# VISCOSITY SUPERSOLUTIONS OF THE EVOLUTIONARY $p$-LAPLACE EQUATION

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## 1. Introduction

Often new proofs of old results give additional insight, besides the simplification offered. We hope that the present study of the diffusion equation

$$\frac{\partial v}{\partial t} = \nabla \cdot (|\nabla v|^{p-2}\nabla v) \tag{1.1}$$

has this character. Even obvious results for this equation may require advanced estimates in the proofs. We refer to the books [DB] and [WZYL] about this equation, which is called the "evolutionary $p$-Laplacian equation," the "$p$-parabolic equation" or even the "non-Newtonian equation of filtration.".

Our objective is to study the regularity of the *viscosity supersolutions* and their spatial gradients. We give a new proof of the existence of $\nabla v$ in Sobolev's sense and of the validity of the equation

$$\iint_\Omega \left(-v\frac{\partial \varphi}{\partial t} + \langle |\nabla v|^{p-2}\nabla v,\ \nabla\varphi\rangle\right) dx\ dt \geq 0 \tag{1.2}$$

for all test functions $\varphi \geq 0$. Here $\Omega$ is the underlying domain in $\mathbb{R}^{n+1}$ and $v$ is a bounded viscosity supersolution in $\Omega$. The first step of our proof is to establish (1.2) for the so-called infimal convolution $v_\epsilon$, constructed from $v$ through a simple formula. The function $v_\epsilon$ has the advantage of being differentiable with respect to all its variables $x_1, x_2, \cdots, x_n$, and $t$, while the original $v$ is merely lower semicontinuous to begin with. The second step is to pass to the limit as $\epsilon \to 0$. It is clear that $v_\epsilon \to v$ but it is delicate to establish a sufficiently good convergence of the $\nabla v_\epsilon$'s.

This has earlier been proved in [KL1] for the so-called $p$-superparabolic functions; according to a theorem in [JLM] they coincide with the viscosity supersolutions. We had better mention that, when it comes to the "supersolutions" several definitions are currently being used. To clarify the concept we mention a few:

- weak supersolutions (test functions under the integral sign);
- viscosity supersolutions (test functions evaluated at points of contact);

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- $p$-superparabolic functions (defined via a comparison principle).

The weak supersolutions are assumed to belong to a Sobolev space; they do not form a good closed class under monotone convergence. The viscosity supersolutions are assumed to be merely lower semicontinuous. So are the $p$-superparabolic functions. As we mentioned, the viscosity supersolutions and the $p$-superparabolic functions coincide. This is an important link in our proof. If they, in addition, are bounded, then they are weak supersolutions satisfying (1.2). Our contribution is a new proof of the last fact. Our use of the $v_\epsilon$'s replace a technically complicated approximation procedure in the old proof in [KL1].

The present proof is not free of technical complications. The corresponding proof for the stationary equation

$$\nabla \cdot \left(|\nabla v|^{p-2}\nabla v\right) = 0,$$

often called the $p$-Laplace equation, is much simpler and more transparent. For the benefit of the reader we have written down also this case, although the original proof in [L] is simple enough. See also [KM].

A final remark about unbounded viscosity solutions is appropriate. The truncated functions $v_k = \min(v, k), k = 1, 2, 3, \cdots$, are viscosity supersolutions and the results above apply to them. Then one may proceed from this as in [KL2], [L], and [KM]. See also [BDGO].

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## 2. Preliminaries

We begin with the $p$-Laplace equation

$$\nabla \cdot \left(|\nabla v|^{p-2}\nabla v\right) = 0$$

in a domain $\Omega$ in $\mathbb{R}^n$. This is the stationary case. We say that $v \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is a *weak supersolution* in $\Omega$, if

$$\int_\Omega \langle |\nabla v|^{p-2}\nabla v,\ \nabla\varphi\rangle dx \geq 0 \tag{2.1}$$

whenever $\varphi \geq 0$ and $\varphi \in C_0^\infty(\Omega)$. If the integral inequality is reversed, we say that $v$ is a *weak subsolution*. We say that a continuous $h \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is a *p-harmonic function*, if

$$\int_\Omega \langle |\nabla h|^{p-2}\nabla h,\ \nabla\varphi\rangle dx = 0 \tag{2.2}$$

for all $\varphi \in C_0^\infty(\Omega)$. By elliptic regularity theory the continuity is a redundant requirement in the definition.

**Definition 1.** *We say that the function $v : \Omega \to (-\infty, \infty]$ is p-superharmonic in $\Omega$, if*

(i) $v \not\equiv +\infty$,
(ii) *v is lower semicontinuous,*
(iii) *v obeys the comparison principle in each subdomain* $D \subset\subset \Omega$ : *if* $h \in C(\overline{D})$ *is p-harmonic in D, then the inequality* $v \geq h$ *on* $\partial D$ *implies that* $v \geq h$ *in D.*

We refer to [L] for this concept. Notice that the definition does not include any hypothesis about $\nabla v$. The next definition is from the modern theory of viscosity solutions.

