Existence of Competitive Equilibrium

Outline

1. Existence of a competitive (Walrasian) equilibrium
The market excess demand correspondence $z : \mathbb{R}_+^L \rightarrow \mathbb{R}^L$ is:

$$z(p) = \sum_{i=1}^{I} (x_i^* - \omega_i) - \sum_{j=1}^{J} y_j^*$$

where $y_j^* \in y_j^*(p)$ for each $j = 1, ..., J$, and $x_i^* \in x_i^*(p)$ for each $i = 1, ..., I$.

An equilibrium price vector $p^* > 0$ satisfies:

$$z_l(p^*) \leq 0 \text{ for all } l, \quad \text{and whenever } z_l(p^*) < 0 \text{ then } p_l^* = 0$$

Walras’ Law: for any price vector $p \in \mathbb{R}_+^L$:

$$p \cdot z(p) = 0$$

**REMARK**

If Walras’ Law holds, $p^* > 0$ is an equilibrium if and only if $z(p^*) \leq 0$. Why?

$$z_l(p^*) \leq 0 \text{ for all } l, \quad \text{and } \sum_{l=1}^{L} p_l^* z_l(p^*) = 0$$

thus $p_i^*$ must be zero if $z_l(p^*) < 0$. 
Equilibrium As A Fixed Point

Summary

An equilibrium price vector $p^*$ must satisfy:

$$z_l(p^*) \leq 0 \text{ for all } l, \quad \text{and whenever } z_l(p^*) < 0 \text{ then } p_l^* = 0$$

A useful observation

Let the function $g : \mathbb{R}^L \rightarrow \mathbb{R}^L$ be defined by

$$g_l(p) = \max \{p_l + z_l(p), 0\} \quad \text{for } l = 1, 2, \ldots, L$$

**Claim**: An equilibrium is a $p^* \geq 0$ such that

$$g(p^*) = p^* \quad \text{or} \quad g_l(p^*) = p_l^* \text{ for all } l$$

At an equilibrium price vector $p^*$:

- either $g_l(p^*) = p_l^* = 0$, and thus either $z_l(p^*) = 0$ or $z_l(p^*) < 0$
- $p_l^* = p_l^* 
eq 0$, and thus $p_l^* + z_l(p^*) = p_l^*$ which implies $z_l(p^*) = 0$.

In both cases we have an equilibrium thus establishing the claim.
An equilibrium price vector $p^*$ must satisfy:

$$z_I(p^*) \leq 0 \text{ for all } I, \quad \text{and whenever } z_I(p^*) < 0 \text{ then } p^*_I = 0$$

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- CLAIM: An equilibrium is a $p^* \geq 0$ such that

$$g(p^*) = p^*$$

At an equilibrium price vector $p^*$:

- either $g_I(p^*) = p^*_I = 0$, and thus either $z_I(p) > 0$ or $z_I(p) = 0$;
- or $g_I(p^*) = p^*_I = 0$, and thus $p^*_I = z_I(p) = 0$, which implies $z_I(p^*) = 0$. 
Equilibrium As A Fixed Point

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An equilibrium price vector $p^*$ must satisfy:
$$z_l(p^*) \leq 0 \text{ for all } l, \quad \text{and whenever } z_l(p^*) < 0 \text{ then } p^*_l = 0$$

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At an equilibrium price vector $p^*$:
- either $g(p^*) = p^* = 0$, and thus either $z_l(p^*) = 0$ or $z_l(p^*) = 0$
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At an equilibrium price vector $p^*$:

- Either $z_l(p^*) = 0$ and thus either $z_l(p^*) = 0$ or $z_l(p^*) = 0$
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An equilibrium price vector $p^*$ must satisfy:

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An equilibrium price vector \( p^* \) must satisfy:
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● Let the function \( g : \mathbb{R}^L \rightarrow \mathbb{R}^L \) be defined by
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● CLAIM: An equilibrium is a \( p^* \geq 0 \) such that
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g(p^*) = p^* \quad \text{or} \quad g_l(p^*) = p_l^* \quad \text{for all} \quad l
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At an equilibrium price vector \( p^* \):
1. either \( g_i(p^*) = p_i^* = 0 \), and thus either \( z_i(p^*) < 0 \) or \( z_i(p^*) = 0 \)
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An equilibrium exists if there exists a \( p^* \) such that
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This is a fixed point: we want to show that \( g(p) \) must have a fixed point.

