# Anscombe & Aumann Subjective Expected Utility

Econ 3030 Fall 2025

Lecture 10

Outline

1. Objective vs. Subjective Probabilities
2. Anscombe & Aumann Acts: Horse Races vs. Roulette Lotteries
3. Subjective Expected Utility

## From last class: Expected Utility Theorem

- $X = \{x_1, x_2, \ldots, x_n\}$ is a finite set.
- The space of all probabilities on $X$ is $\Delta X = \{\pi \in \mathbb{R}^n : \sum_{i=1}^{n} \pi_i = 1 \text{ and } \pi_i \geq 0, \forall i\}$.
- A preference relation $\succsim$ is a binary order on $\Delta X$.
  - Since the $x$-s are fixed, this compares probability distributions.

**Theorem (Expected Utility Theorem, von Neumann and Morgenstern 1947)**

*Let $\Delta X$ be the set of all probabilities on a finite set $X$. The preference relation $\succsim$ on $\Delta X$ is complete, transitive, independent and Archimedean if and only if there exists a function $v : X \to \mathbf{R}$ such that*

$$U(\pi) = \sum_{x \in X} v(x)\pi(x)$$

*represents $\succsim$. This representation is unique up to affine transformations.*

- $U$ represents $\succsim$ means

$$\pi \succsim \rho \Leftrightarrow \sum_{x \in X} v(x)\pi(x) \geq \sum_{x \in X} v(x)\rho(x).$$

# Objective Probabilities

- Let $\Delta X$ be the set of all probabilities on a finite set $X$. The preference relation $\succsim$ on $\Delta X$ is complete, transitive, independent, and Archimedean if and only if there exists $v : X \to \mathbb{R}$ such that $\succsim$ is represented by $U(\pi) = \sum_{x \in X} v(x)\pi(x)$.
  - Preferences rank lotteries over a given set of prizes.
    - Information about the preferences implies existence of a representation with a particular functional form, and pins down the vNM function $v$.
- The only possible randomness is the ones about which of the elements of $X$ the decision maker will get.
  - Since prizes are fixed, decision makers rank pairs of probability distributions.
    - They rank things like "$x_1$ with probability $\frac{1}{2}$, $x_2$ with probability $\frac{1}{4}$, and $x_3$ with probability $\frac{1}{4}$" versus "$x_1$ with probability $\frac{1}{3}$, $x_2$ with probability $\frac{1}{3}$, and $x_3$ with probability $\frac{1}{3}$"
- The probabilities are part of the description of the objects ranked.
- Probabilities and consequences are known primitives of the model.

- But there are also uncertainties that are of a different kind: uncertainties for which the probabilities are not necessarily given to the decision maker:
  - Things like "$x_1$ if tomorrow rains, $x_2$ if tomorrow is cloudy, and $x_3$ if tomorrow is sunny"

# Objective vs. Subjective Probabilities

- $\succsim$ on $\Delta X$ is complete, transitive, independent, and Archimedean if and only if there exists $v : X \to \mathbb{R}$ such that $\succsim$ is represented by $U(\pi) = \sum_{x \in X} v(x)\pi(x)$.
  - Preferences rank lotteries over a given set of prizes.
- If probabilities are known primitives of the model: all decision makers must perceive them as identical: decision makers can only differ in their vNM utility $v$..
- What if the probabilities are not given?

- In these (most natural) cases, probabilities reflect what a decision maker thinks and can differ across individuals.
  - This cannot be done in the model we have seen so far: probabilities are objective.
- To allow for subjective probabilities, we need a model where events' likelihoods are also a characteristic of the agents.
  - Then, the probability distribution of each event and the utility of each consequence are derived from a decision makers' preferences (both are characteristics of an individual).
- This new model has a more general consumption space and a new axiom.

# Anscombe and Aumann Acts

- $\Omega = \{1, 2, \ldots, S\}$ is a finite set of (random) states, with generic element $s \in \Omega$.
- $X = \{x_1, \ldots, x_n\}$ is a finite set of outcomes, with a generic element $x \in X$.
- $H$ is the space of all functions from $\Omega$ to $\Delta X$ (sometimes denoted $(\Delta X)^{\Omega}$).
  - this is a convex subset of the space of all functions from $\Omega$ to $\mathbb{R}^n$.

**Anscombe–Aumann acts**

- An Anscombe–Aumann act $h \in H$ is a function $h : \Omega \to \Delta X$;

- Anscombe–Aumann acts assign a lottery (an element of $\Delta X$) to each state (an element of $\Omega$).
  - For example: if the state is *rain*, the lottery is *umbrella* with probability $\frac{1}{2}$ and *no umbrella* with probability $\frac{1}{2}$; if the state is *no rain*, the lottery is *umbrella* with probability $\frac{1}{3}$, and *no umbrella* with probability $\frac{2}{3}$.