**Definition 2.** *Let* $p \geq 2$. *We say that the function* $v : \Omega \to (-\infty, \infty]$ *is a viscosity supersolution in* $\Omega$, *if*

(i) $v \not\equiv +\infty$,
(ii) *v is lower semicontinuous, and*
(iii) *whenever* $x_0 \in \Omega$ *and* $\varphi \in C^2(\Omega)$ *are such that*

$$\begin{aligned} v(x_0) &= \varphi(x_0), \text{ and} \\ v(x) &> \varphi(x) \text{ when } x \neq x_0, \end{aligned}$$

*we have*

$$\nabla \cdot \left(|\nabla \varphi(x_0)|^{p-2} \nabla \varphi(x_0)\right) \leq 0.$$

According to [JLM] (Theorem 2.5), the viscosity supersolutions and the $p$-superharmonic functions are the same. In other words, Definition 1 and Definition 2 are equivalent.

In [L] the following theorem was proved for the $p$-superharmonic functions.

**Theorem 1.** *Suppose that v is a locally bounded p-superharmonic function in* $\Omega$. *Then the Sobolev derivative*

$$\nabla v = \left(\frac{\partial v}{\partial x_1}, \cdots \frac{\partial v}{\partial x_n}\right)$$

*exists and* $v \in W^{1,p}_{loc}(\Omega)$. *Moreover, v is a weak supersolution, i.e.,*

$$\int_\Omega \langle |\nabla v|^{p-2} \nabla v,\ \nabla \varphi \rangle dx \geq 0$$

*whenever*

$$\varphi \in C_0^\infty(\Omega),\ \varphi \geq 0.$$

We aim at giving a new proof of this theorem, using the viscosity theory. The proof for viscosity supersolutions is given in Section 3.

We now proceed to the parabolic equation

$$\frac{\partial v}{\partial t} = \nabla \cdot (|\nabla v|^{p-2} \nabla v)$$

in a domain $\Omega$, this time in $\mathbb{R}^{n+1}$. We use the notation

$$v = v(x, t) = v(x_1, \cdots, x_n, t).$$

We assume that $p \geq 2$. (The case $p < \frac{2n}{n+2}$ is in doubt.) With obvious modifications, we repeat what was written above, but by paying attention to the time variable. We say that $v$ is a *weak supersolution* in $\Omega$, if $v \in L(t_1, t_2; W^{1,p}(D))$ whenever $D \times (t_1, t_2) \subset\subset \Omega$ and

$$\iint_{\Omega} \left( -v \frac{\partial \varphi}{\partial t} + \langle |\nabla v|^{p-2} \nabla v, \nabla \varphi \rangle \right) dxdt \geq 0 \tag{2.3}$$

for all $\varphi \geq 0, \varphi \in C_0^\infty(\Omega)$. Similarly we define *weak subsolutions*. A continuous function $h$, belonging to the aforementioned space, is called a *p-parabolic* function, if

$$\iint_{\Omega} \left( -h \frac{\partial \varphi}{\partial t} + \langle |\nabla h|^{p-2} \nabla h, \nabla \varphi \rangle \right) dxdt = 0 \tag{2.4}$$

for all test functions $\varphi \in C_0^\infty(\Omega)$.

**Definition 3.** *We say that the function $v : \Omega \to (-\infty, \infty]$ is p-superparabolic in $\Omega$, if*

(i) *$v$ is finite in a dense subset of $\Omega$.*
(ii) *$v$ is lower semicontinuous.*
(iii) *$v$ obeys the comparison principle in each subdomain $D_{t_1,t_2} = D \times (t_1, t_2) \subset\subset \Omega$: if $h \in C(\overline{D_{t_1,t_2}})$ is p-parabolic in $D_{t_1,t_2}$ and if $v \geq h$ on the parabolic boundary of $D_{t_1,t_2}$, then $v \geq h$ in $D_{t_1,t_2}$.*

Recall that the parabolic boundary is the union of $\partial D \times [t_1, t_2]$ and $\overline{D} \times \{t_1\}$. Thus $D \times \{t_2\}$ is excluded. See [KL] for some basic facts. Again there is an equivalent definition in terms of the viscosity theory.

**Definition 4.** *Let $p \geq 2$. Suppose that $v : \Omega \to (-\infty, \infty]$ satisfies (i) and (ii) above. We say that $v$ is a viscosity supersolution, if*

(iii) *whenever $(x_0, t_0) \in \Omega$ and $\varphi \in C^2(\Omega)$ are such that $v(x_0, t_0) = \varphi(x_0, t_0)$ and $v(x, t) > \varphi(x, t)$ when $(x, t) \neq (x_0, t_0)$, we have*

$$\frac{\partial \varphi(x_0, t_0)}{\partial t} \geq \nabla \cdot (|\nabla \varphi(x_0, t_0)|^{p-2} \nabla \varphi(x_0, t_0))$$

Again the test function is touching $v$ from below and the differential inequality is evaluated only at the point of contact. According to Theorem 4.4 in [JLM] Definitions 3 and 4 are equivalent. Moreover, one also obtains an equivalent definition by looking only at points $(x, t)$ such that $t < t_0$, see [J]. In [KL] the following theorem was proved for the $p$-superparabolic functions.

