Core, Pareto Optimality and Social Welfare Maximization

Econ 3030 Fall 2022

Lecture 2, October 24

Outline

1. The Core
2. Utility possibility set
3. Social welfare function
4. Pareto optimality and social welfare maximization
From Last Class

- Firms $j = 1, \ldots, J$ are described by their production set, $Y_j \subset \mathbb{R}^L$.
- Consumers $i = 1, \ldots, I$ are described by: consumption set $X_i \subset \mathbb{R}^L_+$, preferences $\succ_i$, endowments $\omega_i \in X_i$, and shares $\theta_i \in [0, 1]^J$ of the profits of each firm (with $1 = \sum_i \theta_{ij}$).
  - For a while we will not care about firm ownership because there will be no prices and thus no profits to talk about.
- An allocation: $(x, y) \in \mathbb{R}^{L(I+J)}$ where $x_i \in X_i$ for each $i = 1, \ldots, I$, and $y_j \in Y_j$ for each $j = 1, \ldots, J$.
- An allocation $(x, y)$ is feasible if $\sum_{i=1}^I x_i \leq \sum_{i=1}^I \omega_i + \sum_{j=1}^J y_j$

**Definition**

A feasible allocation $(x, y)$ is **Pareto optimal (efficient)** if there is no other feasible allocation $(x', y')$ such that

$$x'_i \succeq_i x_i$$

for all $i = 1, \ldots, I$ and

$$x'_i \succ_i x_i$$

for some $i$
### Proposition

If the following conditions are satisfied, the set of feasible allocations is closed and bounded.\(^a\)

1. \(X_i\) is closed and bounded below for each \(i = 1, \ldots, I\).\(^b\)
2. \(Y_j\) is closed for each \(j = 1, \ldots, J\).
3. \(Y = \sum_j Y_j\) is convex and satisfies
   - \(0_L \in Y\) (inaction),
   - \(Y \cap R^L_+ \subseteq \{0_L\}\) (no free-lunch), and
   - if \(y \in Y\) and \(y \neq 0_L\) then \(-y \notin Y\) (irreversibility).

Moreover, if \(-R^L_+ \subset Y_j\) (free-disposal) and there are \(x_i \in X_i\) for each \(i\) such that \(\sum_i x_i \leq \omega\), then the set of feasible allocations is also non-empty.

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\(^a\) Closed and bounded means there exists an \(r > 0\) such that for each \(l = 1, \ldots, L\), \(i = 1, \ldots, I\), and \(j = 1, \ldots, J\), we have \(|x_{l,i}| < r\) and \(|y_{l,j}| < r\) for each \((x, y) \in A\).

\(^b\) Bounded below means there exists an \(r > 0\) such that \(x_{l,i} > -r\) for each \(l = 1, \ldots, L\).

- When these conditions are not satisfied, the latter in particular, we may have nothing to talk about. So we take them for granted.
Pareto optimality is not fairness

- Pareto optimal allocations are not necessarily fair:
  - when preferences are monotone, the allocations that give the aggregate endowment to a single individual are Pareto efficient.

- Fairness is hard to tackle and we will mostly ignore it.

- Pareto optimal allocations also disregard what each individual owns to begin with.

- The constraint that individuals should not “lose” is easier to tackle: make sure nobody is better-off alone.
Suppose each individual has veto power over allocations.

Observation
- A consumer could veto allocations that do not improve her initial conditions.
- She rejects allocations that are worse than her initial endowment.
  - She can, for example, refuse to participate in the economy in those cases.

Definition
A feasible allocation \((x, y)\) is individually rational if \(x_i \succeq_i \omega_i\) for all \(i\).

An individually rational allocation represents a trade that does not make anyone worse-off relative to their initial endowment.

Pareto optimal allocations are not necessarily individually rational.

Draw a picture of Pareto optimal allocations that are not individually rational.
Pareto Optimality and Individual Rationality
Coalitions and Blocking

- A different notion of ‘making everyone happy’ considers groups of individuals doing what improves the situation for group members.
- This is easier to define for an exchange economy: let $J = 1$, and $Y_J = -\mathbb{R}^L_+$. 

**Definition**

A coalition is a subset of $\{1, ..., I\}$.

