

Expected Utility Over Money and Risk Aversion

Econ 3030

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Lecture 11

Outline

- 1 Expected Utility Of Wealth
- 2 Betting & Insurance
- 3 Risk Aversion
- 4 Certainty Equivalent
- 5 Risk Premium

Probability Distribution On Wealth

- Many applications of expected utility consider preferences over probability distributions of wealth (a continuous variable).
- A probability distribution is characterized by its cumulative distribution function.

Definition

A **cumulative distribution function** (cdf) $F : \mathbb{R} \rightarrow [0, 1]$ satisfies:

- $x \geq y$ implies $F(x) \geq F(y)$ (nondecreasing);
- $\lim_{y \downarrow x} F(y) = F(x)$ (right continuous);^a
- $\lim_{x \rightarrow -\infty} F(x) = 0$ and $\lim_{x \rightarrow \infty} F(x) = 1$.

^aRecall that, if it exists, $\lim_{y \downarrow x} f(y) = \lim_{n \rightarrow \infty} f(x + \frac{1}{n})$.

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Notation

- μ_F denotes the expected value of F , i.e. $\mu_F = \int x dF(x)$.
- δ_x is the degenerate distribution function at x ; i.e. δ_x yields x with certainty:

$$\delta_x(z) = \begin{cases} 0 & \text{if } z < x \\ 1 & \text{if } z \geq x \end{cases}$$

Expected Utility Of Wealth

- As usual, the space of all distribution functions is convex and one can define preferences on it.
- The utility index $v : \mathbb{R} \rightarrow \mathbb{R}$ is defined over **wealth** (NOTE: values can be negative).
- The expected utility is the integral of v with respect to F

$$\int v(x) dF(x) = \int v dF$$

- If F is differentiable, the expectation is computed using the density $f = F'$:
$$\int v(x) dF = \int v(x) f(x) dx.$$

von Neumann and Morgenstern Expected Utility

Under some axioms, there exists a utility function U on **distributions** defined as $U(F) = \int v(x) dF$, for some continuous index $v : \mathbb{R} \rightarrow \mathbb{R}$ over **wealth**, such that

$$F \succsim G \iff \int v(x) dF \geq \int v(x) dG$$

- We do not focus on axioms (need a stronger continuity assumption).
- We assume v as a weakly increasing function (more wealth cannot be bad).

Simple Probability

Definition

A **simple probability** distribution π on $X \subset \mathbb{R}$ is specified by:

- a finite subset of X called the support and denoted $supp(\pi)$, and
- for each $x \in supp(\pi)$, $\pi(x) > 0$ (with $\sum_{x \in supp(\pi)} \pi(x) = 1$), and $\pi(x) = 0$ otherwise.

- If we restrict attention to simple probability distributions, then even if X is infinite, only elements with strictly positive probability matter.
- The expected utility is the expected value of v with respect to π

$$\sum_{x \in supp(\pi)} \pi(x)v(x)$$

- One can write more money is better as: for each $x, y \in X$ such that $x > y$ then $\delta_x \succ \delta_y$.
- We can use this setting to think about many applied choice under uncertainty problems like betting and insurance.

Betting and Expected Utility

A Gamble

- Suppose an individual is offered the following gamble (with $a > 0$ and $x \geq 0$):

win $\$ax$ with probability p

lose $\$x$ with probability $1 - p$

- The expected value of this bet is

$$pax + (1 - p)(-x) = [pa + (1 - p)(-1)]x$$

Definition

A bet is **actuarially fair** if it has expected value equal to zero (i.e. $a = \frac{1-p}{p}$); it is **better than fair** if the expected value is positive and **worse than fair** if it is negative.

How does she evaluate this gamble? Use the expected utility model to find out

- If vNM index is $v(\cdot)$ and initial wealth is $W \geq 0$, expected utility is:

$$\underbrace{\overbrace{p}^{\text{probability of winning}}}_{\text{probability of winning}} \underbrace{v(W + ax)}_{\text{utility of wealth if win}} + \underbrace{\overbrace{(1 - p)}^{\text{probability of losing}}}_{\text{probability of losing}} \underbrace{v(W - x)}_{\text{utility of wealth if lose}}$$

Optimal Betting

How much does she want of this bet? Answer by finding the optimal x .

- The consumer solves

$$\max_x pv(W + ax) + (1 - p)v(W - x)$$

- The FOC is

$$pav'(W + ax) = (1 - p)v'(W - x)$$

- rearranging

$$\frac{pa}{(1 - p)} = \frac{v'(W - x)}{v'(W + ax)}$$

- If the bet is fair, the left hand side is 1. Therefore, at an optimum, the right hand side must also be 1.
- If the vNM utility function is strictly increasing and strictly concave ($v' > 0$ and $v'' < 0$), a fair bet can satisfy the FOC only if

$$W + ax = W - x$$

which implies $x = 0$.