We need a theorem that gives conditions for functions to have a fixed point.
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- An equilibrium exists if there exists a $p^*$ such that
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Brouwer’s Fixed Point Theorem

**Theorem (Brouwer’s Fixed Point Theorem)**

If $X \subseteq \mathbb{R}^L$ is convex and compact and the function $f : X \to X$ is continuous, then there exists an $x \in X$ such that $f(x) = x$ (that is, $f$ has a fixed point).

- **Counterexamples:**
  - $X = [0, 1]$ and $f(x) = \frac{1}{2}$

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  - \( X = [0,1] \) and \( f(x) = \begin{cases} 1 & \text{if } 0 \leq x < 0.5 \\ 0 & \text{if } 0.5 < x \leq 1 \end{cases} \) \( (f \) must be continuous)  

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- $X = \{x \in \mathbb{R}^2 : |x| = 1\}$ and $f(x) = x$  
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- In general, we want to allow for correspondences (aggregate excess demand may not be a function);
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A continuous \( f : S \to S \), where \( S \subseteq \mathbb{R}^L \) is convex and compact, has a fixed point.

How can we use this theorem to show that an equilibrium exist?

Let \( g : \mathbb{R}^L \to \mathbb{R}^L \) be defined by: \( g_l(p) = \max \{ p_l + z_l(p), 0 \} \) for \( l = 1, 2, \ldots, L \). Can we apply Brouwer?

1. Need domain and range to be the same convex and compact subset of \( \mathbb{R}^L \).
2. Normalize prices so that the domain is \( \Delta^L = \{ p \in \mathbb{R}^L : \sum p_l = 1 \} \), and then divide \( g(\cdot) \) by the sum of its elements so that the range is also \( \Delta^L \).
3. \( g(\cdot) \) needs to be continuous.
   Assume preferences that yield continuity of excess demand.
4. \( g(\cdot) \) needs to be a function.
   Assume strictly convex preferences (there is a theorem for correspondences).
5. \( g(\cdot) \) must be well defined even if some prices are zero.
   If preferences are monotone, excess demand can blow up. But we need nonmonotonicity for other properties. Assume this away (for now).
Brouwer’s Theorem and Existence

**Theorem**

A continuous \( f : S \rightarrow S \), where \( S \subseteq \mathbb{R}^L \) is convex and compact, has a fixed point.

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1. Need domain and range to be the same convex and compact subset of $\mathbb{R}^L$.

   Normalize prices so that the domain is $\Delta^{L-1} = \{p \in \mathbb{R}_+^L : \sum_{i=1}^{L} p_i = 1\}$, and then divide $g(\cdot)$ by the sum of its elements so that the range is also $\Delta^{L-1}$.  


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   Normalize prices so that the domain is \( \Delta^{L-1} = \{ p \in \mathbb{R}^L_+ : \sum_{l=1}^L p_l = 1 \} \), and then divide \( g(\cdot) \) by the sum of its elements so that the range is also \( \Delta^{L-1} \).

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Assume that aggregate excess demand $z : \Delta^{L-1} \rightarrow \mathbb{R}^L$ is a continuous function such that $p \cdot z(p) = 0$ for all $p$. Then, there exists $p^* \in \Delta^{L-1}$ such that $z(p^*) \leq 0$.

- If excess demand is a continuous function that satisfies Walras' Law, an equilibrium exists.

Remark

We should prove existence from assumptions on the primitives of the economy (technology, preferences, and endowments), not on excess demand.

The proof goes as follows

1. Define a function of excess demand so that...
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‘Easy’ Existence Proof: Step 1a

Let $g : \Delta^{L-1} \to \mathbb{R}^L$ be defined by

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\begin{align*}
g_l(p) &= \max \{p_l + z_l(p), 0\} \quad \text{with } l = 1, 2, \ldots, L.
\end{align*}
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Remark

Claim: $g(p) \neq 0$.

- By definition, $g_l(p) \geq p_l + z_l(p)$.
- Thus

$$
\begin{align*}
p \cdot g(p) &\geq p \cdot (p + z(p)) \\
&= p \cdot p + p \cdot z(p) \\
&= p \cdot p + 0 \quad \geq 0
\end{align*}
$$

by Walras’ Law because $p \in \Delta^{L-1}$

- Since there must be some good that has a positive price, $g_l(p)$ cannot be equal to zero for all $l$.
- Thus $\max \{p_l + z_l(p), 0\} > 0$ for at least one $l$. 

