First Welfare Theorem

Econ 2100 Fall 2018

Lecture 17, October 29

Outline

1. First Welfare Theorem
2. Preliminaries to Second Welfare Theorem

Past Definitions

- A feasible allocation $(\hat{x}, \hat{y})$ is Pareto optimal if there is no other feasible allocation (x, y) such that

$$x_i \succsim_i \hat{x}_i \qquad \text{for all } i \qquad \text{and} \qquad x_i \succ_i \hat{x}_i \qquad \text{for some } i.$$

- An allocation (x^*, y^*) and a price vector $p^* \in \mathbb{R}_+^L$ form a competitive equilibrium if

1. for each $j = 1, ..., J$: $\quad p^* \cdot y_j \leq p^* \cdot y_j^* \quad$ for all $y_j \in Y_j$;
2. for each $i = 1, ..., I$:
   $x_i^* \succsim_i x_i \quad$ for all $x_i \in \{x_i \in X_i : p^* \cdot x_i \leq p^* \cdot \omega_i + \sum_j \theta_{ij} \left(p^* \cdot y_j^*\right)\}$; and
3. $\sum_i x_i^* \leq \sum_i \omega_i + \sum_j y_j^*$ and if $\sum_{i=1}^I x_{li}^* < \sum_{i=1}^I \omega_{li} + \sum_{j=1}^J y_{lj}^*$ then $p_l^* = 0$.

What is the relationship between competitive equilibrium and Pareto efficiency?

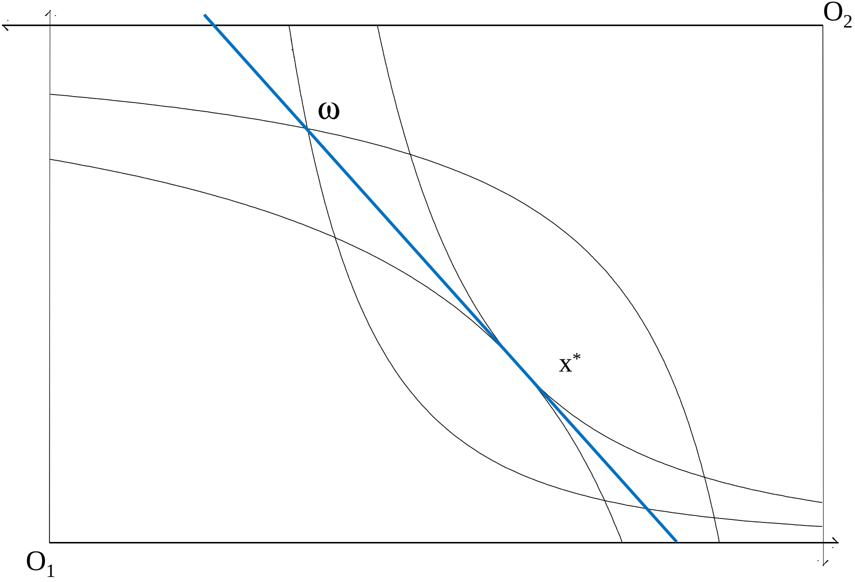
- Is any competitive equilibirum Pareto efficient? First Welfare Theorem.
  - This is about excluding something can Pareto dominate the equilibrium allocation.
- Is any Pareto efficient allocation (part of) a competitive equilibrium? Second Welfare Theorem.
  - This is about finding prices that make the efficient allocation an equilibirum.

