Pareto Optimality and Planner's Problem

Econ 2100 Fall 2018

Lecture 15, October 22

Outline

1. Hyperplanes
2. Minkowski's Separating Hyperplane Theorem
3. Pareto optimality and social welfare maximization

From Last Class

- The utility possibility set is

$$\mathbb{U} = \left\{ (v_1, ..., v_I) \in \mathbb{R}^I : \begin{array}{c} \text{there exists a feasible } (x, y) \\ \text{such that } v_i \leq u(x_i) \text{ for } i = 1, ..., I \end{array} \right\}$$

  The utility possibility frontier is

$$\mathbb{UF} = \{(\bar{v}_1, ..., \bar{v}_I) \in \mathbb{U} : \text{there is no } v \in \mathbb{U} \text{ such that } v > \bar{v}\}$$

- A (linear) social welfare function is a weighted sum of the individuals' utilities:

$$\sum_{i=1}^{I} \lambda_i v_i = \lambda \cdot v \qquad \text{with } \lambda_i \geq 0$$

Theorem

If the allocation $(\hat{x}, \hat{y})$ is feasible for the economy $\mathcal{E} = \left\{ \{u_i, \omega_i\}_{i=1}^{I}, \{Y_j\}_{j=1}^{J} \right\}$ and solves the problem

$$\max_{(x,y) \text{ is a feasible allocation}} \sum_{i=1}^{I} \lambda_i u_i(x_i) \qquad \text{where } \lambda_i > 0 \text{ for all } i$$

then $(\hat{x}, \hat{y})$ is Pareto optimal.

- We are looking for a converse: any Pareto optimal allocation maximizes some social welfare function.
  - This means, find the λs that work.

From Last Class

- Consider the function $W(x,y) = \sum_{i=1}^{I} \lambda_i u_i(x_i)$ where $\lambda_i \geq 0$ for all i and (x,y) is an allocation.
  - Notice this is defined over allocations, not utility vectors.
- Think of it as the composition of
$$U(x,y) = (u_1(x_1), ..., u_I(x_I)) \text{ and } f(v) = \lambda \cdot v$$
where $U : \mathbb{A} \to \mathbb{R}^I$ ($\mathbb{A}$ is the set of allocations) and $f : \mathbb{R}^I \to \mathbb{R}$.
- The image of the set of feasible allocations under U is:
$$\mathbb{V} = \left\{ U(x,y) \in \mathbb{R}^I : (x,y) \text{ is a feasible allocation} \right\}$$
  - $\mathbb{V}$ is not the utility possibility set (it is smaller).
- The allocation $(\hat{x}, \hat{y})$ solves the problem
$$\max_{(x,y) \text{ is a feasible allocation}} W(x,y)$$
if and only if the vector $\hat{v} = U(\hat{x}, \hat{y})$ solves the problem $\max_{v \in \mathbb{V}} \lambda \cdot v$.
  - Any Pareto optimal allocation maximizes $\lambda \cdot v$ over the set $\mathbb{V}$.
- We want to think about the problem $\max_{v \in \mathbb{V}} \lambda \cdot v$ geometrically.

Hyperplanes

Definition

Given $r \in \mathbb{R}$, an hyperplane is defined as $\left\{x \in \mathbb{R}^N : p \cdot x = r\right\}$.

- An hyperplane is orthogonal to p in the sense that:
$$\left\{x \in \mathbb{R}^N : p \cdot x = r\right\} = \left\{x \in \mathbb{R}^N : p \cdot x = 0\right\} + \hat{x}$$
where $\hat{x}$ is any vector such that $p \cdot \hat{x} = r$
- For any $t \in \mathbb{R}$, the vectors tp form a line through the origin in direction p.
- The hyperplane is the boundary between two halves of $\mathbb{R}^N$: one in which $p \cdot x < r$ and the other in which $p \cdot x > r$.

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Hyperplanes and Maximization

- Suppose that $B \subset \mathbb{R}^N$ and that x^* solves $\max_{x \in B} p \cdot x$.
- The hyperplane
$$\left\{x \in \mathbb{R}^N : p \cdot x = p \cdot x^*\right\}$$
is 'tangent' to B at the point x^*, and this is the hyperplane that is furthest in the direction from 0 to p.
- Draw a picture (highlight the half spaces below and above the tangent hyperplane).
- Think about pictures where the tangent is not unique.

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Hyperplanes and Pareto Optimality

- Given this geomety, the content of the last proposition is that if the function $\lambda \cdot v$ is maximized over $\mathbb{V}$ at $\hat{v}$ then the hyperplane
$$\left\{v \in \mathbb{R}^I : \lambda \cdot v = \lambda \cdot \hat{v}\right\}$$
is tangent to $\mathbb{V}$ at $\hat{v}$.

- This hyperplane gives the λs we need to assert that for any Pareto optimal allocation there exists a vector of individual weights such that this allocation maximizes $W(x, y)$.
- To use these ideas, one needs to know what the set $\mathbb{V}$ looks like.
- The utility possibility set $\mathbb{U}$ is equal to the set $\mathbb{V}$ plus all the points dominated by points in $\mathbb{V}$.
- Therefore, if $(\hat{x}, \hat{y})$ maximizes $W(x, y)$ then $\mathbb{U}$ is tangent to the hyperplane $\left\{v \in \mathbb{R}^I : \lambda \cdot v = \lambda \cdot \hat{v}\right\}$ at $\hat{v}$.