**Theorem 2.** *Suppose that $v$ is a locally bounded p-superparabolic function in $\Omega$. Then the Sobolev derivative*

$$\nabla v(x, t) = \left( \frac{\partial v(x, t)}{\partial x_1}, \cdots, \frac{\partial v(x, t)}{\partial x_n} \right)$$

*exists and $\nabla v \in L^p_{loc}(\Omega)$. Moreover, $v$ is a weak supersolution, i.e.,*

$$\iint_{\Omega} \left( -v \frac{\partial \varphi}{\partial t} + \langle |\nabla v|^{p-2} \nabla v, \nabla \varphi \rangle \right) dx\ dt \geq 0$$

*whenever $\varphi \geq 0, \varphi \in C_0^\infty(\Omega)$.*

The interpretation of the time derivative requires caution. It is often merely a measure, as the following example shows. Every function of the form $v(x,t) = g(t)$ is $p$-superparabolic if $g(t)$ is a non-decreasing lower semi-continuous step function. Thus Dirac deltas can appear in $v_t$.

## 3. The Stationary Equation

In this section we prove Theorem 1. Aiming at a local result, we may for the proof assume that $v$ is bounded in the whole $\Omega$. By adding a constant, if needed, we have

$$0 \le v(x) \le L, \quad \text{when} \quad x \in \Omega. \tag{3.1}$$

The approximants

$$v_\epsilon(x) = \inf_{y\in\Omega} \left\{ \frac{|x-y|^2}{2\epsilon} + v(y) \right\}, \quad x \in \Omega, \tag{3.2}$$

have many good properties: they are rather smooth, they form an increasing sequence converging to $v(x)$ as $\epsilon \to 0^+$, and from $v$ they inherit the property of being viscosity supersolutions themselves. Some well-known facts are listed below.

1°) At each $x$ in $\Omega, v_\epsilon(x) \nearrow v(x)$ as $\epsilon \to 0^+$.

2°) The function

$$v_\epsilon(x) - \frac{|x|^2}{2\epsilon}$$

is locally concave in $\Omega$.

3°) The Sobolev gradient $\nabla v_\epsilon$ exists and $\nabla v_\epsilon \in L^\infty_{\text{loc}}(\Omega)$.

In fact, the third assertion follows from the second.

**Proposition 1.** *The approximant $v_\epsilon$ is a viscosity supersolution in the open subset of $\Omega$ where*

$$\textit{dist}\ (x, \partial\Omega) > \sqrt{2L\epsilon}.$$

*Proof.* Choose $x$ in $\Omega$ as required above. Then the infimum in (3.2) is attained at some point $y$ in $\Omega$, say $y = x^*$. Formally, the possibility that $x^*$ escapes to $\partial\Omega$ is prohibited by the inequalities

$$\frac{|x-x^*|^2}{2\epsilon} \le \frac{|x-x^*|^2}{2\epsilon} + v(x^*) = v_\epsilon(x) \le v(x) \le L$$

and

$$|x - x^*| \le \sqrt{2L\epsilon} < \text{dist}(x, \partial\Omega).$$

Fix a point $x_0$ so that $x_0^* \in \Omega$. Assume that the test function $\varphi$ touches $v_\epsilon$ from below at $x_0$. We have

$$\varphi(x_0) = v_\epsilon(x_0) = \frac{|x_0 - x_0^*|^2}{2\epsilon} + v(x_0^*)$$

and

$$\varphi(x) \le v_\epsilon(x) \le \frac{|x-y|^2}{2\epsilon} + v(y)$$

for all $x$ and $y$ in $\Omega$. Using this one can verify that the function

$$\psi(x) = \varphi(x + x_0 - x_0^*) - \frac{|x_0 - x_0^*|^2}{2\epsilon} \tag{3.3}$$

touches the original $v$ from below at the point $x_0^*$. By assumption the inequality

$$\nabla \cdot \big(|\nabla\psi(x_0^*)|^{p-2}\nabla\psi(x_0^*)\big) \geq 0$$

holds since $x_0^*$ is an interior point. Because

$$\nabla\psi(x_0^*) = \nabla\varphi(x_0),\ D^2\psi(x_0^*) = D^2\varphi(x_0),$$

we also have that

$$\nabla \cdot \big(|\nabla\varphi(x_0)|^{p-2}\nabla\varphi(x_0)\big) \geq 0 \tag{3.4}$$

at the original point $x_0$. □

Write

$$\Omega_\epsilon = \Big\{x \in \Omega\colon \text{dist }(x, \partial\Omega) > \sqrt{2\epsilon L}\Big\}.$$

**Theorem 3.** *The approximant $v_\epsilon$ obeys the comparison principle in $\Omega_\epsilon$. In other words, given a domain $D \subset\subset \Omega_\epsilon$ and a $p$-harmonic function $h \in C(\overline{D})$, then the implication*

$$v_\epsilon \geq h \ \text{on}\ \partial D \Rightarrow \ v_\epsilon \geq h \ \text{in}\ D$$

*holds.*

*Proof.* This is Theorem 2.5 in [JLM]. □

The comparison principle implies that $v_\epsilon$ is a weak supersolution with test functions under the integral sign. The proof is based on an obstacle[1]problem in the calculus of variations.