**Definition**

A coalition $S \subset \{1, ..., I\}$ blocks the allocation $x$ if for each $i \in S$ there exist $x'_i \in X_i$ such that

$$x'_i \succ_i x_i \text{ for all } i \in S$$

and

$$\sum_{i \in S} x'_i \leq \sum_{i \in S} \omega_i$$

- A blocking coalition can make all its members better off.
  - One can think of a weaker definition where a coalition benefits at least one of its members strictly, without hurting the others.
- One can define blocking with firms, but things get complicated because one needs to allocate production to coalition members.
The Core

- The core captures the idea that no group of consumers can gain by ‘seceding’ from the economy.

**Definition**

A feasible allocation $x$ is in the core of an economy if there is no coalition that blocks it.

- The idea is that no sub-group of consumers can improve their situation by separating from the economy.
### Easy to Prove Results

- Any allocation in the core of an economy is also Pareto optimal.
  - Obvious since the ‘whole’ (sometimes called ‘grand coalition’ $S = \{1, \ldots, I\}$) is not a blocking coalition.

- Not all Pareto optimal allocations are in the core.
  - Slightly less obvious: $x_i = \omega$ (consumer $i$ gets everything) is Pareto optimal but not in the core.
    - Any assumptions required here?

- In an Edgeworth box, the core is the set of all individually rational Pareto optimal allocations.
  - This is an (easy) homework problem.
  - With more consumers this result does not hold.
    - As the number of consumers grows more coalitions are possible, and more allocations will thus be blocked. Therefore, the core is typically smaller than the set of Pareto optimal allocations that are individually rational.
Characterization of Pareto Optimal Allocations

- How do we know Pareto optimal allocations exist?
- How do we find Pareto optimal allocations?
- To answer these questions, we restrict attention to preference relations that are continuous, complete, and transitive, so that one can work with individuals’ utility functions.
- Existence turns out to be an easy problem.
- Characterization goes through a particular maximization problem: an allocation is Pareto optimal if and only if it maximizes a particular “welfare function” that encompasses everyone’s utility.
Suppose individuals’ preferences are represented by a utility function. Consumer $i$’s utility function is denoted $u_i(x_i)$.

We can rewrite Pareto efficiency and individual rationality as follows.

**Definitions with utility functions**

- A feasible allocation $(x, y)$ is **Pareto optimal** if there is no other feasible allocation $(x', y')$ such that
  
  $u_i(x'_i) \geq u_i(x_i)$ for all $i$ and $u_i(x'_i) > u_i(x_i)$ for some $i$.

- A feasible allocation $(x, y)$ is **individually rational** if
  
  $u_i(x_i) \geq u_i(\omega_i)$ for all $i$. 

Pareto Optimality: Edgeworth Box
In the picture, we find a Pareto optimal allocation whenever the indifference curves have the same slope.

This is the condition that the marginal rates of substitution of the two consumers are equal.

If $u_A(x_{1A}, x_{2A})$ and $u_B(x_{1B}, x_{2B})$ are the consumers’ utility functions, then an allocation is Pareto optimal if

$$\frac{\partial u_A}{\partial x_1} \frac{\partial u_A}{\partial x_2} = \frac{\partial u_B}{\partial x_1} \frac{\partial u_B}{\partial x_2}$$
Pareto Optimality: Representative Agent
In the picture, we find a Pareto optimal allocation when the agent’s indifference curves are tangent to the production possibility set.

This is the condition that the consumer’s marginal rate of substitution is equal to the firm’s marginal rate of transformation.

Let $u_A(x_{1A}, x_{2A})$ be the agent’s utility function, and let $Y = \{(y_L, y_F) \in \mathbb{R}^2 : y_L \leq 0, \text{ and } y_F \leq f(-y_L)\}$ be the production possibility set.

An allocation is Pareto optimal if

$$\frac{\partial u_A}{\partial x_1} \frac{\partial u_A}{\partial x_2} = \frac{df(-y_L)}{dy_L}$$
Do Pareto Optimal Allocations Exist?

- Pareto optimal allocations exist if the set of feasible allocations is well behaved.

**Theorem**

Any economy such that the set of feasible allocations is non-empty, closed, and compact, and such that each $\succeq_i$ is complete, transitive, and continuous, has a Pareto efficient allocation.

**Proof.**

- Let $\mathbb{F}$ be the set of feasible allocations; let $U : \mathbb{F} \rightarrow \mathbb{R}$ be defined by

\[
U(x, y) = \sum_{i=1}^{I} u_i(x_i)
\]

- This is well defined because, by Debreu’s theorem, each $\succeq_i$ is represented by a continuous function $u_i$.
- $U$ is the sum of continuous functions, thus it is also continuous.

