NOTES FOR THE COURSE: NONLINEAR SUBELLIPTIC EQUATIONS ON CARNOT GROUPS "ANALYSIS AND GEOMETRY IN METRIC SPACES"

JUAN J. MANFREDI

Abstract. The main objective of this course is to present an extension of Jensen's uniqueness theorem for viscosity solutions of second order uniformly elliptic equations to Carnot groups. This is done via an extension of the comparison principle for semicontinuous functions of Crandall-Ishii-Lions. We first present the details for the Heisenberg group, where the ideas and the calculations are easier to appreciate. We then consider the general case of Carnot groups and present applications to the theory of convex functions and to minimal Lipschitz extensions. The last lecture is devoted to Cordes estimates in the Heisenberg group.

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Special Note:

The analysis of non-linear subelliptic equations is going through a burst of activity! During the preparation of these notes at least three preprints have appeared that contain substantial improvements in the field.

- Changyou Wang [W1] has found a new approach to Jensen's theorem on Carnot groups that is more powerful than ours. He extends also Jensen's uniqueness theorem for infinite harmonic functions to general Carnot groups, previously known only for the Heisenberg group [Bi].
- Cristian Gutiérrez and Anna Maria Montanari [GM1], [GM2] have recently identified the Monge-Ampère measure of a convex function on the Heisenberg group and proved the twice pointwise differentiability a. e. of convex functions (subelliptic Aleksandrov theorem). Another presentation of similar results has been given by Nicola Garofalo and Federico Tournier [GTo].

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Nevertheless, the responsibility for any mistakes and omissions is solely the author's.

1. Lecture I: The Euclidean case

We first present an elementary discussion in $\mathbb{R}^n$ starting at the maximum principle for smooth functions and arriving to the notion of viscosity solutions. This notion is based on using jets or pointwise generalized derivatives obtained from considering expansions for semi-continuous functions similar to Taylor expansions for smooth functions.

1.1. **Calculus.** Let Ω be a (bounded, smooth) domain in $\mathbb{R}^n$. Let $f\colon \Omega \to \mathbb{R}$ be a C^2-function that has a local maximum at a point $x_0 \in \Omega$. It follows from Taylor's theorem that:

$$Df(x_0) = 0$$

and the second derivative is negative semi-definite

$$D^2 f(x_0) \le 0.$$

This last inequality must be interpreted in the matrix sense: for every $\xi \in \mathbb{R}^n$ we have

$$\langle D^2 f(x_0)\xi, \xi\rangle \le 0.$$

Our first version of the maximum principle is just a restatement of this result for the difference of two functions.

Theorem 1. Maximum Principle, First version: *If the difference $u-v$ has a local maximum at a point x_0, then*

$$Du(x_0) = Dv(x_0) \tag{1.1}$$

and

$$D^2u(x_0) \le D^2v(x_0). \tag{1.2}$$

Let us see how to use this theorem to prove a uniqueness principle for smooth solutions of elliptic partial differential equations. Consider the equation

$$-\Delta u + f(Du) + u = 0, \tag{1.3}$$

where f is a smooth function. Suppose that u and v are smooth solutions of (1.3) on Ω having identical boundary values. Consider the function $(u-v)^+$. If it is positive at a point, the $u-v$ has a positive interior local maximum, say at the point x_0. From Theorem 1 and (1.3) we get the following relations:

$$Du(x_0) = Dv(x_0), \tag{1.4}$$

$$D^2u(x_0) \le D^2v(x_0), \tag{1.5}$$

$$0 = -\Delta u(x_0) + f(Du(x_0)) + u(x_0) \tag{1.6}$$

$$0 = -\Delta v(x_0) + f(Dv(x_0)) + v(x_0) \tag{1.7}$$

Subtracting (1.7) from (1.6) we get

$$0 = -\Delta u(x_0) + \Delta v(x_0) + u(x_0) - v(x_0).$$

From (1.5) if follows that $-\Delta u(x_0) + \Delta v(x_0) \geq 0$. Since we clearly have $u(x_0) - v(x_0) > 0$ we get a contradiction. Therefore $(u - v)^+ \leq 0$, or equivalently $u \leq v$. A symmetric argument gives $v \leq u$.

Therefore, we have proved uniqueness for the Dirichlet problem for the equation (1.3).