**Theorem 4.** *The approximant $v_\epsilon$ is a weak supersolution in $\Omega_\epsilon$, i.e.,*

$$\int_\Omega \langle |\nabla v_\epsilon|^{p-2}\nabla v_\epsilon, \nabla\varphi\rangle dx \geq 0 \tag{3.5}$$

*whenever $\varphi \in C_0^\infty(\Omega_\epsilon)$ and $\varphi \geq 0$.*

*Proof.* Let $D \subset\subset \Omega_\epsilon$ be a regular domain. We regard $v_\epsilon$ as an obstacle and consider the class consisting of all functions $w$ such that

$$\begin{cases} w \in C(\bar{D}) \cap W^{1,p}(D), \\ w \geq v_\epsilon \text{ in } D, \text{ and} \\ w = v_\epsilon \text{ on } \partial D. \end{cases}$$

The problem of minimizing the variational integral $\int |\nabla w|^p dx$ has a unique solution $w_\epsilon$ in this class. In other words,

$$\int_D |\nabla w_\epsilon|^p dx \leq \int_D |\nabla w|^p dx$$

[1]It is not clear, whether the obstacle problem can be totally avoided in the passage to (3.5).

for all $w$ in the aforementioned class. We refer to [MZ] for the continuity. By a standard argument, the minimizer is weak supersolution, i.e.,

$$\int_D \langle |\nabla w_\epsilon|^{p-2}\nabla w_\epsilon, \nabla\varphi\rangle dx \geq 0$$

whenever

$$\varphi \in C_0^\infty(D),\ \varphi \geq 0.$$

The theorem follows from the claim $w_\epsilon = v_\epsilon$ in $D$. To prove the claim, we notice that $w_\epsilon \geq v_\epsilon$. In the open set $A_\epsilon = \{w_\epsilon > v_\epsilon\}$ one knows that $w_\epsilon$ is $p$-harmonic. On the boundary $\partial A_\epsilon$ we have $w_\epsilon = v_\epsilon$. The comparison principle (Definition 1) implies that $v_\epsilon \geq w_\epsilon$ in $A_\epsilon$. It follows that $A_\epsilon$ is empty and $w_\epsilon = v_\epsilon$. This was the claim. □

The next lemma contains a bound that is independent of $\epsilon$.

**Lemma 1.** (Caccioppoli) *We have*

$$\int_\Omega \zeta^p|\nabla v_\epsilon|^p dx \leq p^p\, L^p \int_\Omega |\nabla\zeta|^p dx \tag{3.6}$$

*whenever* $\zeta \in C_0^\infty(\Omega_\epsilon)$ *and* $\zeta \geq 0$.

*Proof.* Use the test function

$$\varphi = (L - v_\epsilon)\zeta^p$$

in (3.5) to obtain this well-known estimate. □

**Corollary 1.** *The Sobolev derivative* $\nabla v$ *exists and* $\nabla v \in L^p_{loc}(\Omega)$.

*Proof.* Use Lemma 1 and a standard compactness argument. □

In order to proceed to the limit under the integral sign in (3.5) we need more than the weak convergence:

$$\nabla v_\epsilon \to \nabla v$$

locally weakly in $L^p(\Omega)$. Actually, the convergence is strong.

**Lemma 2.** *We have that* $\nabla v_\epsilon \to \nabla v$ *strongly in* $L^p_{loc}(\Omega)$.

*Proof.* Let $\theta \in C_0^\infty(\Omega)$ and $\theta \geq 0$. Use the test function $\varphi = (v - v_\epsilon)\theta$ in (3.5). The inequality can be written as

$$\int_\Omega \theta\langle |\nabla v|^{p-2}\nabla v - |\nabla v_\epsilon|^{p-2}\nabla v_\epsilon,\ \nabla v - \nabla v_\epsilon\rangle\, dx$$

$$+ \int_\Omega (v - v_\epsilon)\langle |\nabla v|^{p-2}\nabla v - |\nabla v_\epsilon|^{p-2}\nabla v_\epsilon, \nabla\theta\rangle\, dx$$

$$\leq \int_\Omega \langle |\nabla v|^{p-2}\nabla v,\ \nabla((v - v_\epsilon)\theta)\rangle\, dx$$

The last integral approaches zero as $\epsilon \to 0^+$, because of the weak convergence. We obtain

$$\left| \int_\Omega (v - v_\epsilon) \langle |\nabla v|^{p-2} \nabla v - |\nabla v_\epsilon|^{p-2} \nabla v_\epsilon, \nabla\theta \rangle \, dx \right|$$

$$\leq \left( \int_\Omega (v - v_\epsilon)^p dx \right)^{\frac{1}{p}} \|\nabla\theta\|_{L^\infty} \left\{ \left( \int_{\theta \neq 0} |\nabla v|^p \, dx \right)^{\frac{p-1}{p}} + \left( \int_{\theta \neq 0} |\nabla v_\epsilon|^p \, dx \right)^{\frac{p-1}{p}} \right\}$$

$$\to 0 \text{ as } \epsilon \to 0^+.$$

We conclude that

$$\limsup_{\epsilon \to 0} \int_\Omega \theta \langle |\nabla v|^{p-2} \nabla v - |\nabla v_\epsilon|^{p-2} \nabla v_\epsilon, \nabla v - \nabla v_\epsilon \rangle \, dx = 0.$$

The integrand is non-negative. For $p \geq 2$ the elementary inequality

$$2^{2-p} |b - a|^p \leq \langle |b|^{p-2} b - |a|^{p-2} a, b - a \rangle$$

yields the desired result. □

Now we can take the limit under the integral sign in (3.5). Thus (2.1) follows. This concludes our proof of Theorem 1.