- Conclusion: she will take no part of a fair bet.
- What happens with a better than fair bet?
 - At an optimum $v'(W - x) > v'(W + ax)$, hence $W - x < W + ax$ and $x > 0$.
 - What if $v'' = 0$, or $v'' > 0$?

Insurance and Expected Utility

An Insurance Problem

An individual faces a potential accident:

the loss is $\$L$ with probability π

nothing happens with probability $1 - \pi$

Definition

An **insurance contract** establishes an initial premium $\$P$ and then reimburses an amount $\$Z$ if and only if the loss occurs.

Definition

Insurance is **actuarially fair** when its expected cost is zero; it is less than fair when its expected cost is positive.

- The expected cost (to the individual) of an insurance contract equals the premium minus the expected reimbursement

$$P - [\pi(Z) + (1 - \pi)(0)] = P - \pi Z$$

- Fair insurance means

$$P = \pi Z$$

Optimal Insurance

An individual with current wealth $\$W$ and utility function $v(\cdot)$ faces a potential accident $\$L$ with probability π .

Expected Utility and Insurance

If she buys insurance, her expected utility is:

$$\pi v(\underbrace{W - L - P + Z}_{\text{wealth if loss}}) + (1 - \pi) v(\underbrace{W - P}_{\text{wealth if no loss}})$$

- For example, if the loss is fully reimbursed ($Z = L$), this becomes

$$\pi v(W - P) + (1 - \pi) v(W - P) = v(W - P)$$

- She will buy insurance if

$$\underbrace{\pi v(W - L - P + Z) + (1 - \pi) v(W - P)}_{\text{expected utility with insurance}} \geq \underbrace{\pi v(W - L) + (1 - \pi) v(W)}_{\text{expected utility without insurance}}$$

- Find how much coverage she wants (if any) by finding the optimal Z .
 - The answer depends on the premium set by the insurance company P (which could depend on Z) as well as the curvature of the utility function v .

Optimal Insurance and Indifference Curves

Curvature of the Utility Function

The answers to the previous problems depend on the curvature of the utility function v .

- The curvature of v captures important characteristics of preferences in many applied situations.

Risk Aversion

Definitions

The preference relation $\succsim$ is

- **risk averse** if, for all cumulative distribution functions F , $\delta_{\mu_F} \succsim F$.
- **risk loving** if, for all cumulative distribution functions F , $F \succsim \delta_{\mu_F}$.
- **risk neutral** if it is both risk averse and risk loving: $\delta_{\mu_F} \sim F$.

- DM is risk averse if she weakly prefers the finite expected value μ_F for sure to the uncertain distribution F .
- This definition does not depend on the expected utility representation (or any other).

Remark

Risk attitudes are defined directly from preferences.

Risk Aversion: An example

Exercise

Let $\succsim$ be a preference relation on the space of all non-degenerate cumulative distribution functions represented by the following utility function:

$$U(F) = \begin{cases} x & \text{if } F = \delta_x \text{ for some } x \in \mathbb{R} \\ 0 & \text{otherwise} \end{cases}$$

- True or false: $\succsim$ is risk averse.
 - **False:** If $\mu_F < 0$, then $U(F) = 0 > \mu_F = U(\delta_{\mu_F})$. Therefore: $F \succ \delta_{\mu_F}$.
- True or false: $\succsim$ is risk loving.
 - **False:** If $\mu_F > 0$, then...
- Does $\succsim$ admit an expected utility representation?

Certainty Equivalent

Definition

Given a strictly increasing and continuous vNM index v over wealth, the **certainty equivalent** (CE) of F , denoted $c(F, v)$, is defined by

$$v(c(F, v)) = \int v(\cdot) dF.$$

- The certainty equivalent of F is the amount of wealth c such that $\delta_c \sim F$.
 - DM is indifferent between a distribution and the certainty equivalent of that distribution.
 - The certainty equivalent is constructed to satisfy this indifference.
- One can compare two lotteries by comparing their certainty equivalents.
- The certainty equivalent can also be defined from $\succsim$ directly by $\delta_{c(F)} \sim F$. This is well defined if preferences are strictly monotone and continuous.
 - Under expected utility, it is calculated from the representation and it is invariant to strictly positive affine transformations.
- The value of the certainty equivalent is related to risk aversion.

Risk Premium

Definition

Given a strictly increasing and continuous vNM index v over wealth, the **risk premium** of F , denoted $r(F, v)$ is defined by

$$r(F, v) = \mu_F - c(F, v).$$

- This measures the difference between the expected value of a particular distribution and its certainty equivalent.
- The risk premium can also equivalently defined from $\succsim$.
- The risk premium is also related to risk aversion.

