Existence of Walrasian Equilibrium

Econ 2100 Fall 2018

Lecture 20, 7 November

Outline

1. Existence of a competitive (Walrasian) equilibirum

From Last Class

- The market excess demand correspondence $z : \mathbb{R}^L_+ \to \mathbb{R}^L$ is:

$$z(p) = \sum_{i=1}^{I} (x_i^* - \omega_i) - \sum_{j=1}^{J} y_j^*$$

where $y_j^* \in y_j^*(p)$ for each $j = 1, ..., J$, and $x_i^* \in x_i^*(p)$ for each $i = 1, ..., I$.

- An equilibrium price vector $p^* > 0$ must satisfy:

$$z_l\left(p^*\right) \leq 0 \text{ for all } l, \quad \text{and} \quad \text{whenever } z_l\left(p^*\right) < 0 \text{ then } p_l^* = 0$$

- Walras' Law: for any price vector $p \in \mathbb{R}^L_+$:

$$p \cdot z(p) = 0$$

- If Walras' Law holds, $p^* > 0$ is an equilibrium if and only if $z\left(p^*\right) \leq 0$. Why?

$$z_l\left(p^*\right) \leq 0 \text{ for all } l, \text{ and } \sum_{l=1}^{L} p_l z_l\left(p^*\right) = 0$$

thus p_l must be zero if $z_l\left(p^*\right) < 0$.

Equilibrium As A Fixed Point

Summary

- An equilibrium price vector p^* must satisfy:

$$z_l(p^*) \leq 0 \text{ for all } l, \qquad \text{and} \qquad \text{whenever } z_l(p^*) < 0 \text{ then } p_l^* = 0$$

A useful observation

- Let the function $g : \mathbb{R}^L \to \mathbb{R}^L$ be defined by

$$g_l(p) = \max\{p_l + z_l(p), 0\} \qquad \text{for } l = 1, 2, .., L$$

- CLAIM: An equilibrium is a $p^* \geq 0$ such that

$$g(p^*) = p^* \qquad \text{or} \qquad g_l(p^*) = p_l^* \text{ for all } l$$

At an equilibrium price vector p^*:

1. either $g_l(p^*) = p_l^* = 0$ and thus either $z_l(p^*) < 0$ or $z_l(p^*) = 0$
2. $g_l(p^*) = p_l^* \neq 0$ and thus $p_l^* + z_l(p^*) = p_l^*$ which implies $z_l(p^*) = 0$.

- In both cases we have an equilibrium thus establishing the claim.

Equilibrium As A Fixed Point

Summary

- An equilibirum exists if there exists a p^* such that
$$g\left(p^*\right) = p^*$$
- This is a fixed point: we want to show that $g(p)$ must have a fixed point.

- We need a theorem that gives conditions for functions to have a fixed point.

Brouwer's Fixed Point Theorem

Theorem (Brouwer's Fixed Point Theorem)

If $X \subseteq \mathbb{R}^L$ is convex and compact and the function $f : X \to X$ is continuous, then there exists an $x \in X$ such that $f(x) = x$ (that is, f has a fixed point).

- Counterexamples:
  - $X = (0, 1]$ and $f(x) = \frac{x}{2}$ (X must be closed)
  - $X = (0, \infty]$ and $f(x) = x + 1$ (X must be bounded)
  - $X = \left\{x \in \mathbb{R}^2 : \|x\| = 1\right\}$ and $f(x) = -x$ (X must be convex)
  - $X = [0, 1]$ and $f(x) = \begin{cases} 1 & \text{if} \quad 0 \leq x \leq 0.5 \\ 0 & \text{if} \quad 0.5 < x \leq 1 \end{cases}$ (f must be continuous)
- In general, we want to allow for correspondences (aggregate excess demand may not be a function); fortunately, there is similar theorem for correspondences (Kakutani's fixed point theorem).

Brouwer's Theorem and Existence

Theorem

A continuous $f : S \to S$, where $S \subseteq \mathbb{R}^L$ is convex and compact, has a fixed point.

How can we use this theorem to show that an equilibirum exist?

Let $g : \mathbb{R}^L \to \mathbb{R}^L$ be defined by: $g_l(p) = \max\{p_l + z_l(p), 0\}$ for $l = 1, 2, .., L$. Can we apply Brower?