## 4. The Parabolic Case

For the proof of Theorem 2 we may assume that the viscosity supersolution $v$ of the evolutionary $p$-Laplacian equation is bounded in the domain $\Omega$ in $\mathbb{R}^{n+1}$. Suppose that

$$0 \leq v(x,t) \leq L \text{ when } (x,t) \in \Omega. \tag{4.1}$$

The approximants

$$v_\epsilon(x,t) = \inf_{(y,\tau) \in \Omega} \left\{ \frac{|x-y|^2 + (t-\tau)^2}{2\epsilon} + v(y,\tau) \right\}, \quad \epsilon > 0, \tag{4.2}$$

play a central role in our study. Some useful properties are

1°) At each point $(x,t)$ in $\Omega, v_\epsilon(x,t) \nearrow v(x,t)$ as $\epsilon \to 0^+$.

2°) The function

$$v_\epsilon(x,t) - \frac{|x|^2 + t^2}{2\epsilon}$$

is locally concave in $\Omega$.

3°) The Sobolev derivatives $\frac{\partial v_\epsilon}{\partial t}$ and $\nabla v_\epsilon$ exist and belong to $L^\infty_{\text{loc}}(\Omega)$.

Given a point $(x,t)$ in $\Omega$, the infimum in (4.2) is attained at some point $(x^*, t^*)$ in $\Omega$ provided that

$$\text{dist}\,((x,t), \partial\Omega) > \sqrt{2L\epsilon}. \tag{4.3}$$

Formally, the inequalities

$$\begin{aligned} \frac{|t-t^*|^2 + |x-x^*|^2}{2\epsilon} &\leq \frac{|t-t^*|^2 + |x-x^*|^2}{2\epsilon} + v(x^*,t^*) \\ &= v_\epsilon(x,t) \leq v(x,t) \leq L, \end{aligned} \tag{4.4}$$

and

$$\sqrt{(t-t^*)^2+|x-x^*|^2} \leq \sqrt{2L\epsilon} < \operatorname{dist}\left((x,t),\partial\Omega\right),$$

and the semincontinuity guarantee this. For simplicity, we denote the open set defined by (4.3) as $\Omega_\epsilon$. We then have $\Omega_\epsilon \subset\subset \Omega$ and $\lim_{\epsilon\to 0^+}\Omega_\epsilon = \Omega$.

**Proposition 2.** *The approximant $v_\epsilon$ is a viscosity supersolution in $\Omega_\epsilon$.*

*Proof.* Fix a point $(x_0,t_0)$ in $\Omega_\epsilon$. Then the infimum (4.2) is attained at some *interior* point $(x_0^*,t_0^*)$ in $\Omega$. Select an arbitrary test function $\varphi$ that touches $v$ from below at $(x_0,t_0)$. The inequalities

$$\varphi(x_0,t_0) = v_\epsilon(x_0,t_0) = \frac{(t_0-t_0^*)^2+|x_0-x_0^*|^2}{2\epsilon} + v(x_0^*,t_0^*),$$

$$\varphi(x,t) \leq v_\epsilon(x,t) \leq \frac{(t-\tau)^2+|x-y|^2}{2\epsilon} + v(y,\tau)$$

are at our disposal for all $(x,t)$ and $(y,\tau)$ in $\Omega$. Manipulating these inequalities, one can verify that the function

$$\psi(x,t) = \varphi(x+x_0-x_0^*, t+t_0-t_0^*) - \frac{(t_0-t_0^*)^2+|x_0-x_0^*|^2}{2\epsilon}$$

touches $v$ from below at the point $(x_0^*,t_0^*)$. It will do as a test function. Because $v$ is a viscosity supersolution, the inequality

$$\frac{\partial\psi}{\partial t} \leq \nabla\cdot\left(|\nabla\psi|^{p-2}\nabla\psi\right)$$

holds at the point $(x_0^*,t_0^*)$. The partial derivatives of $\psi$ evaluated at $(x_0^*,t_0^*)$ coincide with those of $\varphi$ evaluated at the original point $(x_0,t_0)$:

$$\psi_t(x_0^*,t_0^*) = \varphi_t(x_0,t_0), \nabla\psi(x_0^*,t_0^*) = \nabla\varphi(x_0,t_0), \ldots$$

Hence the desired inequality

$$\frac{\partial\varphi}{\partial t} \leq \nabla\cdot\left(|\nabla\varphi|^{p-2}\nabla\varphi\right)$$

holds at $(x_0,t_0)$. □

**Theorem 5.** *The approximant $v_\epsilon$ obeys the comparison principle in $\Omega_\epsilon$. In other words, given a domain $D_{t_1,t_2} = D\times(t_1,t_2) \subset\subset \Omega_\epsilon$ and a $p$-parabolic function $h \in C(\overline{D_{t_1,t_2}})$ then $v_\epsilon \geq h$ on the parabolic boundary of $D_{t_1,t_2}$ implies that $v_\epsilon \geq h$ in $D_{t_1,t_2}$.*

*Proof.* This was proved for viscosity supersolutions in Theorem 4.4, p. 712 of [JLM] □

The *parabolic* comparison principle allows comparison in space-time cylinders. We need domains of a more general shape but we do not need to distinguish the parabolic boundary. It turns out that parabolic comparison implies the following *elliptic* comparison principle:

**Proposition 3.** *Given a domain $\Upsilon \subset\subset \Omega$ and a $p$-parabolic function $h \in C(\overline{\Upsilon})$, then $v_\epsilon \geq h$ on $\partial\Upsilon$ implies that $v_\epsilon \geq h$ in $\Upsilon$.*

Now $\Upsilon$ does not have to be a space-time cylinder and $\partial\Upsilon$ is the total boundary in $\mathbb{R}^{n+1}$.