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- Since there must be some good that has a positive price, \( g_l(p) \) cannot be equal to zero for all \( l \).
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- Since there must be some good that has a positive price, $g_l(p)$ cannot be equal to zero for all $l$.
  - Thus $\max \{ p_l + z_l(p), 0 \} > 0$ for at least one $l$. 
Define \( h : \Delta^{L-1} \rightarrow \Delta^{L-1} \) as
\[
h(p) = \frac{g(p)}{\sum_{i=1}^{L} g_i(p)}
\]

- \( h(\cdot) \) is well defined.
- \( h(\cdot) \) is continuous because \( z(\cdot) \) is continuous and thus \( g(\cdot) \) is continuous.
- \( h(\cdot) \) maps from \( \Delta^{L-1} \), a convex and compact set, to itself.

Therefore
\( h(\cdot) \) is a continuous function from a compact and convex set to itself; by Brouwer's theorem, it has a fixed point.
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‘Easy’ Existence Proof: Step 1b

Define $h : \Delta^{L-1} \rightarrow \Delta^{L-1}$ as

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- $h(\cdot)$ is continuous because $z(\cdot)$ is continuous and thus $g(\cdot)$ is continuous.
- $h(\cdot)$ maps from $\Delta^{L-1}$, a convex and compact set, to itself.

**Therefore**

$h(\cdot)$ is a continuous function from a compact and convex set to itself; by Brouwer’s theorem, it has a fixed point.
By Brouwer’s fixed point theorem, there exists a $p^*$ such that

$$p^* = h(p^*) = \frac{g (p^*)}{\sum_{l=1}^{L} g_l (p^*)}$$

- Rewrite this as

$$g (p^*) = \left( \sum_{l=1}^{L} g_l (p^*) \right) p^* = \gamma p^*$$

for some real number $\gamma$.

- Observe that $\gamma \neq 0$ because $g (p) \neq 0$.

- Next, we show that $\gamma = 1$. 
‘Easy’ Existence Proof: Step 2

By Brouwer’s fixed point theorem, there exists a $p^*$ such that

$$ p^* = h(p^*) = \frac{g(p^*)}{\sum_{l=1}^{L} g_l(p^*)} $$

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for some real number $\gamma$.

- Observe that $\gamma \neq 0$ because $g(p) \neq 0$.

- Next, we show that $\gamma = 1$. 
Claim: for each $l = 1, \ldots, L$

$$p^*_l g_l (p^*) = p^*_l (p^*_l + z_l (p^*))$$

- This is easy to show:

  if $g_l (p^*) \neq p^*_l + z_l (p^*) \implies g_l (p^*) = 0$

  by definition

- Summing over $l$, we obtain:

  $$p^* \cdot g (p^*) = p^* \cdot (p^* + z (p^*)) = p^* \cdot p^* + p^* \cdot z (p^*) = p^* \cdot p^* + 0$$

  by Walras’ Law

- By the existence of a fixed point (last slide), we know that $g (p^*) = p^* \gamma$;
  taking the dot product with $p^*$ on both sides:

  $$p^* \cdot g (p^*) = p^* \cdot p^* \gamma$$

- Since we have just shown that $p^* \cdot g (p^*) = p^* \cdot p^*$, we have

  $$p^* \cdot p^* = p^* \cdot p^* \gamma$$

  and thus $\gamma = 1$ as desired.
Claim: for each \( l = 1, \ldots, L \)

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p^*_l g_l (p^*) = p^*_l (p^*_l + z_l (p^*))
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This is easy to show:

if \( g_l (p^*) \neq p^*_l + z_l (p^*) \) \( \Rightarrow \) \( g_l (p^*) = 0 \)

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Thus: either \( g_l (p^*) = p^*_l + z_l (p^*) \) or \( p^*_l = 0 \); in both cases the claim holds.

Summing over \( l \), we obtain:

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‘Easy’ Existence Proof: Step 3a

**Claim:** for each $l = 1, \ldots, L$

$$p_l^* g_l (p^*) = p_l^* (p_l^* + z_l (p^*))$$

- This is easy to show:
  
  if $g_l (p^*) \neq p_l^* + z_l (p^*)$  \implies  $g_l (p^*) = 0$  \implies  $p_l^* = 0$
  
  by definition
  
  by $\gamma \neq 0$

- Thus: either $g_l (p^*) = p_l^* + z_l (p^*)$ or $p_l^* = 0$; in both cases the claim holds.