Note however that the above argument does not apply directly to the Laplace equation $-\Delta u = 0$. A careful examination of the previous argument shows that we can, in fact, prove the following:

Theorem 2. Comparison Principle, Early Version: *Assume that $u \leq v$ on $\partial\Omega$, that u is a subsolution of (1.3)*

$$-\Delta u + f(Du) + u \leq 0$$

and that v is a supersolution of (1.3)

$$-\Delta v + f(Dv) + v \geq 0,$$

then we have

$$u \leq v$$

in Ω.

Note that the first order terms play no essential role in the maximum or comparison principles in this section.

The more flexible Theorem 2 allows us to expand the class of equations for which we can prove uniqueness for the Dirichlet problem. In the case of the Laplace equation, we can proceed as follows. Let u and v be harmonic functions ($-\Delta u = 0$, $-\Delta v = 0$) on a bounded domain Ω, continuous on $\overline{\Omega}$, and with the same boundary values on $\partial\Omega$. For $\epsilon > 0$ small define $u_\epsilon = u + \epsilon x_1^2$ and $v_\epsilon = v - \epsilon x_1^2$. Here is the basis of the argument. For $\epsilon > 0$ we can always find $\lambda > 0$ small enough so that u_ϵ is a subsolution and v_ϵ is a supersolution of the equation $-\Delta u + \lambda u = 0$. Therefore, by the comparison principle Theorem (2), the difference $u_\epsilon - v_\epsilon$ attains its maximum in Ω at a point in $\partial\Omega$. Next, compute the maximum

$$\max_{\partial\Omega} (u_\epsilon - v_\epsilon) \leq \max_{\partial\Omega}(2\epsilon x_1^2) \leq \text{Constant } \epsilon$$

and observe that

$$u_\epsilon - v_\epsilon \leq \text{Constant } \epsilon$$

in Ω. Letting $\epsilon \to 0$ we obtain $u \leq v$.

Modifications of these simple ideas can be used to prove comparison principles for many elliptic equations of the form

$$F(x, u, Du, D^2u) = 0$$

as long as F is increasing in u and decreasing in D^2u, but the hypothesis of C^2 regularity seems quite necessary. The main purpose of this introduction is to present a version of theorems 1 and 2 for non-smooth functions in $\mathbb{R}^n$.

We need a notion of generalized derivative in order to find the analogue of (1.1) and (1.2) for non-smooth functions. Of course, there is already an extensive theory of generalized derivatives or distributions that has proven extremely useful for understanding linear and quasilinear equations. However, distributions are not always useful in nonlinear problems. Take as an example the ∞-Laplace equation

$$\sum_{i,j=1}^{n} \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} = 0.$$

It is not clear at all what a distributional solution should be.

1.2. **Jets.** The notion of generalized derivative that we shall use is based on the Taylor theorem. Suppose that $u : \mathbb{R}^n \mapsto \mathbb{R}$ is a C^2-function. Taylor at 0 says:

$$u(x) = u(0) + \langle Du(0), x\rangle + \frac{1}{2}\langle D^2u(0)x, x\rangle + o(|x|^2)$$

as $x \to 0$. If u is not necessarily smooth, we could define a generalized first derivative as the vector η and a generalized second derivative as an $n \times n$ symmetric matrix X if

$$u(x) = u(0) + \langle \eta, x\rangle + \frac{1}{2}\langle X \cdot x, x\rangle + o(|x|^2) \tag{1.8}$$

as $x \to 0$. This turns out to be very strict for non-smooth functions. The key idea is to split (1.8) in two parts. Since we still need to evaluate functions pointwise, the class of semicontinuous functions (upper and lower) is the natural class of non-smooth functions to be consider. Recall that upper-semicontinuous functions are locally bounded above and lower-semicontinuous functions are locally bounded below.

Definition 1. *A pair (η, X), where $\eta \in \mathbb{R}^n$ and X is an $n \times n$ symmetric matrix, belongs to the second order superjet of an upper semicontinuous function u at the point x_0 if*

$$u(x_0 + h) \le u(x_0) + \langle \eta, h\rangle + \frac{1}{2}\langle X \cdot h, h\rangle + o\left(|h|^2\right)$$

as $h \to 0$. The collection of all of these pairs, is denoted by $J^{2,+}u(x_0)$. (Later on, when we need to emphasize the Euclidean structure we will write $J^{2,+}_{eucl}u(x_0)$.)

