{V}$.

- Under what conditions can we say that for any Pareto optimal allocation there exists a vector λ such that $\hat{v}$ maximizes the desired dot product in that set.
- The tool that yields this result is (a version of) the separating hyperplane theorem.
- But to get there we need a math refresher.

Next Week

- Separating Hyperplane Theorem.
- Any Pareto optimal allocation maximizes the social welfare function.
- Markets and Competitive Equilibrium.