- Summing over $l$, we obtain:
  
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  $$p^* \cdot g (p^*) = p^* \cdot (p^* + z (p^*)) = p^* \cdot p^* + p^* \cdot z (p^*) = p^* \cdot p^* + 0$$
  
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- By the existence of a fixed point (last slide), we know that $g (p^*) = p^*$.
  
  taking the dot product with $p^*$ on both sides:
  
  $$p^* \cdot g (p^*) = p^* \cdot p^*$$

  Since we have just shown that $p^* \cdot g (p^*) = p^* \cdot p^*$, we have
  
  $$p^* \cdot p^* = p^* \cdot p^*$$

  and thus $\gamma = 1$ as desired.
Claim: for each \( l = 1, \ldots, L \)

\[
p^*_l g_l (p^*) = p^*_l (p^*_l + z_l (p^*))
\]

- This is easy to show:
  
  if \( g_l (p^*) \neq p^*_l + z_l (p^*) \) \( \Rightarrow \) \( g_l (p^*) = 0 \) \( \Rightarrow \) \( p^*_l = 0 \)

  - by definition
  - by \( \gamma \neq 0 \)

- Thus: either \( g_l (p^*) = p^*_l + z_l (p^*) \) or \( p^*_l = 0 \); in both cases the claim holds.

Summing over \( l \), we obtain:

\[
p^* \cdot g (p^*) = p^* \cdot (p^* + z (p^*)) = p^* \cdot p^* + p^* \cdot z (p^*) = p^* \cdot p^* + 0
\]

- By Walras' Law

By the existence of a fixed point (last slide), we know that \( g (p^*) = p^* \gamma \); taking the dot product with \( p^* \) on both sides:

\[
p^* \cdot g (p^*) = p^* \cdot p^*
\]

Since we have just shown that \( p^* \cdot g (p^*) = p^* \cdot p^* \), we have

\[
p^* \cdot p^* = p^* \cdot p^*
\]

and thus \( \gamma = 1 \) as desired.
Claim: for each $l = 1, \ldots, L$

$$p_l^* g_l (p^*) = p_l^* (p_l^* + z_l (p^*))$$

- This is easy to show:
  
  $$\text{if } g_l (p^*) \neq p_l^* + z_l (p^*) \Rightarrow g_l (p^*) = 0 \Rightarrow p_l^* = 0$$

  by definition

  by $\gamma \neq 0$

- Thus: either $g_l (p^*) = p_l^* + z_l (p^*)$ or $p_l^* = 0$; in both cases the claim holds.

- Summing over $l$, we obtain:

$$p^* \cdot g (p^*) = p^* \cdot (p^* + z (p^*)) = p^* \cdot p^* + p^* \cdot z (p^*) = p^* \cdot p^* + 0$$

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Claim: for each $l = 1, \ldots, L$

$$p^*_l g_l (p^*) = p^*_l (p^*_l + z_l (p^*))$$

- This is easy to show:
  
  if $g_l (p^*) \neq p^*_l + z_l (p^*)$  \[ \Rightarrow \]  $g_l (p^*) = 0$  \[ \Rightarrow \]  $p^*_l = 0$

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- Thus: either $g_l (p^*) = p^*_l + z_l (p^*)$ or $p^*_l = 0$; in both cases the claim holds.

Summing over $l$, we obtain:

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**Claim:** for each \( l = 1, \ldots, L \)

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p_i^* g_l (p^*) = p_i^* (p_i^* + z_l (p^*))
\]

- This is easy to show:

\[
\text{if } g_l (p^*) \neq p_i^* + z_l (p^*) \quad \Rightarrow \quad g_l (p^*) = 0 \quad \Rightarrow \quad p_i^* = 0
\]

  by definition \hspace{1cm} \text{by } \gamma \neq 0

- Thus: either \( g_l (p^*) = p_i^* + z_l (p^*) \) or \( p_i^* = 0 \); in both cases the claim holds.

Summing over \( l \), we obtain:

\[
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Claim: for each \( l = 1, \ldots, L \)

\[
p^*_l g_l (p^*) = p^*_l (p^*_l + z_l (p^*))
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This is easy to show:

\[
\text{if } g_l (p^*) \neq p^*_l + z_l (p^*) \quad \Rightarrow \quad g_l (p^*) = 0 \quad \Rightarrow \quad p^*_l = 0
\]

by definition \( \gamma \neq 0 \)

Thus: either \( g_l (p^*) = p^*_l + z_l (p^*) \) or \( p^*_l = 0 \); in both cases the claim holds.

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\[
p^* \cdot p^* = p^* \cdot p^* \gamma
\]

and thus \( \gamma = 1 \) as desired.
Summary
By the fixed point theorem, \( g(p^*) = p^*\gamma \), and since \( \gamma = 1 \): \( g(p^*) = p^* \).