*Proof.* For the proof of the necessity, it is enough to realize that the proof is immediate when $\Upsilon$ is a finite union of space-time cylinders $D_j \times (a_j, b_j)$. To verify this, just start with the earliest cylinder(s). Then the general case follows by exhausting $\Upsilon$ with such unions. Indeed, given $\alpha > 0$ the compact set $\{h(x,t) \geq v_\epsilon(x,t) + \alpha\}$ is contained in an open finite union

$$\bigcup D_j \times (a_j, b_j)$$

comprised in $\Omega$ so that $h < v_\epsilon + \alpha$ on the (Euclidean) boundary of the union. It follows that $h \leq v_\epsilon + \alpha$ in the union. Since $\alpha$ was arbitrary, we conclude that $v_\epsilon \geq h$ in $\Upsilon$. □

The above *elliptic* comparison principle does not acknowledge the parabolic boundary. The reasoning can easily be slightly modified so that the latest boundary part is exempted.[2] Suppose that $t < T$ for all $(x,t) \in \Upsilon$. (In this case $\partial\Upsilon$ may have a plane portion with $t = T$.) It is sufficient to verify that

$$v_\epsilon \geq h \text{ on } \partial\Upsilon \text{ when } t < T$$

in order to conclude that $v_\epsilon \geq h$ in $\Upsilon$.

This variant of the comparison principle is convenient for the following conclusion.

**Lemma 3.** *The approximant $v_\epsilon$ is a weak supersolution in $\Omega_\epsilon$. That is, we have*

$$\iint_\Omega \left( -v_\epsilon \frac{\partial\varphi}{\partial t} + \langle |\nabla v_\epsilon|^{p-2}\nabla v_\epsilon, \nabla\varphi\rangle \right) dxdt \geq 0 \tag{4.5}$$

*for all $\varphi \in C_0^\infty(\Omega_\epsilon)$, $\varphi \geq 0$.*

*Proof.* We show that in a given domain $D_{t_1,t_2} = D \times (t_1,t_2) \subset\subset \Omega_\epsilon$ our $v_\epsilon$ coincides with the solution of an obstacle problem. The solutions of the obstacle problem are *per se* weak supersolutions. Hence, so is $v_\epsilon$. Consider the class of all functions

$$\begin{cases} w \in C(\overline{D_{t_1,t_2}}) \cap L^p(t_1,t_2,W^{1,p}(D)), \\ w \geq v_\epsilon \text{ in } D_{t_1,t_2}, \text{ and} \\ w = v_\epsilon \text{ on the parabolic boundary of } D_{t_1,t_2}. \end{cases}$$

The function $v_\epsilon$ itself acts as an obstacle and induces the boundary values. There exists a (unique) weak supersolution $w_\epsilon$ in this class satisfying the variational inequality

$$\int_{t_1}^{t_2}\int_D \left[ (\psi - w_\epsilon)\ \frac{\partial\psi}{\partial t} + \langle |\nabla w_\epsilon|^{p-2}\nabla w_\epsilon, \nabla(\psi - w_\epsilon)\rangle \right] dxdt$$

[2] Another way to see this is to use $v_\epsilon(x,t) + \alpha/(T-t)$ in the place of $v_\epsilon$ and then let $\alpha \to 0^+$.

$$\geq \frac{1}{2}\int_D (\psi(x,t_2) - w_\epsilon(x,t_2))^2 dx$$

for all smooth $\psi$ in the aforementioned class. Moreover, $w_\epsilon$ is $p$-parabolic in the open set $A_\epsilon = \{w_\epsilon > v_\epsilon\}$. We refer to [C].

On the boundary $\partial A_\epsilon$ we know that $w_\epsilon = v_\epsilon$ except possibly when $t = t_2$. By the "elliptic" comparison principle we have $v_\epsilon \geq w_\epsilon$ in $A_\epsilon$. On the other hand $w_\epsilon \geq v_\epsilon$. Hence $w_\epsilon = v_\epsilon$.

Let $\varphi \in C_0^\infty(D_{t_1,t_2})$, $\varphi \geq 0$, and choose $\psi = w_\epsilon + \varphi = v_\epsilon + \varphi$ above. An easy manipulation yields (4.5.) □

Recall that $0 \leq v \leq L$. Then also $0 \leq v_\epsilon \leq L$. An estimate for $\nabla v_\epsilon$ is provided in the well-known lemma below.

**Lemma 4.** (Caccioppoli) *We have*

$$\begin{aligned}\iint_\Omega \zeta^p |\nabla v_\epsilon|^p \, dxdt \quad &\leq \quad CL^2 \iint_\Omega \left|\frac{\partial \zeta^p}{\partial t}\right| dxdt \\ &+ \quad CL^p \iint_\Omega |\nabla \zeta|^p \, dxdt\end{aligned} \tag{4.6}$$

*whenever* $\zeta \in C_0^\infty(\Omega_\epsilon), \zeta \geq 0$. *Here* $C$ *depends only on* $p$.