1. Need domain and range to be the same convex and compact subset of $\mathbb{R}^L$. They are not, but since only relative prices matter we can normalize them. Take the domain to be $\Delta^{L-1} = \{p \in \mathbb{R}^L_+ : \sum_{l=1}^{L} p_l = 1\}$, and divide $g(\cdot)$ by the sum of its elements so that the range is also Δ^{L-1}. ✓
2. $g(\cdot)$ needs to be continuous.
   Assume preferences that yield continuity of excess demand. ✓
3. $g(\cdot)$ needs to be a function.
   Assume strictly convex preferences (there is a theorem for correspondences). ✓
4. $g(\cdot)$ must be well defined even if some prices are zero.
   If preferences are monotone, excess demand can blow up. But we need monotonicity for other properties. Assume this away (for now). ☒

'Easy' Existence Theorem

Theorem

Assume that aggregate excess demand $z : \Delta^{L-1} \to \mathbb{R}^L$ is a continuous function such that $p \cdot z(p) = 0$ for all p. Then, there exists $p^* \in \Delta^{L-1}$ such that $z(p^*) \leq 0$.

- If excess demand is a continuous function that satisfies Walras' Law, an equilibrium exists.

Remark

- We should prove existence from assumptions on the primitives of the economy, i.e.preferences and endowments, not on excess demand.
    - Although many assumptions in this theorem can be written from preferences way (use the results from last class), the fact that excess demand is well defined and continuous over its entire domain cannot (need to rule out zero prices).

The proof goes as follows

1. Define a function of excess demand so that...
2. ... Brouwer's fixed point theorem applies, and a fixed point of this function exists.
3. Complete the proof by showing this fixed point is an equilibrium.

'Easy' Existence Proof: Step 1a

Let $g : \Delta^{L-1} \to \mathbb{R}^L$ *be defined by*

$$g_l(p) = \max\{p_l + z_l(p), 0\} \quad \textit{with } l = 1, 2, .., L.$$

Remark

Claim: $g(p) \neq 0$.

- By definition, $g_l(p) \geq p_l + z_l(p)$.
- Thus

$$\begin{aligned} p \cdot g(p) &\geq p \cdot (p + z(p)) \\ &= p \cdot p + p \cdot z(p) \\ &= p \cdot p + \underbrace{0}_{\text{By Walras' Law}} \quad \underbrace{> 0}_{\text{by } p \in \Delta^{L-1}} \end{aligned}$$

- Since there must be some good that has a positive price, $g_l(p)$ cannot be equal to zero for all l.
  - Thus $\max\{p_l + z_l(p), 0\} > 0$ for at least one l.

'Easy' Existence Proof: Step 1b

Define $h : \Delta^{L-1} \to \Delta^{L-1}$ *as*

$$h(p) = \frac{g(p)}{\sum_{l=1}^{L} g_l(p)}$$

- $h(\cdot)$ is well defined.
- $h(\cdot)$ is continuous because $z(\cdot)$ is continuous and thus $g(\cdot)$ is continuous.
- $h(\cdot)$ maps from Δ^{L-1}, a convex and compact set, to itself.

Therefore

$h(\cdot)$ is a continuous function from a compact convex set to itself; by Brouwer's theorem, it has a fixed point.

'Easy' Existence Proof: Step 2

By Brouwer's fixed point theorem, there exists a p^* such that

$$p^* = h(p^*) = \frac{g(p^*)}{\sum_{l=1}^{L} g_l(p^*)}$$

- Rewrite this as

$$g(p^*) = p^* \left(\sum_{l=1}^{L} g_l(p^*) \right) = p^* \gamma$$

  for some real number γ.
- Observe that $\gamma \neq 0$ because $g(p) \neq 0$.
- Next, we show that $\gamma = 1$.