The fixed point is an equilibrium

Claim: \( p^* \) is an equilibrium.

Proof.
- The equation above implies:
  \[
p_i^* = g_i(p^*)
  \]
Summary

By the fixed point theorem, \( g(p^*) = p^* \gamma \), and since \( \gamma = 1 \): \( g(p^*) = p^* \).

The fixed point is an equilibrium

Claim: \( p^* \) is an equilibrium.

Proof.

- The equation above implies:
  \[
  p_i^* = g_i(p^*) = \max \left\{ p_i^* + z_i(p^*), 0 \right\}
  \]
  for \( l = 1, 2, \ldots, L \)
  
  by definition of \( g(\cdot) \)
By the fixed point theorem, \( g(p^*) = p^* \gamma \), and since \( \gamma = 1 \): \( g(p^*) = p^* \).

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**Claim:** \( p^* \) is an equilibrium.

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- The equation above implies:
  \[
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  for \( i = 1, 2, ..., L \)
  
  by definition of \( g(\cdot) \)

- Therefore:
  \[
  z_i(p^*) \leq 0 \quad \text{for} \quad i = 1, 2, ..., L
  \]
Summary

By the fixed point theorem, \( g(p^*) = p^* \gamma \), and since \( \gamma = 1 \): \( g(p^*) = p^* \).

The fixed point is an equilibrium

Claim: \( p^* \) is an equilibrium.

Proof.

- The equation above implies:
  \[
  p^*_l = g_l(p^*) = \max \{ p^*_l + z_l(p^*), 0 \}
  \]
  for \( l = 1, 2, \ldots, L \)
  by definition of \( g(\cdot) \)

- Therefore:
  \[
  z_l(p^*) \leq 0
  \]
  for \( l = 1, 2, \ldots, L \)

- If not, there exists some \( k \) such that
  \[
  z_k(p^*) > 0
  \]
  and
  \[
  p^*_k = \max \{ p^*_k + z_k(p^*), 0 \}
  \]
Summary
By the fixed point theorem, \( g(p^*) = p^*\gamma \), and since \( \gamma = 1 \): \( g(p^*) = p^* \).

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Claim: \( p^* \) is an equilibrium.

Proof.

- The equation above implies:
  \[ p^*_l = g_l(p^*) = \max \{ p^*_l + z_l(p^*), 0 \} \quad \text{for } l = 1, 2, \ldots, L \]
  by definition of \( g(\cdot) \)

- Therefore:
  \[ z_l(p^*) \leq 0 \quad \text{for } l = 1, 2, \ldots, L \]

  - If not, there exists some \( k \) such that
    \[ z_k(p^*) > 0 \]
    and
    \[ p^*_k = \max \{ p^*_k + z_k(p^*), 0 \} \]
‘Easy’ Existence Proof: Step 3b

Summary
By the fixed point theorem, \( g(p^*) = p^* \gamma \), and since \( \gamma = 1 \): \( g(p^*) = p^* \).

The fixed point is an equilibrium

**Claim:** \( p^* \) is an equilibrium.

**Proof.**

- The equation above implies:
  \[
  p_i^* = g_I(p^*) = \max \{p_i^* + z_I(p^*), 0\} \quad \text{for } l = 1, 2, \ldots, L
  \]
  by definition of \( g(\cdot) \)

- Therefore:
  \[
  z_l(p^*) \leq 0 \quad \text{for } l = 1, 2, \ldots, L
  \]

- If not, there exists some \( k \) such that
  \[
  z_k(p^*) > 0
  \]
  and
  \[
  p_k^* = \max \{p_k^* + z_k(p^*), 0\} = p_k^* + z_k(p^*)
  \]
  an impossibility.
‘Easy’ Existence Proof: Step 3b

Summary
By the fixed point theorem, \( g(p^*) = p^*\gamma \), and since \( \gamma = 1 \): \( g(p^*) = p^* \).

The fixed point is an equilibrium

Claim: \( p^* \) is an equilibrium.

Proof.
- The equation above implies:
  \[
p_i^* = g_i(p^*) = \max \{ p_i^* + z_i(p^*), 0 \}
  \] for \( i = 1, 2, \ldots, L \)
  by definition of \( g(\cdot) \)

- Therefore:
  \[
z_i(p^*) \leq 0 \quad \text{for} \quad i = 1, 2, \ldots, L
  \]

- If not, there exists some \( k \) such that
  \[
z_k(p^*) > 0
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Summary

By the fixed point theorem, \( g(p^*) = p^* \gamma \), and since \( \gamma = 1 \): \( g(p^*) = p^* \).