First Welfare Theorem: Examples

Things seem easy

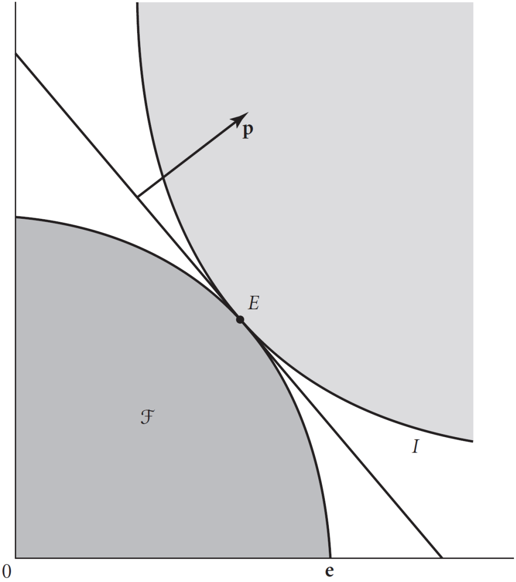
First Welfare Theorem: Edgeworth Box Example

Things seem easy



First Welfare Theorem: Robinson Crusoe Example

Things seem easy



First Welfare Theorem: Counterexample

An Edgeworth Box Economy

- Consider a two-person, two-good exchange economy.
- a's utility function $U_a(x_{1a}, x_{2a}) = 7$;
  b's utility function $U_b(x_{1b}, x_{2b}) = x_{1b}x_{2b}$.
- The initial endowments are $\omega_a = (2, 0)$ and $\omega_b = (0, 2)$.
- CLAIM: $x_a^* = (1, 1)$, $x_b^* = (1, 1)$, and prices $p^* = (1, 1)$ form a competitive equilibirum.
  - a's utility is maximized.
  - b's utility when her income equals 2 is maximized (this is a Cobb-Douglas utility function with equal exponents, so spending half her income on each good is optimal).
  - $x_a^* + x_b^* = (2, 2) = \omega_a + \omega_b$.
- Is this allocation Pareto optimal? No:
  - $\hat{x}_a = (0, 0)$ and $\hat{x}_b = (2, 2)$ Pareto dominates x_a^*, x_b^* since consumer a has the same utility while consumer b's utility is higher.

- How do we rule examples like this out?
- Need consumers preferences to be locally non satiated (there is always something nearby that makes the consumer better off).

Local Non Satiation

Definition

A preference ordering $\succsim_i$ on X_i is satiated at y if there exists no x in X_i such that $x \succ_i y$.

Definition

The preference relation $\succsim_i$ on X_i is locally non-satiated if for every x in X_i and for every $\varepsilon > 0$ there exists an x' in X_i such that $\|x' - x\| < \varepsilon$ and $x' \succ_i x$.

- Remember: $\|y - z\| = \sqrt{\sum_{l=1}^{L} (y_l - z_l)^2}$ is the Euclidean distance between two points.

Remark

- If $\succsim_i$ is continuous and locally non-satiated there exist a locally non-satiated utility function; then, any closed consumption set must be unbounded (or there would be a global satiation point).

Local Non Satiation and Walrasian Demand

Lemma

Suppose $\succsim_i$ is locally non-satiated, and let x_i^* be defined as:

$$x_i^* \succsim_i x_i \qquad \text{for all } x_i \in \{x_i \in X_i : p \cdot x_i \leq w_i\}.$$

Then

$$x_i \succsim_i x_i^* \qquad \text{implies} \qquad p \cdot x_i \geq w_i$$

and

$$x_i \succ_i x_i^* \qquad \text{implies} \qquad p \cdot x_i > w_i$$

- If a consumption vector is weakly preferred to a maximal consumption bundle (i.e. an element of the Walrasian demand correspondence), it cannot cost strictly less.
- If a consumption vector is strictly preferred to a maximal bundle, it must not be affordable
  - If not the consumer would have chosen it and been better-off.
- Make sure you prove this as an exercise (easy, but still worth doing).

First Welfare Theorem

Theorem (First Fundamental Theorem of Welfare Economics)

Suppose each consumer's preferences are locally non-satiated. Then, any allocation x^*, y^* that with prices p^* forms a competitive equilibrium is Pareto optimal.

- The invisible hand is Pareto efficient.
- This is true under pretty mild conditions on each preference relation.
- Local non-satiation has bite: there is always a more desirable commodity bundle nearby.
- There is another assumption implicit in our framework: lack of externalities (more later).