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Hyperplanes and Pareto Optimality: An Example

- Draw a picture for an Edgeworth box economy.
- $\hat{v} = (u_A(\hat{x}_A), u_B(\hat{x}_B))$, and the Pareto optimal allocation $\hat{x}$ maximizes the social welfare function

$$\lambda_A u_A(x_A) + \lambda_B u_B(x_B)$$

- The vector $\hat{v}$ is a solution to

$$\max_{v \in \mathbb{U}} \lambda_A v_A + \lambda_B v_B$$

- We want the converse: if $(\tilde{x}, \tilde{y})$ is Pareto optimal then there exists some vector $\tilde{\lambda}$ such that the hyperplane $\tilde{\lambda} \cdot \tilde{v}$ is tangent to $\mathbb{U}$ at $\tilde{v}$.
- This clearly fails if $\mathbb{U}$ is not convex. Draw an example.

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Separating Hyperplane Theorem

Definition

Given two sets $X, Y \subset \mathbb{R}^N$, a vector $p \in \mathbb{R}^N$ is said to separate these sets if $p \neq 0$ and $p \cdot x \leq p \cdot y$ for all $x \in X$ and $y \in Y$.

Draw a picture

- If the vector p separates X from Y , then a straight line H perpendicular to p lies between X and Y.
- If the vector p separates X from Y, then there is an hyperplane $\{x \in \mathbb{R}^N : p \cdot x = r\}$ that comes between X and Y , though it may touch one or both sets on their boundaries.

Theorem (Separating Hyperplane (Minkowski))

Let $X, Y \subset \mathbb{R}^N$ be convex sets; suppose that $intX$ is not empty, and that $Y \cap intX = \emptyset$. Then, there exists a $p \neq 0$ that separates X from Y.

- Draw counterexamples: convexity, empty interior (and show empty interior may work).

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Pareto Efficient Allocations and Planner's Problem

Theorem

Assume the economy $\mathcal{E} = \left\{ \{u_i, \omega_i\}_{i=1}^{I}, \{Y_j\}_{j=1}^{J} \right\}$ *is such that*

1. *for all* $j = 1, ..., J$ $\quad Y_j$ *is a convex set*
2. *for all* $i = 1, ..., I$ $\quad u_i : \mathbb{R}^N \to \mathbb{R}$ *is concave*

If $(\hat{x}, \hat{y})$ *is a Pareto optimal allocation, then there exists* $\lambda \in \mathbb{R}_+^I$ *with* $\lambda \neq 0$ *such that* $(\hat{x}, \hat{y})$ *solves*

$$\max_{(x,y) \text{ is a feasible allocation}} \sum_{i=1}^{I} \lambda_i u_i(x_i)$$

- For any Pareto efficient allocation, there is a vector of weights that makes that allocation the solution to the planner's problem.
- Notice that in this theorem some λs can be zero, so this is not quite a converse of the theorem from last class.

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Geometric idea of proof: start from a Pareto optimal allocation, and show that there is an hyperplane through this allocation that separates the utility possibility set from the set of utility vectors that improve on that allocation.

Proof Part I

Let $\mathbb{U}$ be the utility possibility set for $\mathcal{E}$ and given a Pareto optimal allocation $(\hat{x}, \hat{y})$ let Γ be defined as

$$\Gamma = \left\{ v \in \mathbb{R}^I : v \geq \hat{v} \right\} \quad \text{where } \hat{v} = u(\hat{x}, \hat{y})$$

- Clearly, $\hat{v}$ belongs to $\mathbb{U}$ and Γ.
- Claim: no other point belongs to $\mathbb{U}$ and Γ.

- Draw a picture.

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        - $w \in \mathbb{U} \Longrightarrow \exists$ a feasible (x, y) such that $u_i(x_i) \geq w_i \; \forall i$
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    - Therefore: $u_i(x_i) \geq u_i(\hat{x}_i)$ for all i and $u_i(x_i) > u_i(\hat{x}_i)$ for some i
    - This contradicts Pareto optimality of $(\hat{x}, \hat{y})$

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- Consider, for simplicity, an exchange economy.

Proposition

In an exchange economy where each $i = 1, ..., I$, $\succsim_i$ can be represented by a strictly increasing and continuous utility function $u_i(x_i)$, an allocation $\hat{x}$ is Pareto optimal if and only if it solves the following

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- Notice that this does not say for any $\{\bar{u}_i\}_{i \neq j}$ (easy to find a counterexample).
- Proof is an homework problem.

Pareto Efficiency: An Example

Finding Pareto Optimal Allocations: An Example

- Consider an economy: where

consumer 1	consumer 2
$\omega_1 = (\omega_{11}, \omega_{22})$	$\omega_2 = (\omega_{12}, \omega_{22})$
$u_1(x_1) = u_1(x_{11}, x_{21})$	$u_2(x_2) = u_2(x_{12}, x_{22})$

- To find the set of Pareto optimal allocations for an Edgeworth box economy in which consumers' utility functions are

$$u_1(x_{11}, x_{21}) = (x_{11})^{\alpha}(x_{21})^{1-\alpha} \quad \text{and} \quad u_2(x_{12}, x_{22}) = (x_{12})^{\beta}(x_{22})^{1-\beta}$$

- We must solve the following planner's problem:

$$\max_{x_{11}, x_{21}, x_{12}, x_{22}} (x_{11})^{\alpha}(x_{21})^{1-\alpha} \quad \text{subject to} \quad \begin{array}{c} (x_{12})^{\beta}(x_{22})^{1-\beta} \geq \bar{u} \\ x_{11} + x_{12} = \omega_{11} + \omega_{12} \\ x_{21} + x_{22} = \omega_{21} + \omega_{22} \\ x_{11}, x_{21}, x_{12}, x_{22} \geq 0 \end{array}$$

for some $\bar{u}$.

- In this case, the utility functions are differentiable, so we can write the Lagrangean, the first order conditions, and then solve.

Pareto Efficiency: An Example

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Pareto Efficiency: An Example

Finding Pareto Optimal Allocations: An Example

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Next Class

- Competitive equilibrium