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Claim: \( p^* \) is an equilibrium.

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  \]
  
  by definition of \( g(\cdot) \)

- Therefore:
  
  \[ z_l(p^*) \leq 0 \quad \text{for } l = 1, 2, \ldots, L \]

  - If not, there exists some \( k \) such that
    
    \[ z_k(p^*) > 0 \]

  - and
    
    \[ p_k^* = \max \{ p_k^* + z_k(p^*), 0 \} = p_k^* + z_k(p^*) \]

    an impossibility.

  Since \( p^* \) is an equilibrium if and only if \( z(p^*) \leq 0 \), we are done.
'Easy' Existence Proof: Step 3b

Summary
By the fixed point theorem, \( g(p^*) = p^* \gamma \), and since \( \gamma = 1 \): \( g(p^*) = p^* \).

The fixed point is an equilibrium

Claim: \( p^* \) is an equilibrium.

Proof.
- The equation above implies:
  \[
  p_i^* = g_i(p^*) = \max \left\{ p_i^* + z_i(p^*), 0 \right\}
  \]
  for \( l = 1, 2, \ldots, L \)
  by definition of \( g(\cdot) \).
- Therefore:
  \[
  z_l(p^*) \leq 0 \quad \text{for } l = 1, 2, \ldots, L
  \]
- If not, there exists some \( k \) such that
  \[
  z_k(p^*) > 0
  \]
  and
  \[
  p_k^* = \max \left\{ p_k^* + z_k(p^*), 0 \right\} = p_k^* + z_k(p^*)
  \]
  an impossibility.
- Since \( p^* \) is an equilibrium if and only if \( z(p^*) \leq 0 \), we are done.
In 1950, John Nash proved existence of the (Nash) equilibrium of a game.
This was the breakthrough needed by Arrow, Debreu, and McKenzie to prove existence of a competitive equilibrium shortly thereafter.
A “game” is a situation in which an individual’s payoff may depend on others’ choices.
Games and Nash Equilibrium

- A game is defined by describing choices and payoffs for each player.

- Each $i = 1, ..., I$ chooses a strategy in the set $S_i$;
  
  Let $S = S_1 \times S_2 \times ... \times S_I$, and $s = (s_1, ..., s_I) \in S$ is a strategy profile (an action for each player).

- $u_i(s)$ is player $i$'s payoff function from $s$.

- A Nash equilibrium is a strategy profile such that: each player maximizes her payoff given the other players' strategies.

- Thus, a Nash Equilibrium is an $s^*$ such that for all $i$
  \[ u_i(s_1^*, ..., s_i^*, ..., s_I^*) \geq u_i(s_1^*, ..., s_{i-1}^*, t, s_{i+1}^*, ..., s_I^*) \quad \text{for any} \ t \in S_i, \]
  or
  \[ u_i(s_i^*, s_{-i}^*) \geq u_i(t, s_{-i}^*) \quad \text{for any} \ t \in S_i. \]

- Nash proved that such a strategy profile exists in any game when (i) the strategy sets are compact and convex, and (ii) payoff are strictly quasi-concave and continuous functions. How? Using Brower's Theorem.
A game is defined by describing choices and payoffs for each player.

Each player $i = 1, ..., I$ chooses a strategy in the set $S_i$,

- let $S = S_1 \times S_2 \times \ldots \times S_I$, and $s = (s_1, ..., s_I) \in S$ is a strategy profile (an action for each player);

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A Nash equilibrium is a strategy profile such that: each player maximizes her payoff given the other players' strategies.

Thus, a Nash Equilibrium is an $s^*$ such that for all $i$

$$u_i(s_1^*, ..., s_i^*, s_{i+1}^*, \ldots) \geq u_i(s_1^*, \ldots, s_i^*_{-1}, t, s_{i+1}^*, \ldots, s_i^*)$$

for any $t \in S_i$.

or

$$u_i(s_i^*, s_{-i}^*) \geq u_i(t, s_{-i}^*)$$

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Nash proved that such a strategy profile exists in any game when (i) the strategy sets are compact and convex, and (ii) payoff are strictly quasi-concave and continuous functions. How? Using Brower’s Theorem.
A game is defined by describing choices and payoffs for each player.