There is no apriori reason why the set $J^{2,+}u(x_0)$ should be non-empty for an arbitrary upper-semicontinuous function. However, a simple argument translating paraboloids shows that indeed $J^{2,+}u(x_0)$ is non-empty for a dense set of points x_0. For lower semi-continuous functions, we consider subjets:

Definition 2. *A pair* (η, X)*, where* $\eta \in \mathbb{R}^n$ *and* X *is an* $n \times n$ *symmetric matrix, belongs to the second order subjet of a lower semicontinuous function* v *at the point* x_0 *if*

$$v(x_0 + h) \geq v(x_0) + \langle \eta, h \rangle + \frac{1}{2} \langle X \cdot h, h \rangle + o\left(|h|^2\right)$$

as $h \to 0$*. The collection of all of these pairs, is denoted by* $J^{2,-}v(x_0)$*. (Later on, when we need to emphasize the Euclidean structure we will write* $J^{2,-}_{eucl}v(x_0)$*.)*

If u is continuous and $J^{2,+}(u, x_0) \cap J^{2,-}(u, x_0) \neq \emptyset$, then it contains only one pair (η, X). Moreover, the function u is differentiable at x_0, the vector $\eta = Du(x_0)$ and we say that u is twice pointwise differentiable at x_0 and write $D^2u(x_0) = X$.

A special class of jets is determined by smooth functions φ that touch a function u from above or below at a point x_0. Denote by $K^{2,-}(u, x_0)$ the collection of pairs

$$\left(\nabla\varphi(x_0), D^2\varphi(x_0)\right)$$

where $\varphi \in C^2(\Omega)$ touches u from below at x_0; that is, $\varphi(x_0) = u(x_0)$ and $\varphi(x) < u(x)$ for $x \neq x_0$. Similarly, we define $K^{2,+}(u, x_0)$ using smooth test functions that touch a function u from above. In fact, all jets come from test functions.

Theorem 3. (Crandall, Ishii, [C]) *We always have*

$$K^{2,+}(u, x_0) = J^{2,+}(u, x_0)$$

and

$$K^{2,-}(u, x_0) = J^{2,-}(u, x_0).$$

1.3. **Viscosity Solutions.** Consider equations of the form

$$F(x, u, Du, D^2u) = 0$$

for a continuous F increasing in u and decreasing in D^2u. We say that an upper-semicontinuous function u is a **viscosity subsolution** of the above equation, if whenever $x_0 \in \Omega$ and $(\eta, X) \in J^{2,+}u(x_0)$ we have

$$F(x_0, u(x_0), \eta, X) \leq 0.$$

Similarly, we define viscosity supersolutions by using second order subjets of lower-semicontinuos functions. A viscosity solution is both a viscosity subsolution and a viscosity supersolution. With this notion of weak solution, we have increased enormously the class of functions possibly qualifying as solutions. However, a remarkable result of Jensen [J1] states that for second order uniformly elliptic equations there is at most one viscosity solution.

The proof of this fundamental uniqueness result is based on a powerful non-linear regularization technique from Control Theory involving first order Hamilton-Jacobi equations - see [E] for an enlightening discussion. Once you have the comparison principle, one can also adapt techniques from classical potential theory to prove existence (Perron's method).

1.4. **The Maximum Principle for Semicontinuous Functions.**

Theorem 4. (Crandall-Ishii-Jensen-Lions, 1992, [CIL]**):** *Let the function u be upper semicontinuous in a bounded domain Ω. Let the function v be lower semi-continuous in Ω. Suppose that for $x \in \partial\Omega$ we have*

$$\limsup_{y\to x} u(y) \le \liminf_{y\to x} v(y),$$

where both sides are not $+\infty$ or $-\infty$ simultaneously. If $u-v$ has an interior local maximum

$$\sup_{\Omega}(u-v) > 0$$

then we have:

For $\tau > 0$ we can find points $p^\tau, q^\tau \in \Omega$ such that

i)
$$\lim_{\tau\to\infty} \tau\psi(p_\tau - q_\tau) = 0,$$
where $\psi(p) = |p|^2$,

ii) *there exists a point $\hat{p} \in \Omega$ such that $p_\tau \to \hat{p}$ (and so does q_τ by (i)) and*
$$\sup_{\Omega}(u-v) = u(\hat{p}) - v(\hat{p}) > 0,$$

iii) *there exist $n \times n$ symmetric matrices X_τ, Y_τ and vectors η_τ so that*
$$(\eta_\tau, X_\tau) \in \overline{J}^{2,+}(u, p_\tau),$$