*Proof.* The test function

$$\varphi(x,t) = (L - v_\epsilon(x_1,t))\zeta(x,t)$$

leads to this estimate. □

Keeping $0 \leq v \leq L$, we can conclude from the Caccioppoli estimate that $\nabla v$ exists and $\nabla v \in L^p_{\mathrm{loc}}(\Omega)$. Moreover, we have

$$\nabla v_\epsilon \to \nabla v \text{ weakly in } L^p_{\mathrm{loc}}(\Omega),$$

at least for a subsequence. This proves the first part of the main theorem. The second part follows, if we can pass to the limit under the integral sign in

$$\iint_\Omega \Big( - v_\epsilon \, \frac{\partial \varphi}{\partial t} + \langle |\nabla v_\epsilon^{p-2} \nabla v_\epsilon, \nabla \varphi\rangle \Big) \, dxdt \geq 0 \tag{4.7}$$

as $\epsilon \to 0+$. When $p \neq 2$ the weak convergence alone does not directly justify such a procedure. Strong local convergence in $L^p$ is, as it were, difficult to achieve. The difficulty is that no good bound on $\frac{\partial v_\epsilon}{\partial t}$ is available. In fact, calculations with the example

$$v(x,t) = \begin{cases} 1 & \text{when } t > 0 \\ 0, & \text{when } t \leq 0 \end{cases}$$

reveal that simple adaptations of the proof given in the stationary case fail. However, the elementary vector inequality

$$\Big| |b|^{p-2}b - |a|^{p-2}a \Big| \leq (p-1)|b-a|(|b|+|a|)^{p-2}$$

valid for $p \geq 2$, implies that strong convergence in $L^{p-1}_{\text{loc}}$ is sufficient for the passage to the limit. This is more accessible. Thus the theorem follows from

**Lemma 5.** *We have that $\nabla v_\epsilon \to \nabla v$ strongly in $L^{p-1}_{loc}(\Omega)$, when $p \geq 2$.*

**Remark:** The same proof yields strong convergence in $L^q_{\text{loc}}(\Omega)$, where $q < p$. The method fails for $q = p$, except when the original $v$ is continuous.

This lemma is a special case of the next theorem.

**Theorem 6.** *Suppose that $v_1, v_2, v_3, \dots$ is a sequence of Lipschitz continuous weak supersolutions satisfying*

$$0 \leq v_k \leq L \text{ in } Q_T = Q \times (0,T)$$

*and*

$$v_k \to v \text{ in } L^p(Q_T).$$

*Then $\nabla v_1, \nabla v_2, \nabla v_3, \dots$ is a Cauchy sequence in $L^{p-1}_{loc}(Q_T)$.*

*Proof.* Let $\delta > 0$. The idea is that a good estimate can be obtained integrated over the set where $|v_j - v_k| \leq \delta$. The exceptional set where $|v_j - v_k| > \delta$ requires an extra consideration based on the fact that it is of small measure[3] for large indices, to wit less than $\delta^{-p}\|v_j - v_k\|_p^p$. To this end, let $\theta \in C_0^\infty(Q_T)$, where $0 \leq \theta \leq 1$. Recall that

$$\iint_{\theta \neq 0} |\nabla v_k|^p \, dxdt \leq A^p, \quad k = 1, 2, 3, \dots, \tag{4.8}$$

by the Caccioppoli estimate from Lemma 4. The constant $A$ depends on $L$ and on the support of $\theta$.

In the equation

$$\iint (\langle |\nabla v_j|^{p-2} \nabla v_j, \nabla \varphi \rangle - v_j \frac{\partial \varphi}{\partial t}) \, dxdt \geq 0$$

the test function $\varphi = (\delta - w_{j_k})\theta$ is admissible[4] where

$$w_{jk} = \begin{cases} \delta, & \text{when} \quad v_j - v_k > \delta, \\ v_j - v_k, & \text{when} \quad |v_j - v_k| \leq \delta, \\ -\delta, & \text{when} \quad v_j - v_k < -\delta. \end{cases}$$

Notice that $|w_{jk}| \leq \delta$ and $\varphi \geq 0$. In the corresponding equation for $v_k$ we use the test function $(\delta + w_{jk})\theta$. Subtracting the resulting equations and arranging terms, we arrive at

[3] The $L^p$-convergence assumption can be replaced by convergence in measure.

[4] We seize the opportunity to mention that the parameter $\delta$ is missing from the test function $(v^* - v_k)\theta$ in [KL1], which should be $(v^* - v_k + \delta)_+\theta$. To correct the error there the Egorov theorem is convenient.