Preferences rank Anscombe-Aumann acts

# Horse Lotteries and Roulette Lotteries

- Two uncertanties: states of the world ($\Omega$) and lotteries over consequences ($\Delta X$).
- How can we think about them?

## Interpretation of $H$

- Lotteries over consequences are bets on a "roulette" spin:
  - outcomes' probabilities are objectively determined (everyone agrees on them).
- A state of the world represent the event that a specific "horse" named $s$ wins a race among the field of horses $\Omega$:
  - the decision maker subjectively assesses each horse's strength (different decision makers can evaluate each horse differently).
- So, first the horses run the race, and afterwards the roulette is spun.
  - The roulette's payoff can depend on which horse wins.

- The theory's aim is to identify the decision maker's personal assessment of the probability that horse $s$ will win the race using her preferences.
- To perform this identification, we set the payout on horse $s$ equal to a lottery that depends on the outcome of some roulette spin.

# Notation

**Anscombe–Aumann acts**

- An Anscombe–Aumann act $h \in H$ is a function $h : \Omega \to \Delta X$;
- Anscombe–Aumann acts assign a lottery (an element of $\Delta X$) to each state (an element of $\Omega$).

- Let $h_s = h(s) \in \Delta X$, and denote $h_s(x) = [h(s)](x) \in [0, 1]$.
  - This is the probability of $x$ conditional on $s$, given the act $h$: $h_s(x) = \mathrm{Pr}(x|s, h)$.

*How can we describe elements of $H$ (functions from $\Omega$ to $\Delta X$)?*

# First description of H

**First.** *The original mathematical interpretation where* $H = (\Delta X)^{\Omega}$.

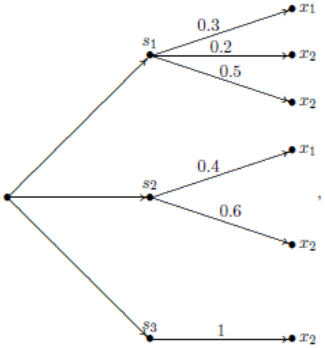
- Suppose $\Omega = \{s_1, s_2, s_3\}$ and $X = \{x_1, x_2, x_3\}$.
- Then a particular $h : \Omega \to \Delta X$ would be the following:

$$\begin{aligned} h(s_1) &= (0.3, 0.2, 0.5) \\ h(s_2) &= (0.4, 0.6, 0) \\ h(s_3) &= (0, 1, 0) \end{aligned}$$

# Second description of H

**Second.** *H as a set of compound lotteries.*

- The subjective first stage lottery is over which state $s \in \Omega$ obtains, and the objective second stage lottery (conditional on $s$) is over which $x \in X$ obtains.
- These compound lotteries can be written as probability trees.
- For example: $h$ is



# Third description of H

**Third.** *H is the set of weakly positive* $m \times n$ *matrices where*

$$\sum_{j=1}^{n} h_{s,j} = 1 \textit{ for each } s = 1, \ldots S.$$

- Then the $h$ on the previous slides can be represented as

$$h = \begin{bmatrix} & \Pr(x = x_1) & \Pr(x = x_2) & \Pr(x = x_3) \\ s_1 & 0.3 & 0.2 & 0.5 \\ s_2 & 0.4 & 0.6 & 0 \\ s_3 & 0 & 1 & 0 \\ \end{bmatrix}.$$

- We can then write $h_s(x) = h_{s,x}$; for example $h_{s_1}(x_2) = h_{1,2} = 0.2$.

# Summing Anscombe-Aumann Acts

## Anscombe–Aumann acts

- An Anscombe–Aumann act $h \in H$ is a function $h : \Omega \to \Delta X$;
- Anscombe–Aumann acts assign a lottery (an element of $\Delta X$) to each state (an element of $\Omega$).

## Summing

Given $H \subset (\mathbb{R}^n)^\Omega$, if $f, g : \Omega \to \mathbb{R}^n$, then the function $\alpha f + \beta g : \Omega \to \mathbb{R}^n$ is defined by

$$[\alpha f + \beta g](s) = \alpha f(s) + \beta g(s).$$

- Summing is all about the objective lotteries (not about the horse race).
- This definition is crucial: Archimedean and Independence axioms sum acts.

# Objective Lotteries Are Anscombe-Aumann Acts

- A $\pi \in \Delta X$ denotes a constant act $f : \Omega \to \Delta X$ s.t. $f(s) = \pi$ for all $s \in \Omega$.