Proof of the First Welfare Theorem (by contradiction)

Suppose not: there exists a feasible allocation x, y *such such that* $x_i \succsim_i x_i^*$ *for all* i*, and* $x_i \succ_i x_i^*$ *for some* i*.*

- *By local non satiation,* $x_i \succsim_i x_i^*$ *implies* $p^* \cdot x_i \geq p^* \cdot \omega_i + \sum_j \theta_{ij}(p^* \cdot y_j^*)$
  $x_i \succ_i x_i^*$ *implies* $p^* \cdot x_i > p^* \cdot \omega_i + \sum_j \theta_{ij}(p^* \cdot y_j^*)$
- *Therefore, summing over consumers*
$$\sum_{i=1}^{I} p^* \cdot x_i > \sum_{i=1}^{I} p^* \cdot \omega_i + \sum_{i=1}^{I}\sum_{j=1}^{J} \theta_{ij}(p^* \cdot y_j^*) \overset{accounting}{=} p^* \cdot \left(\sum_{i=1}^{I} \omega_i\right) + \sum_{j=1}^{J} p^* \cdot y_j^*$$
- *Since each* y_j^* *maximizes profits at prices* p^**, we also have*
$$\sum_{j=1}^{J} p^* \cdot y_j^* \geq \sum_{j=1}^{J} p^* \cdot y_j.$$
- *Substituting this into the previous inequality:*
$$\sum_{i=1}^{I} p^* \cdot x_i > p^* \cdot \left(\sum_{i=1}^{I} \omega_i\right) + \sum_{j=1}^{J} p^* \cdot y_j^* \geq p^* \cdot \left(\sum_{i=1}^{I} \omega_i\right) + \sum_{j=1}^{J} p^* \cdot y_j$$
- *But then* x, y *cannot be feasible since if it were we would have*
$$\sum_{i=1}^{I} x_i \leq \sum_{i=1}^{I} \omega_i + \sum_{j=1}^{J} y_j \qquad \Rightarrow \qquad \sum_{i=1}^{I} p^* \cdot x_i \leq p^* \cdot \left(\sum_{i=1}^{I} \omega_i\right) + \sum_{j=1}^{J} p^* \cdot y_j$$
  which contradicts the expression above.

First Welfare Theorem

Theorem (First Fundamental Theorem of Welfare Economics)

Suppose each consumer's preferences are locally non-satiated. Then, any allocation x^*, y^* that with prices p^* forms a competitive equilibrium is Pareto optimal.

- The theorem says that as far as Pareto optimality goes the social planner cannot improve welfare upon a competitive equilibrium.

Conjecture

The theorem needs only a seemingly weak assumption to obtain a pretty strong conclusion.

- On the other hand, the important assumption of absence of externalities is implicit in the way we set up the model.
- An externality is present when preferences or profit depend on more than one's choices.

Externalities: An Example

An Edgeworth Box Economy

- Consider a two-person, two-good exchange economy.
- $u_A(x_{1A}, x_{2A}, x_{1B}) = x_{1A}x_{2A} - x_{1B}$; $u_B(x_{1B}, x_{2B}) = x_{1B}x_{2B}$.
  - A suffers from B's consumption of the first good.

- CLAIM: $x_A^* = (1,1)$, $x_B^* = (1,1)$, and $p^* = (1,1)$ is a competitive equilibirum.
  - A cannot choose x_{1B} so that is a constant in her utility function. Then A's utility is maximized by x_A^* at prices p^* since this is a Cobb-Douglas utility function with equal exponents and spending half your income on each good is optimal.
  - B's utility when her income equals 2 is maximized (this is a Cobb-Douglas utility function with equal exponents, so spending half her income on each good is optimal).
  - $x_a^* + x_b^* = (2,2) = \omega_A + \omega_B$.
- Is this allocation Pareto optimal? No: $(\hat{x}_A, \hat{x}_B) = \left((\frac{5}{4}, \frac{2}{3}), (\frac{3}{4}, \frac{4}{3})\right)$ is a feasible Pareto improvement:

$$\begin{aligned} U_A\left(\hat{x}_A, \hat{x}_{1B}\right) &= \frac{5}{4}\frac{2}{3} - \frac{3}{4} = \frac{1}{12} > U_A\left(x_A^*, x_{1B}^*\right) = 1 - 1 = 0 \\ U_B(\hat{x}_B) &= \frac{3}{4}\frac{4}{3} = 1 = U_B(x_B^*) \end{aligned}$$