- Since $\mathbb{F}$ is compact and nonempty, the Extreme Value Theorem implies that there exists a feasible allocation that maximizes $U$.

- This allocation must be Pareto optimal because if another feasible allocation Pareto dominates it, that allocation must have a larger $U$, implying that someone reaches a higher utility value for that allocation.

$\square$
Utility Possibility Set and Frontier

**Definitions**

The **utility possibility set** is

\[
U = \left\{ (v_1, \ldots, v_I) \in \mathbb{R}^I : \text{there exists a feasible } (x, y) \text{ such that } v_i \leq u(x_i) \text{ for } i = 1, \ldots, I \right\}
\]

The **utility possibility frontier** is

\[
UF = \left\{ (\bar{v}_1, \ldots, \bar{v}_I) \in U : \text{there is no } v \in U \text{ such that } v > \bar{v}\right\}
\]

- Draw a picture for an Edgeworth box economy.
Utility Possibility Set and Frontier: Edgeworth Box
Pareto Efficiency and Utility Possibility Set

**Definitions**

The utility possibility set is

\[ U = \left\{ (v_1, ..., v_I) \in \mathbb{R}^I : \text{there exists a feasible } (x, y) \text{ such that } v_i \leq u(x_i) \text{ for } i = 1, ..., I \right\} \]

The utility possibility frontier is

\[ UF = \{ (\bar{v}_1, ..., \bar{v}_I) \in U : \text{there is no } v \in U \text{ such that } v > \bar{v} \} \]

- The utility possibility frontier is the boundary of \( U \); a Pareto optimal allocation must belong to the frontier.
- In the picture, a point on the frontier can be characterized as the solution to the following optimization problem

\[
\max_{(x,y) \text{ is a feasible allocation}} u_i(x_i) \text{ such that } u_j(x_j) \geq v_j \text{ for } j \neq i
\]

- If an allocation is Pareto efficient then it maximizes the utility of one consumer subject to the constraint that all others get at least some fixed (feasible) amount.
Let $\mathbb{A}$ denote the set of allocations.

**Definition**

A (linear) social welfare function $W : \mathbb{A} \rightarrow \mathbb{R}$ is a weighted sum of the individuals’ utilities:

$$W(x, y) = \sum_{i=1}^{l} \lambda_i u_i(x_i) \quad \text{with} \quad \lambda_i \geq 0$$

- The social welfare maximization problem is

$$\max \sum_{i=1}^{l} \lambda_i u_i(x_i)$$

This is sometimes called the “planner’s problem”.

- Contrast with another planner’s problem: maximize the utility of one consumer, subject to all other consumers getting a predefined utility value.
Pareto Efficiency and Social Welfare

One can use the planner’s problem to find Pareto optimal allocations.

**Theorem**

If the allocation \((\hat{x}, \hat{y})\) is feasible for the economy \(\mathcal{E} = \left\{ \{u_i, \omega_i\}_{i=1}^I, \{Y_j\}_{j=1}^J \right\}\) and solves the problem

\[
\max_{(x,y) \text{ is a feasible allocation}} \sum_{i=1}^I \lambda_i u_i(x_i) \quad \text{where } \lambda_i > 0 \text{ for all } i
\]

then \((\hat{x}, \hat{y})\) is Pareto optimal.

**Proof.**

By contradiction. Suppose \((\hat{x}, \hat{y})\) is not Pareto optimal. Then, it is Pareto dominated by some feasible allocation \((x, y)\).

- Since the \(\lambda_i\) are all strictly positive, we must have
  \[
  \sum_{i=1}^I \lambda_i u_i(x_i) > \sum_{i=1}^I \lambda_i u_i(\hat{x}_i)
  \]

- But this means \((\hat{x}, \hat{y})\) does not maximize \(\sum_{i=1}^I \lambda_i u_i(x_i)\) which is a contradiction.
Pareto Efficiency and Social Welfare

**Theorem**

If the allocation \((\hat{x}, \hat{y})\) is feasible for the economy \(E = \{\{u_i, \omega_i\}_{i=1}^l, \{Y_j\}_{j=1}^I\}\) and solves the problem

\[
\max_{(x,y) \text{ is a feasible allocation}} \sum_{i=1}^l \lambda_i u_i (x_i)
\]

where \(\lambda_i > 0\) for all \(i\), then \((\hat{x}, \hat{y})\) is Pareto optimal.