## Notation

- One can identify $\Delta X$ as a subset of $H$.
  - Any probability distributions on $X$ is an Anscombe-Aumann act that, regardless of the state, gives the same lottery over outcomes.
  - The set of all constant acts is denoted $H_c$ (this subset of $H$ is isomorphic to $\Delta X$)

$$\Delta X \approx H_c = \{f \in H : f(s) = f(s') \text{ for all } s \in \Omega\}$$

- One can also identify $X$ as a subset of $H$.
  - Elements of $X$ are Dirac lotteries (degenerate probability distributions) in $\Delta X$, denoted $\delta_x$.
  - Thus, $X$ is a subset of $H_c$ defined as follows

$$X = \{f \in H : f(s) = f(s') \ \forall \ s \in \Omega, \text{ and } f(s) = \delta_x \text{ for some } x \in X\}$$

- Anscombe-Aumann acts generalize von Neumann and Morgenstern's setting.

# Subjective Expected Utility (SEU): The Idea

- Starting from preferences, identify a unique probability distribution $\mu \in \Delta\Omega$ and a utility index $v : X \to \mathbb{R}$ such that a utility representation of these preferences is

$$U(h) = \sum_{s \in \Omega} \mu(s) \left[ \sum_{x \in X} v(x) h_s(x) \right]$$

## Some Accounting Details

- $U(h)$ is the subjective expected value of $v$ given act $h$:

$$\overbrace{\sum_s \mu(s) \underbrace{\left[ \sum_x v(x) h_s(x) \right]}_{\mathbf{E}_{h(s)}[v(x)]}}^{\mathbf{E}_\mu\left[\mathbf{E}_{h(s)}[v(x)]\right]} = \sum_x v(x) \underbrace{\left[ \sum_s \overbrace{\mu(s)}^{\Pr(s)} \overbrace{h_s(x)}^{\Pr(x|s,h)} \right]}_{\Pr(x)}.$$

  - $\sum_s \mu(s) h_s(x)$ is the total or unconditional subjective probability of receiving consequence $x$ under the function $h$, denoted $\Pr(x)$.
  - Therefore $\Pr(x) = \sum_s \Pr(s)\Pr(x|s,h)$, since $\mu(s)h_s(x) = \Pr(s)\Pr(x|s,h)$.

## State Dependent Expected Utility

- Without extra assumptions, we can use the Mixture Space Theorem on $H$.

**Theorem**

*The preference relation $\succsim$ on $H$ is complete, transitive, independent, and Archimedean if and only if there exists a set of vNM indices $v_1, \ldots, v_S : X \to \mathbb{R}$ such that*

$$U(h) = \sum_{s \in \Omega} \sum_{x \in X} v_s(x) h_s(x)$$

*is a utility representation.*

- This follows from the Mixture Space Theorem applied to $H$ (a convex set).
- Sufficiency proof is similar to vNM: find a function $U : \mathbb{R}^{S \times n} \to \mathbb{R}$ representing $\succsim$;
  - this linear function is uniquely determined by a vector in $\mathbb{R}^{S \times n}$.

**Remark**

The utility of each consequence depends on the state in which it obtains (formally, $v_s(x)$ depends on $s$); this is a state-dependent additive representation.

# State Dependent Expected Utility

**Theorem**

*The preference relation $\succsim$ on $H$ is complete, transitive, independent, and Archimedean if and only if there exists a set of vNM indices $v_1, \ldots, v_S : X \to \mathbb{R}$ such that*

$$U(h) = \sum_{s \in \Omega} \sum_{x \in X} v_s(x) h_s(x)$$

*is a utility representation.*

**Remark**

- One cannot pin down unique probabilities using state dependent expected utility.

- Because of this, the theorem does not identify a unique probability over the state space.
- For this identification to be possible, one needs a "state-independent" representation in which the function $v$ does not change across states.

# Null States

## Notation

Given an act $h \in (\Delta X)^\Omega$, a state $s \in \Omega$, and a lottery $\pi \in \Delta X$, define the new act $(h_{-s}, \pi) : \Omega \to \Delta X$ by $(h_{-s}, \pi) = (h_1, \ldots, h_{s-1}, \pi, h_{s+1}, \ldots, h_m)$. So

$$[(h_{-s}, \pi)](t) = \begin{cases} \pi & \text{if } t = s \\ h(t) & \text{if } t \neq s \end{cases};$$

- $(h_{-s}, \pi)$ replaces $h_s$ (the lottery that act $h$ assigns to state $s$) with the lottery $\pi$ while the remainder of $h$ stays the same.