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- let \( S = S_1 \times S_2 \times \ldots \times S_I \), and \( s = (s_1, \ldots, s_I) \in S \) is a strategy profile (an action for each player);

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\[
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A game is defined by describing choices and payoffs for each player.

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A **Nash equilibrium** is a strategy profile such that: each player maximizes her payoff given the other players' strategies.

Thus, a Nash Equilibrium is an \( s^* \) such that for all \( i \)
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    u_i(s_1^*, ..., s_i^*, ..., s_I^*) \geq u_i(s_1^*, ..., s_{i-1}^*, t, s_{i+1}^*, ..., s_I^*) \quad \text{for any } t \in S_i.
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Nash proved that such a strategy profile exists in any game when (i) the strategy sets are compact and convex, and (ii) payoff are strictly quasi-concave and continuous functions. How? Using Brouwer's Theorem.
A game is defined by describing choices and payoffs for each player.

Each player, $i = 1, ..., l$, chooses a strategy in the set $S_i$.

Let $S = S_1 \times S_2 \times \ldots \times S_l$, and $s = (s_1, s_2, \ldots, s_l) \in S$ is a strategy profile (an action for each player);

$u_i(s)$ is player $i$'s payoff function from $s$.

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Nash proved that such a strategy profile exists in any game when (i) the strategy sets are compact and convex, and (ii) payoff are strictly quasi-concave and continuous functions. How? Using Brower's Theorem.
Nash’s Existence Theorem

A Nash equilibrium is a fixed point

- For each $i$, let $BR_i : S_{-i} \to S_i$ be defined as follows:
  $$BR_i(s_{-i}) = \{s_i \in S_i : u_i(s_i, s_{-i}) \geq u_i(t, s_{-i}) \quad \forall t \in S_i\}$$

- Define the mapping $BR : S \to S$ as
  $$BR(s) = BR_1(s_{-1}) \times \ldots \times BR_I(s_{-I})$$

- $s^*$ is a Nash equilibrium if and only if $s^* = BR(s^*)$: a fixed point.

Assumptions (i) and (ii) from the previous slide are enough to use Brouwer’s Theorem.
- We need $S$ closed, bounded, and convex; so, assume each $S_i$ has those properties.
- We need $BR_i(s)$ to be a continuous function; so, assume each $u_i(s)$ is continuous and strictly quasi-concave, so that each $BR_i(s)$ is a continuous function.

Under these assumption, every game has a Nash equilibrium.
Nash’s Existence Theorem

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  \[ BR_i(s_{-i}) = \{ s_i \in S_i : u_i(s_i, s_{-i}) \geq u_i(t, s_{-i}) \ \forall t \in S_i \} \]
  - This function describes $i$’s ‘best response’ to the other players strategy $s_{-i}$.

- Define the mapping $BR : S \to S$ as
  \[ BR(s) = BR_1(s_{-1}) \times \ldots \times BR_I(s_{-I}) \]
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  \]
  - this function describes \( i \)'s ‘best response’ to the other players strategy \( s_{-i} \).

- Define the mapping \( BR : S \to S \) as
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  BR (s) = BR_1 (s_{-1}) \times \ldots \times BR_I (s_{-I})
  \]
  - \( s^* \) is a Nash equilibrium if and only if \( s^* = BR (s^*) \): a fixed point.

Assumptions (i) and (ii) from the previous slide are enough to use Brouwer’s Theorem.

- we need \( S \) closed, bounded, and convex, so, assume each \( S_i \) has these properties.
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A Nash equilibrium is a fixed point

- For each $i$, let $BR_i : S_{-i} \rightarrow S_i$ be defined as follows:
  
  $$BR_i(s_{-i}) = \{s_i \in S_i : u_i(s_i, s_{-i}) \geq u_i(t, s_{-i}) \quad \forall t \in S_i\}$$

  - this function describes $i$’s ‘best response’ to the other players’ strategy $s_{-i}$.

- Define the mapping $BR : S \rightarrow S$ as
  
  $$BR(s) = BR_1(s_{-1}) \times \ldots \times BR_i(s_{-i})$$

- $s^*$ is a Nash equilibrium if and only if $s^* = BR(s^*)$: a fixed point.

  - In a Nash equilibrium, every player chooses a best response.

Assumptions (i) and (ii) from the previous slide are enough to use Brouwer’s Theorem:

- we need $S$ closed, bounded, and convex, so, assume each $S_i$ has those properties.

- we need $BR(s)$ to be a continuous function, so, assume each $u_i(s)$ is continuous and strictly quasi-concave, so that each $BR_i(s)$ is a continuous function.