First Welfare Theorem: Externalities

- In the previous example, the first welfare theorem fails because A's utility depends on B's consumption.
- This is called a (negative) externality: the more B consumes of the good, the worse-off A becomes.
- Among the assumptions implicit in our definition of preferences, one is important for the first welfare theorem: there are no externalities in consumption.
- There can be also externalities in production.
- Also, externalities can also be positive.

Competitive Equilibrium and the Core

Theorem

Any competitive equilibrium is in the core.

Proof.

Homework. This is very very similar to the proof of the First Welfare Theorem. □

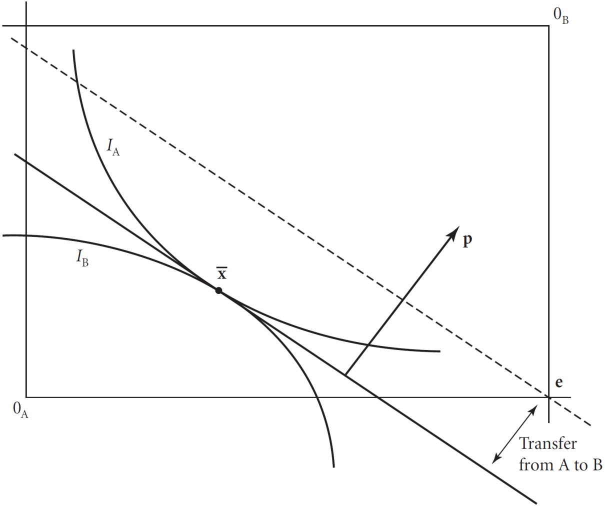
- A 'converse' can be established in cases in which the economy is "large", that is, it contains many individuals.
  - That is called the core convergence theorem and I do not think we will have time for it.

Second Welfare Theorem: Preliminaries

- This is a converse to the First Welfare Theorem.
- The statement goes something like this: under some conditions, any Pareto optimal allocation is part of a competitive equilibrium.
  - Next, we try to understand what these conditions must be. We state and prove the theorem next class.
- In order to prove a Pareto optimal allocation is part of an equilibrium one needs to find the price vector that 'works' for that allocation, since an equilibrium must specify and allocation **and** prices.
- First, we see a simple sense in which this cannot work: Pareto optimality disregards the budget constraints.
  - This is fixed by adjusting the definition of equilibrium.
- Then we see two counterexamples that stress the need for convexities.
  - These are fixed by assuming production sets and better-than sets are convex.
- Finally, we see an example showing that boundary issues can pose problems.
  - This is fixed by, again, adjusting the definition of equilibrium.

Second Welfare Theorem: Need Transfers

Picture



- There is only one candidate price vector (the one tangent to both indifference curves).
- However, this cannot work (B is not rich enough).
- One can fix this by making B richer, giving her some money.

Equilibrium With Transfers

Definition

Given an economy $\left(\{X_i, \succsim_i, \omega_i\}_{i=1}^{I}, \{Y_j\}_{j=1}^{J}\right)$, an allocation x^*, y^* and a price vector p^* constitute a price equilibrium with transfers if there exists a vector of wealth levels

$$w = (w_1, w_2, \ldots, w_I) \qquad \text{with} \qquad \sum_{i=1}^{I} w_i = p^* \cdot \sum_{i=1}^{I} \omega_i + \sum_{j=1}^{J} p^* \cdot y_j^*$$

such that:

1. For each $j = 1, \ldots, J$: $\quad p^* \cdot y_j \leq p^* \cdot y_j^* \quad$ for all $y_j \in Y_j$.
2. For each $i = 1, \ldots, I$: $\quad x_i^* \succsim_i x_i \quad$ for all $x_i \in \{x_i \in X_i : p^* \cdot x_i \leq w_i\}$
3. $\sum_{i=1}^{I} x_i^* \leq \sum_{i=1}^{I} \omega_i + \sum_{j=1}^{J} y_j^*$, with $p_l = 0$ if the inequlity is strict

- Aggregate wealth is divided among consumers so that the budget constrants are satisfied.
- How is the wealth of each consumer effected? They get a positive or negative transfer relative to the value of their endowment at the equilibrium prices.
- A competitive equilibrium satisfies this: set $w_i = p^* \cdot \omega_i + \sum_{j=1}^{J} \theta_{ij}(p^* \cdot y_j^*)$.

Equilibrium With Transfers

Remark

The income transfers (across consumers) that achieve the budget levels in the previous definition are:

$$T_i = w_i - \left[p^* \cdot \omega_i + \sum_{j=1}^{J} \theta_{ij} \left(p^* \cdot y_j^* \right) \right]$$

- Summing over consumers, we get

$$\begin{aligned} \sum_i T_i &= \sum_{i=1}^{I} w_i - \left[\sum_i p^* \cdot \omega_i + \sum_{i=1}^{I} \sum_{j=1}^{J} \theta_{ij} \left(p^* \cdot y_j^* \right) \right] \\ &= \sum_{i=1}^{I} w_i - \left[p^* \cdot \omega + \sum_{j=1}^{J} p^* \cdot y_j^* \right] \\ &= 0 \end{aligned}$$

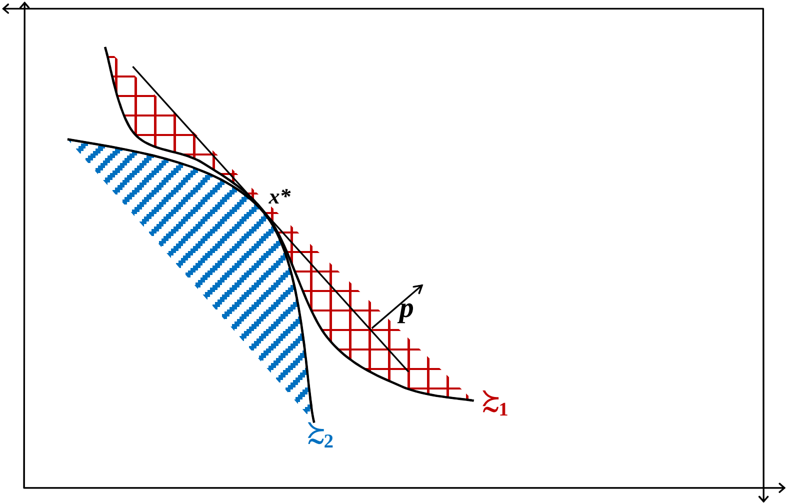
- Transfers redistribute income so that the ‘aggregate budget’ balances.
- This is important: in a general equilibirum model nothing should be ‘outside’ the economy

Second Welfare Theorem: Need Convex Preferences and Production Sets

Pictures

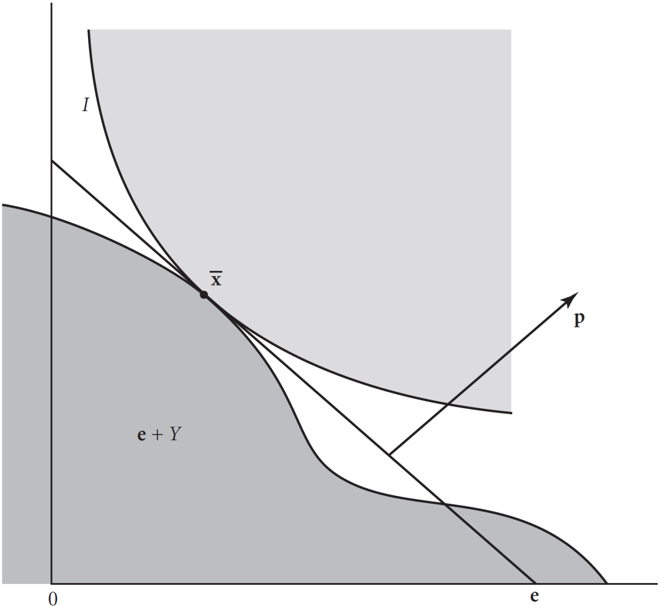
Counterexample I to Second Welfare Theorem