Certainty Equivalent and Risk Premium: Figure

Risk Aversion, Certainty Equivalent, and Risk Premium

- If preferences satisfy the vNM axioms, risk aversion is characterized by concavity of the utility index and a non-negative risk-premium.

Proposition

Suppose $\succsim$ has an expected utility representation, and that the corresponding von Neumann and Morgenstern utility index over money v is strictly increasing and continuous. The following are equivalent:

- ① $\succsim$ is risk averse;
- ② v is concave;
- ③ $r(F, v) \geq 0$ for all F ;

- The proof uses Jensen's inequality.

Jensen's Inequality

Reminder: a function $f : \mathbb{R} \rightarrow \mathbb{R}$ is concave if for any $x, y \in \mathbb{R}$ and any $\alpha \in [0, 1]$

$$f(\alpha x + (1 - \alpha)y) \geq \alpha f(x) + (1 - \alpha)f(y)$$

Jensen's inequality

A function g is concave if and only if for all distributions F

$$g\left(\int x dF\right) \geq \int g(x) dF$$

- This says

$$g(\mathbf{E}(X)) \geq \mathbf{E}(g(X))$$

Consequences of Jensen's inequality

- Hence, $v(\cdot)$ is concave if and only if for all distributions F

$$\underbrace{v\left(\int x dF\right)}_{\text{utility of the expected value of } F} \geq \underbrace{\int v dF}_{\text{expected utility of } F}$$

Risk Aversion, CE, and Risk Premium

$$\underbrace{\succsim \text{ is risk averse}}_{(1)} \Leftrightarrow \underbrace{v \text{ is concave}}_{(2)} \Leftrightarrow \underbrace{r(F, v) \geq 0}_{(3)}$$

We prove $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (1)$. Start with $(1) \Rightarrow (2)$ (and note that $(2) \Rightarrow (1)$ is immediate).

Proof.

$\succsim$ is risk averse, hence $\delta_{\mu_F} \succsim F$ for all F , and thus $v(\mu_F) \geq \int v dF$.

- For any $x, y \in \mathbb{R}$ and $\alpha \in [0, 1]$, let the discrete random variable X be such that $P(X = x) = \alpha$ and $P(X = y) = 1 - \alpha$. Let $F_{x,y}^\alpha$ be the associated cumulative distribution.
- By risk aversion we have:

$$\begin{aligned} v(\mu_{F_{x,y}^\alpha}) &\geq \int v(z) dF_{x,y}^\alpha(z) \\ &\Rightarrow \\ v(\alpha x + (1 - \alpha)y) &\geq \sum_z v(z) P(X = z) = \alpha v(x) + (1 - \alpha)v(y) \end{aligned}$$

- Thus v is concave.



Risk Aversion, CE, and Risk Premium

$$\underbrace{\succsim \text{ is risk averse}}_{(1)} \Leftrightarrow \underbrace{v \text{ is concave}}_{(2)} \Leftrightarrow \underbrace{r(F, v) \geq 0}_{(3)}$$

Now prove that (2) $\Rightarrow$ (3)

Proof.

Let v be concave, and X be a random variable with cdf F .

- By Jensen's inequality:

$$v(\mathbf{E}(X)) \geq \mathbf{E}(v(X))$$

or

$$v(\mu_F) \geq \int v(x) dF(x) = v(c(F, v))$$

- Since v is an increasing function, we have

$$\mu_F \geq c(F, v)$$

- Thus

$$\mu_F - c(F, v) = r(F, v) \geq 0$$



Risk Aversion, CE, and Risk Premium

$$\underbrace{\succsim \text{ is risk averse}}_{(1)} \Leftrightarrow \underbrace{v \text{ is concave}}_{(2)} \Leftrightarrow \underbrace{r(F, v) \geq 0}_{(3)}$$

(3) $\Rightarrow$ (1)

Proof.

Let $r(F, v) \geq 0$ for all F .

- Then we have

$$\mu_F \geq c(F, v)$$

- which in turn implies that

$$v(\mu_F) \geq v(c(F, v)) = \int v(x) dF(x)$$

- Hence $\delta_{\mu_F} \succsim F$ for all F ; therefore $\succsim$ is risk averse. ■

We have shown that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1), thus the proof is complete.

Corollary: The Other Risk Attitudes

Corollary

If $\succsim$ has an expected utility representation with v strictly increasing and continuous.

- $\succsim$ is *risk loving* $\iff v$ is convex $\iff r(F, v) \leq 0$ for all F .

Proof.

This is the same as the three proofs, with every inequality reversed: Jensen's inequality flips for convex v , and strict monotonicity turns $v(\mu_F) \leq v(c(F, v))$ into $\mu_F \leq c(F, v)$. ■

Remark

- A v that is concave on one region and convex on another is not risk-averse, not risk-loving, and not risk-neutral.

Next Class

- Relative Risk Aversion
- Stochastic Dominance
- Random Consumption Bundles