$$
\begin{aligned}
\iint_{|v_j-v_k|\le\delta} & \theta\langle|\nabla v_j|^{p-2}\nabla v_j-|\nabla v_k|^{p-2}\nabla v_k,\ \nabla v_j-\nabla v_k\rangle\,dxdt\\
\le\ & \delta\int_0^T\!\!\int_Q\langle|\nabla v_j|^{p-2}\nabla v_j+|\nabla v_k|^{p-2}\nabla v_k,\nabla\theta\rangle\,dxdt\\
-\ & \int_0^T\!\!\int_Q w_{jk}\langle|\nabla v_j|^{p-2}\nabla v_j-|\nabla v_k|^{p-2}\nabla v_k,\nabla\theta\rangle\,dxdt \qquad (4.9)\\
+\ & \int_0^T\!\!\int_Q (v_j-v_k)\,\frac{\partial}{\partial t}(\theta w_{jk})dxdt-\delta\int_0^T\!\!\int_Q(v_j+v_k)\,\frac{\partial\theta}{\partial t}\,dxdt\\
=\ & I+II+III+IV.
\end{aligned}
$$

We need an estimate that is free of the time derivatives $\frac{\partial v_j}{\partial t}$ and $\frac{\partial v_k}{\partial t}$, now present in term III. Thus we write this term as

$$
\begin{aligned}
III &= \int_0^T\!\!\int_Q\theta\frac{\partial}{\partial t}\left(\frac{w_{jk}^2}{2}\right)dxdt+\int_0^T\!\!\int_Q(v_j-v_k)w_{jk}\,\frac{\partial\theta}{\partial t}\,dxdt\\
&= -\frac{1}{2}\int_0^T\!\!\int_Q w_{jk}^2\,\frac{\partial\theta}{\partial t}dxdt+\int_0^T\!\!\int_Q(v_j-v_k)w_{jk}\,\frac{\partial\theta}{\partial t}\,dxdt\\
&\le \frac{1}{2}\delta^2\|\theta_t\|_1+2L\delta\|\theta_t\|_1\le\delta C_3,
\end{aligned}
$$

where $C_3$ is independent of $j$ and $k$. We also have

$$IV\le 2\delta L\|\theta_t\|_1=\delta C_4.$$

Next we turn to the first term. Hölder's inequality yields

$$
\begin{aligned}
I &\le \delta\,\|\nabla\theta\|_p\Big(\|\nabla v_j\|_p^{p-1}+\|\nabla v_k\|_p^{p-1}\Big)\\
&\le 2A^{p-1}\delta\|\nabla\theta\|_p\\
&= \delta C_1,
\end{aligned}
$$

and since $|w_{jk}|\le\delta$ we also obtain

$$II\le\delta C_1.$$

Summing up, we have the estimate

$$I+II+III+IV\le C\delta$$

with $C$ independent of $j$ and $k$.

The elementary inequality

$$2^{2-p}|b-a|^p\le\langle|b|^{p-2}b-|a|^{p-2}a,b-a\rangle$$

valid for vectors yields a minorant for the left-hand side. It follows that

$$\iint_{|v_j-v_k|\le\delta} \theta|\nabla v_j-\nabla v_k|^p\,dxdt \le 2^{p-2}\delta\,C=\mathcal{O}(\delta)$$

and, *a fortiori*,

$$\iint_{|v_j-v_k|\le\delta} \theta|\nabla v_j-\nabla v_k|^{p-1}\;dxdt = \mathcal{O}(\delta^{(p-1)/p}) \tag{4.10}$$

Recall the bound $\delta^{-p}||v_j-v_k||_p^p$ (Chebychev's inequality) for the measure of the set where $|v_j-v_k|>\delta$. It follows from Hölder's inequality that

$$\begin{aligned}\iint_{|v_j-v_k|>\delta} \theta|\nabla v_j-\nabla v_k|^{p-1}\;dxdt &\le \delta^{-1}\|v_j-v_k\|_p\Big(\|\nabla v_j\|_p+\|\nabla v_k\|_p\Big)^{p-1}\\ &\le (2A)^{p-1}\delta^{-1}\|v_j-v_k\|_p\\ &\to 0 \text{ as } j,k\to\infty. \end{aligned}\tag{4.11}$$

Adding up the estimates (4.10) and (4.11) we finally arrive at

$$\int_0^T\!\!\int_Q \theta|\nabla v_j-\nabla v_k|^{p-1}\;dxdt \le \mathcal{O}(\delta^{(p-1)/p})+(2A)^{p-1}\delta^{-1}\|v_j-v_k\|_p.$$

The theorem follows since the left-hand side is independent of $\delta$. □

**Remark:** We have locally that $\nabla v_\epsilon\to\nabla v$ strongly in each fixed $L^q$-norm with $q<p$. The claim in [KL1] that this convergence also holds for $q=p$ has not been rigorously proved, so far as we know (the error is described in the footnote on page 12.)

**Epilogue:** The use of the *infimal convolutions* suggests a problem in Analysis. We state it in its simplest form. Suppose that $v\in W^{1,2}(\mathbb{R}^n)=H^1(\mathbb{R}^n)$ is a lower semicontinuous and bounded function of compact support. Again, define

$$v_\epsilon(x)=\min_{y\in\mathbb{R}^{\mathbf{n}}}\left\{\frac{|x-y|^2}{2\epsilon}+v(y)\right\}$$

where $\epsilon>0$. Assume that $\nabla v_\epsilon \rightharpoonup \nabla v$ weakly in $L^2(\mathbb{R}^{\mathbf{n}})$. Does it follow that $\nabla v_\epsilon\to\nabla v$ strongly in $L^2(\mathbb{R}^n)$? If not, what about strong convergence in some $L^q(\mathbb{R}^n)$? (Notice that now $v$ is not necessarily a viscosity supersolution.)

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