'Easy' Existence Proof: Step 3a

Claim: *for each $l = 1, .., L$*

$$p_l^* g_l(p^*) = p_l^* \left(p_l^* + z_l(p^*)\right)$$

- *This is easy to show:*

$$\text{if } g_l(p^*) \neq p_l^* + z_l(p^*) \underbrace{\Rightarrow}_{\text{by definition}} g_l(p^*) = 0 \underbrace{\Rightarrow}_{\text{by } \gamma \neq 0} p_l^* = 0$$

- *Thus: either $g_l(p^*) = p_l^* + z_l(p^*)$ or $p_l^* = 0$; in both cases the claim holds.*

- Summing over l, we obtain:

$$p^* \cdot g(p^*) = p^* \cdot (p^* + z(p^*)) = p^* \cdot p^* + p^* \cdot z(p^*) = p^* \cdot p^* + \underbrace{0}_{\text{by Walras' Law}}$$

- By the existsence of a fixed point (last slide), we know that $g(p^*) = p^* \gamma$; taking the inner product with p^* on both sides:

$$p^* \cdot g(p^*) = p^* \cdot p^* \gamma$$

- Since we have just shown that $p^* \cdot g(p^*) = p^* \cdot p^*$, we have

$$p^* \cdot p^* = p^* \cdot p^* \gamma$$

and thus $\gamma = 1$ as desired.

'Easy' Existence Proof: Step 3b

Summary

By the fixed point theorem, $g(p^*) = p^*\gamma$, and since $\gamma = 1$: $g(p^*) = p^*$.

The fixed point is an equilibrium

Claim: p^* is an equilibrium.

Proof.

- The equation above implies:

$$p_l^* = g_l(p^*) = \underbrace{\max\{p_l^* + z_l(p^*), 0\}}_{\text{by definition of } g(\cdot)} \qquad \text{for } l = 1, 2, .., L$$

- Therefore:

$$z_l(p^*) \leq 0 \qquad \text{for } l = 1, 2, .., L$$

  - If not, there exists some k such that

$$z_k(p^*) > 0$$

  - and

$$p_k^* = \max\{p_k^* + z_k(p^*), 0\} = p_k^* + z_k(p^*)$$

    an impossibility.

- Since p^* is an equilibirum if and only if $z(p^*) \leq 0$, we are done. □

What Took So Long

- Before 1950 economists focused on proving existence by showing that the system of $L-1$ equations in $L-1$ unknowns given by the condition that excess demand equals zero had a solution.
- This went nowhere.
- In 1950, John Nash proved existence of the (Nash) equilibrium of a game.
- This was the breakthrough needed by Arrow, Debreu, and McKenzie to prove existence of a competitive equilibrium shortly thereafter.
- A "game" is a situation in which an individual payoffs may depend on others' choices.

Games and Nash Equilibrium

- A game is a defined by describing choices and payoffs for each player.

- each $i = 1, ..., I$ chooses a strategy in the set S_i;
  - let $S = S_1 \times S_2 \times .. \times S_I$, and $s = (s_1, .., s_I) \in S$ is a strategy profile (an action for each player);
- $u_i(s)$ is player i's payoff function from s.
- A Nash equilibrium is a strategy profile such that: each player maximizes her payoff given the other players' strategies.
- Thus, a Nash Equilibrium is an s^* such that for all i

$$u_i(s_1^*, .., s_i^*, .., s_I^*) \geq u_i\left(s_1^*, .., s_{i-1}^*, t, s_{i+1}^*, .., s_I^*\right) \qquad \text{for any } t \in S_i.$$

  or

$$u_i\left(s_i^*, s_{-i}^*\right) \geq u_i\left(t, s_{-i}^*\right) \qquad \text{for any } t \in S_i.$$

- Nash proved that such a strategy profile exists in any game when (i) the strategy sets are compact and convex, and (ii) payoff are strictly quasi-concave and continuous functions. How? Using Brower's Theorem.

Nash's Existence Theorem

A Nash equilibirum is a fixed point

- For each i, let $BR_i : S_{-i} \to S_i$ be defined as follows:
$$BR_i\left(s_{-i}\right) = \left\{s_i \in S_i : \; u_i\left(s_i, s_{-i}\right) \geq u_i\left(t, s_{-i}\right) \quad \forall t \in S_i\right\}$$
  - this function describes i's 'best response' to the other players strategy s_{-i}.
- Define the mapping $BR : S \to S$ as
$$BR\left(s\right) = BR_1\left(s_{-1}\right) \times ... \times BR_I\left(s_{-I}\right)$$
- s^* is a Nash equilibrium if and only if $s^* = BR\left(s^*\right)$: a fixed point.
  - In a Nash equilibrium every player chooses a best response.