- Notice that if a feasible allocation maximizes \(\sum_{i=1}^l \lambda_i u_i (x_i)\) it also maximizes \(K \sum_{i=1}^l \lambda_i u_i (x_i)\) where \(K\) is a strictly positive number.
- So, without loss of generality, we can consider the social welfare function

\[
\sum_{i=1}^l \frac{\lambda_i}{\sum_{i=1}^l \lambda_i} u_i (x_i) = \sum_{i=1}^l \hat{\lambda}_i u_i (x_i)
\]

where \(\hat{\lambda}_i \geq 0\) for all \(i\) and \(\sum_{i=1}^l \hat{\lambda}_i = 1\).
- In other words, a Pareto optimal allocation maximizes a weighted sum of individual utilities.
Planner’s Problem: Edgeworth Box Economy

- Draw the utility possibility set and solve the planner’s problem in an Edgeworth box economy.
  - Note that the objective function is linear.
Planner’s Problem: Example

- Draw the utility possibility set and solve the planner’s problem in a Representative Agent economy.
- The objective function is...
Pareto Efficiency and Constrained Planner Problem

**Theorem**

If the allocation \((\hat{x}, \hat{y})\) is feasible for the economy \(E = \left\{ \{u_i, \omega_i\}_{i=1}^l, \{Y_j\}_{j=1}^J \right\}\) and solves the problem

\[
\max_{(x, y) \text{ is a feasible allocation}} \sum_{i=1}^l \lambda_i u_i (x_i) \quad \text{where } \lambda_i > 0 \text{ for all } i
\]

then \((\hat{x}, \hat{y})\) is Pareto optimal.

- In order for this to be a full characterization result, one would like to prove a converse: if an allocation is Pareto optimal, then it must solve some planner's problem.

- This means proving that for any Pareto optimal allocation one can always find some weights that make it a solution to some planner's problem.

- For that kind of result we will need more assumptions, and even with those, one cannot guarantee that everyone gets a non-zero weight in the social welfare function.
Social Welfare Maximization and Pareto Efficiency

- Consider the social welfare function \( W(x, y) = \sum_{i=1}^{I} \lambda_i u_i(x_i) \) where \( \lambda_i \geq 0 \) for all \( i \) and \( (x, y) \) is an allocation.

- Think of \( W(x, y) \) as the composition of two functions:
  - \( U: \mathbb{A} \rightarrow \mathbb{R}^I \), defined as \( U(x, y) = (u_1(x_1), ..., u_I(x_I)) \), where \( \mathbb{A} \) is the set of allocations, and
  - \( f: \mathbb{R}^I \rightarrow \mathbb{R} \), defined as \( f(v) = \sum_{i=1}^{I} \lambda_i v_i \).
    - Using the dot product notation, \( f(v) = \lambda \cdot v \).
  - Then, \( W(x, y) = f(U(x, y)) \).

- The image of the set of feasible allocations under the mapping \( U \) is given by:
  \[ \mathbb{V} = \{ U(x, y) \in \mathbb{R}^I : (x, y) \text{ is a feasible allocation} \} \]

**Remark**

- \( \mathbb{U} \) and \( \mathbb{V} \) are not the same set:
  The utility possibility set is equal to \( \mathbb{V} \) plus all the points dominated by points in \( \mathbb{V} \)
- \( \mathbb{U} \) is a larger set.
Proposition

The allocation \((\hat{x}, \hat{y})\) solves the problem

\[
\max_{(x,y) \text{ is a feasible allocation}} W(x, y)
\]

if and only if the vector \(\hat{v} = U(\hat{x}, \hat{y})\) solves the problem

\[
\max_{\forall \in \mathbb{V}} \lambda \cdot \nu.
\]

Proof.

Problem set 2, Question 2.

- Given this proposition, any Pareto optimal allocation maximizes \(\lambda \cdot \nu\) over the set \(\mathbb{V}\).

- Under what conditions can we say that for any Pareto optimal allocation there exists a vector \(\lambda\) such that \(\hat{v}\) maximizes the desired dot product in that set?

- The tool that yields this result is (a version of) the separating hyperplane theorem.
Next Class

- Separating Hyperplane Theorem refresher.
- Any Pareto optimal allocation maximizes some social welfare function.