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Nash’s Existence Theorem

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From Nash to Walras

- Arrow, Debreu, and McKenzie saw Nash’s paper and used it to solve the existence problem. How?

Build a ‘game’ that describes a Walrasian (competitive) equilibrium

- The players are consumers, firms, and a “Price Player”.
- The price player chooses a price vector $p$.
- Each $i$’s payoff function is $u_i(x_i)$, consumer $i$’s utility function.
- Each $j$’s payoff function is $p_j y_j$.
- The price player’s payoff is the value of the aggregate excess demand:

$$u_{PP}(p, x) = p \cdot x = p \cdot \left( \sum_i (x_i^d - x_i) \right) - \sum_i x_i^d$$

- The price player’s best response is the price that maximizes the value of that excess demand.
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  u_{PP}(p, x) = p \cdot z = p \cdot \left( \sum_{i=1}^{I} (x_i^* - \omega_i) - \sum_{j=1}^{J} y_{1}^* \right)
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  - This must be continuous with respect to $x_i$.
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A Nash equilibrium is a fixed point

- By Nash’s existence theorem, there exist a $p^* \in \Delta^{L-1}$, an $x^*_i(p^*)$ for each $i$, and an $y^*_j(p^*)$ for each $j$ such that
  - each $y^*_j(p^*)$ maximizes profits given $p^*$,
  - each $x^*_i(p^*)$ maximizes individuals’ utility given $p^*$ and $y^*_j$, and
  - $p^*$ maximizes the value of aggregate excess demand (given $x^*_i$ and $y^*_j$).

Claim: this is a Walrasian equilibrium

Why is $p^*$ a competitive equilibrium?

Remarks

- To use Brouwer’s theorem, best responses must be functions.
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  - even if aggregate excess demand is a function, $BR_{pp} (\cdot)$ can be multi-valued.
  - We need fixed point existence for correspondences: Kakutani’s theorem.
A Nash equilibrium is a fixed point

- By Nash’s existence theorem, there exist a $p^* \in \Delta^{L-1}$, an $x_i^*(p^*)$ for each $i$, and an $y_j^*(p^*)$ for each $j$ such that
  - each $y_j^*(p^*)$ maximizes profits given $p^*$,
  - each $x_i^*(p^*)$ maximizes individuals’ utility given $p^*$ and $y_j^*$, and
  - $p^*$ maximizes the value of aggregate excess demand (given $x_i^*$ and $y_j^*$).

Claim: this is a Walrasian equilibrium

- Why is $p^*$ a competitive equilibrium?
  - Walras’ Law ($0 = p \cdot z(p)$) implies $p^* \cdot z(p^*) = 0$.
  - The price player maximization implies $p^* \cdot z(p^*) \geq p \cdot z$ for all $p \in \Delta^{L-1}$.
  - These together imply $z(p^*) \leq 0$ (make sure you convince yourself of this).
  - We already proved that $z(p^*) \leq 0$ implies $p^*$ is an equilibrium.

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An General Existence Theorem

**Theorem**

Suppose an economy satisfies the following properties.

1. For each $i$: $X_i \subset \mathbb{R}^L$ is closed and convex.
2. For each $i$: $\succ_i$ satisfies local non-satiation, and convexity $\omega_i \succeq \hat{x}_i$ for some $\hat{x}_i \in X_i$.
3. For each $j$: $Y_j \subset \mathbb{R}^L$ is closed, convex, includes the origin, and satisfies free-disposal.
4. The set of feasible allocations is compact.

Then a Walrasian quasi-equilibrium exists (if $\omega_i \gg \hat{x}_i$ for all $i$ then an equilibrium exists).

**Issues a proof needs to take care of.**

- When some prices are zero and individual's preferences are locally non satiated, their demand can explode.
  - This is because the budget set is not compact.
- Demand (and therefore excess demand) is not necessarily continuous at zero prices.
  - Think about how could this happen.
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Proving a Not So Simple Existence Theorem

- This is a sketch of the proof for existence of a competitive equilibrium.

  1. Truncate the economy, so that all choices must belong to a compact set.
  2. Construct a ‘game’ with $I + J + 1$ players: consumers, firms, and the ‘price player’.
  3. Show that each player’s best response is a non-empty, convex, and upper hemi-continuous correspondence.
  4. Hence the ‘product’ best-response correspondence that describes this game inherits those properties.
  5. Use Kakutani’s fixed point theorem to show this correspondence has a fixed point.
  6. This fixed point is an equilibrium of the truncated economy.
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