- Assumptions (i) and (ii) from the previous slide are enough to use Brouwer's Theorem.
  - we need S closed, bounded, and convex; so, assume each S_i has those proerties.
  - we need $BR\left(s\right)$ to be a continuous function; so, assume each $u_i(s)$ is continuous and strictly quasi-concave, so that each $BR_i\left(s\right)$ is a continuous function.
- Under these assumption, every game has a Nash equilibrium.

From Nash to Walras

- Arrow, Debreu, and McKenzie saw Nash's paper and used it to solve the existence problem. How?

Build a 'game' that describes a Walrasian (competitive) equilibirum

- The players are consumers, firms, and a 'Price Player'.
- The price player chooses a price vector p.
- Each i's payoff function is $u_i(x_i)$, consumer i's utility function;
  - i's best response to is an element of Walrasian demand x_i^*.
- Each j's payoff function is $p \cdot y_j$;
  - j's best response is an element of the supply y_j^*.
- The Price Player's payoff is the value of the aggregate excess demand:

$$u_{PP}(p, x) = p \cdot z = p \cdot \left(\sum_{i=1}^{I} (x_i^* - \omega_i) - \sum_{j=1}^{J} y_j^* \right)$$

- The price player's best response is the price that maximizes the value of that excess demand.
- A Nash equilibrium is given by a price vector and consumption choices such that all players choose a best response.

From Nash to Walras

A Nash equilibirum is a fixed point

- By Nash's existence theorem, there exist a $p^* \in \Delta^{L-1}$,an $x_i^*(p^*)$ for each i, and an $y_j^*(p^*)$ for each j such that
  - each $y_j^*(p^*)$ maximizes profits given p^*,
  - each $x_i^*(p^*)$ maximizes individuals' utility given p^* and y_j^*, and
  - p^* maximizes the value of aggregate excess demand (given x_i^* and y_j^*).

Claim: this is a Walrasian equilibrium

- Why is p^* a competitive equilibrium?
  - Walras' Law ($0 = p \cdot z(p)$) implies $p^* \cdot z(p^*) = 0$.
  - The price player maximization implies $p^* \cdot z(p^*) \geq p \cdot z$ for all $p \in \Delta^{L-1}$.
  - These together imply $z(p^*) \leq 0$ (make sure you convince yourself of this).
  - We already proved that $z(p^*) \leq 0$ implies p^* is an equilibrium.

Remarks

- To use Brouwer's theorem, best responses must be functions.
  - even if aggregate excess demand is a function, $BR_{PP}(\cdot)$ can be multi-valued.
- We need fixed point existence for correspondences: Kakutani's theorem.

An General Existence Theorem

Theorem

Suppose an economy satisfies the following properties.

1. *For each i:* $X_i \subset \mathbb{R}^L$ *is closed and convex*
   $\succsim_i$ *satisfies local non-satiation, and convexity*
   $\omega_i \geq \widehat{x}_i$ *for some* $\widehat{x}_i \in X_i$.
2. *For each j:* $Y_j \subset \mathbb{R}^L$ *is closed, convex,*
   includes the origin, and satisfies free-disposal.
3. *The set of feasible allocations is compact.*

Then a Walrasian quasi-equilibrium exists (if $\omega_i \gg \widehat{x}_i$ for all i then an equilibrium exists).

- Issues a proof needs to take care of.
  - When some prices are zero and individual's preferences are locally non satiated, their demand can explode.
    - This is because the budget set is not compact.
  - Demand (and therefore excess demand) is not necessarily continuous at zero prices.
    - Think about how could this happen.
  - We need all prices to be strictly positive.

Proving a Not So Simple Existence Theorem

- This is a sketch of the proof for existence of a competitive equilibrium.

1. Truncate the economy, so that all choices must belong to a compact set.
2. Construct a 'game' with $I + J + 1$ players: consumers, firms, and the 'price player'.
3. Show that each player's best response is a non-empty, convex, and upper hemi-continuous correspondence.
4. Hence the 'product' best-response correspondence that describes this game inherits those properties.
5. Use Kakutani's fixed point theorem to show this correspondence has a fixed point.
6. This fixed point is an equilibrium of the truncated economy.
7. Prove that the truncation does not matter: an equilibrium of the truncated economy is a quasi-equilibrium of the whole economy.
8. DONE.

- Some of the tricky issues have to do with getting strictly positive prices so that the correspondences are upper hemi-continuous. See a book for the proof.

Next Week

- Uniqueness of Equilibrium.
- How to model uncertainty and time.