- States the decision maker never cares about should be irrelevant.

## Definitions

- A state $s \in \Omega$ is null if, for all $h \in (\Delta X)^\Omega$ and $\pi, \rho \in \Delta X$, $(h_{-s}, \pi) \sim (h_{-s}, \rho)$.
- A state $s \in \Omega$ is non-null if it is not null, i.e. if there exist $h \in (\Delta X)^\Omega$ and $\pi, \rho \in \Delta X$ such that $(h_{-s}, \pi) \succ (h_{-s}, \rho)$.

- A null state never affects rankings.

# State Independence

**Definition**

The binary relation $\succsim$ on $H$ is state-independent if, for all non-null states $s, t \in \Omega$, for all acts $h, g \in H$, and for all lotteries $\pi, \rho \in \Delta X$,

$$(h_{-s}, \pi) \succsim (h_{-s}, \rho) \quad \Rightarrow \quad (g_{-t}, \pi) \succsim (g_{-t}, \rho).$$

- In words, the ranking of roulette lotteries does not depend on the state in which they obtain.
  - This only needs to hold in states the decision maker cares about.
- Implies the utility index over consequences is the same across states.

# Subjective Expected Utility Theorem

**Theorem (Expected Utility Theorem, Anscombe and Aumann)**

*A preference relation $\succsim$ on $H$ is complete, transitive, independent, Archimedean, and state-independent if and only if there exists a vNM index $v : X \to \mathbb{R}$ and a probability $\mu \in \Delta\Omega$ such that*

$$U(h) = \sum_{s\in\Omega} \mu(s) \left[ \sum_{x\in X} v(x) h_s(x) \right]$$

*is a utility representation of $\succsim$. Moreover, this representation is unique up to affine transformations provided at least two acts can be ranked strictly.*

- Therefore: $h \succsim g \Leftrightarrow \sum_{s\in\Omega} \mu(s) \left[ \sum_{x\in X} v(x) h_s(x) \right] \geq \sum_{s\in\Omega} \mu(s) \left[ \sum_{x\in X} v(x) g_s(x) \right]$
- The second part says: if there exist $h, g \in H$ such that $h \succ g$, then $U'(h) = \sum_s \mu'(s) \left[ \sum_x v'(x) h_s(x) \right]$ also represents $\succsim$ if and only if $\mu' = \mu$ and $v' = av + b$ for some $a > 0$ and $b \in \mathbb{R}$.

**Remark**

- The probability distribution $\mu$ is unique.

**Theorem (Expected Utility Theorem, Anscombe and Aumann)**

*A preference relation $\succsim$ on $H$ is independent, Archimedean, and state-independent if and only if there exists a vNM index $v : X \to \mathbb{R}$ and a probability $\mu \in \Delta\Omega$ such that*

$$U(h) = \sum_{s \in \Omega} \mu(s) \left[ \sum_{x \in X} v(x) h_s(x) \right] = \sum_{x \in X} v(x) \underbrace{\left[ \sum_{s \in \Omega} \overbrace{\mu(s)}^{\Pr(s)} \overbrace{h_s(x)}^{\Pr(x|s)} \right]}_{\Pr(x)}$$

*is a utility representation of $\succsim$. This representation is unique up to affine transformations.*

- Preferences identify two things:
  - the utility index over consequences $v : X \to \mathbb{R}$ and
  - the probability distribution over states $\mu \in \Delta\Omega$.
- Different preferences may imply different beliefs $\mu$ on $\Omega$.
- von Neumann–Morgenstern's Theorem only identifies the utility index $v$.

*For this reason, this is called Subjective Expected Utility.*

## Anscombe & Aumann Proof

- We will not do it in detail, but here is a breakdown of what would happen.

**Proof.**

- First, convert each act $h$ to a vector in $[0,1]^S$, by assigning each dimension $s \in \{1, ..., S\}$ a vNM expected utility for $h_s$
  - this gives $[\sum_x v(x) h_s(x)]$ in each state $s$;
  - monotonicity, a consequence of state-independence, is essential for this step.
- Then, construct an independent and Archimedean preference relation on $[0,1]^S$ and the measure $\mu$ is the dual of the affine utility on $[0,1]^S$.
- The necessity and uniqueness parts are as usual.

□

- Notice that this argument involves using the mixture space theorem (or vNM theorem) twice.
  - the first time to find $[\sum_x v(x) h_s(x)]$ in each state $s$ (this is $\mathbf{E}_{h(s)}[v(x)]$);
  - the second to find the measure $\mu$ as the dual of the affine utility on $[0,1]^S$.

# Next Class

- Expected Utility Over Money