Need convex preferences for the Second Welfare Theorem



x^* is Pareto optimal, but one can see it is not an equilibrium at prices p

Counterexample II to Second Welfare Theorem



Convexity

Definition

A preference relation $\succsim$ on X is convex if the set $\succsim(x) = \{y \in X \mid y \succsim x\}$ is convex for every x.

- If x' and x'' are weakly preferred to x so is any convex combination.
- Convexity implies existence of an hyperplane that 'supports' a consumer's better than set.

Definition

In an exchange economy, an allocation x is supported by a non-zero price vector p if: for each $i = 1, ..., I$

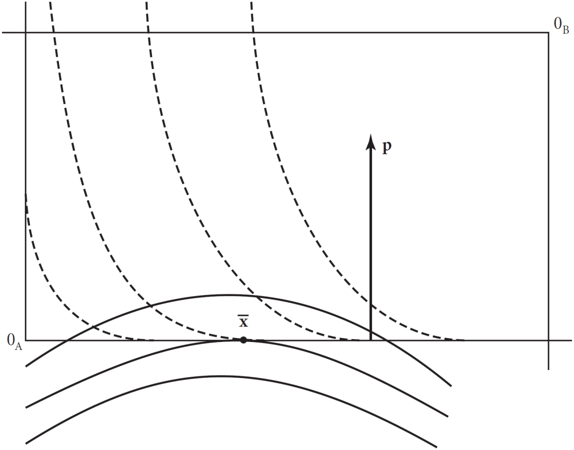
$$x' \succsim_i x_i \qquad \Longrightarrow \qquad p \cdot x' \geq p \cdot x_i$$

- Convexity also yields an hyperplane that 'supports' all producers' better than set at the same time.
- This hyperplane is the price vector that makes a Pareto optimal allocation an equilibirum.

Need Interior Allocations

Picture

Counterexample III to Second Welfare Theorem



- $\bar{x}$ is Pareto optimal, hence we need prices to make it an equilibrium. The unique price vector that can support it as equilibrium (normalizing the price of the first good to 1) is $p = (1, 0)$.
- The corresponding wealth is $w_B = (1, 0) \cdot (0, \omega_B) = 0$.
- However, $\bar{x}_B$ is not maximial for consumer 1 at p since she would like more of good 2 (it has zero price, hence she can afford it).

Quasi-Equilibrium

- To fix the 'existence at the boundary' problem, we make a small change to the definition of equilibrium.

Definition

Given an economy $\{X_i, \succsim_i\}_{i=1}^{I}, \{Y_j\}_{j=1}^{J}, \omega$, an allocation x^*, y^* and a price vector p^* constitute a quasi-equilibrium with transfers if there exists a vector of wealth levels

$$w = (w_1, w_2, ..., w_I) \qquad \text{with} \qquad \sum_{i=1}^{I} w_i = p^* \cdot \omega + \sum_{j=1}^{J} p^* \cdot y_j^*$$

such that:

1. For each $j = 1, ..., J$: $\quad p^* \cdot y_j \leq p^* \cdot y_j^* \quad$ for all $y_j \in Y_j$.
2. For every $i = 1, ..., I$:
$$\text{if} \quad x \succ_i x_i^* \quad \text{then} \quad p^* \cdot x \geq w_i$$
3. $\sum_{i=1}^{I} x_i^* = \sum_{i=1}^{I} \omega_i + \sum_{j=1}^{J} y_j^*$

- Make sure you see why this deals with the problem in the previous slide.
- Any equilibrium with transfers is a quasi-equilibrium (make sure you check this).

Next Class

- Proof of the Second Welfare Theorem