iv)
$$(\eta_\tau, Y_\tau) \in \overline{J}^{2,-}(u, q_\tau),$$
and

v)
$$X_\tau \leqq Y_\tau.$$

The last statement means that if $\xi \in \mathbb{R}^n$ we have

$$\langle X_\tau \xi, \xi\rangle \le \langle Y_\tau \xi, \xi\rangle.$$

In fact, a stronger statement than (v) holds, which we shall need later on. Set

$$A = D^2_{p,q}(\phi(p-q))$$

and

$$C = \tau(A^2 + A).$$

Then for all $\xi, \gamma \in \mathbb{R}^n$ we have

$$\langle X_\tau \gamma, \gamma\rangle - \langle Y_\tau \chi, \chi\rangle \le \langle C\gamma \oplus \chi, \gamma \oplus \chi\rangle. \tag{1.9}$$

Statement (v) follows from 1.9 by setting $\gamma = \chi$ and noting that $\langle A\xi, \xi\rangle = 0$ for $\xi \in \mathbb{R}^{2n}$.

A minor technical detail: we have used the closures of the second order super and subjets, $\bar{J}^{2,+}(u, p_\tau)$ and $\bar{J}^{2,-}(v, p_\tau)$. These are defined by taking pointwise limits as follows: A pair $(\eta, X) \in \bar{J}^{2,+}(u, p)$ if there exist sequences of points $p_m \to p$, vectors $\eta_m \to \eta$ and matrices $X_m \to X$ as $m \to \infty$ such that $u(p_m) \to u(p)$ and $(\eta_m, X_m) \in J^{2,+}(u, p_m)$.

1.5. **Extensions and generalizations.** The notion of viscosity solution is so robust that it makes sense on Lie groups, Riemannian and sub-Riemannian manifolds and even more general structures generated by vector fields. The extension of the Crandall-Ishii-Lions-Jensen machinery to these more general structures is the objective of this course.

2. Lecture II: The Heisenberg group

In this section we consider the simplest Carnot group, the Heisenberg group $\mathcal{H} = (\mathbb{R}^3, \cdot)$, where $\cdot$ is the group operation given by

$$(x, y, z) \cdot (x', y', z') = (x + x', y + y', z + z' + \frac{1}{2}(xy' - yx')).$$

$\mathcal{H}$ is a Lie group with Lie algebra h generated by the left-invariant vector fields

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x} - \frac{y}{2} \frac{\partial}{\partial z} \\ X_2 &= \frac{\partial}{\partial y} + \frac{x}{2} \frac{\partial}{\partial z} \\ X_3 &= \frac{\partial}{\partial z} \end{aligned}$$

The only non-trivial commuting relation is $X_3 = [X_1, X_2]$. Thus $\mathcal{H}$ is a nilpotent Lie group of step 2. The horizontal tangent space at a point $p = (x, y, z)$ is $T_h(p) = \text{linear span}\{X_1(p), X_2(p)\}$. A piecewise smooth curve $t \to \gamma(t) \in \mathcal{H}$ is horizontal if $\gamma'(t) \in T_h(\gamma(t))$ whenever $\gamma'(t)$ exists. Given two points $p, q \in \mathcal{H}$ denote by

$$\Gamma(p, q) = \{\text{horizontal curves joining } p \text{ and } q\}.$$

There are plenty of horizontal curves. We have the classical:

Theorem 5. *(Chow)* $\Gamma(p, q) \neq \phi$.

For convenience, fix an ambient Riemannian metric in $\mathbb{R}^3$ so that

$$\{X_1, X_2, X_3\}$$

is an orthonormal frame and

$$\text{Riemannian vol. element} = \text{Haar measure of } \mathcal{H} = \text{ Lebesgue meas. in } \mathbb{R}^3.$$

The Carnot-Carathèodory metric is then defined by

$$d_{cc}(p, q) = \inf\{\text{length}(\gamma) : \gamma \in \Gamma(p, q)\}.$$

It depends only on the restriction of the ambient Riemannian metric to the horizontal distribution generated by the horizontal tangent spaces.

Since horizontal curves are preserved by left translations d_{cc} is left-invariant. The exponential mapping $\exp : h \to \mathcal{H}$ is a global diffeomorphism. We use exponential coordinates of the first kind:

$$p = (x, y, z) = \exp(xX_1 + yX_2 + zX_3).$$

There is a family of non-isotropic dilations δ_t, parametrized by $t > 0$

$$\delta_t(x, y, z) = (tx, ty, t^2 z)$$

that are group homomorphisms.

References for Carnot groups and Carnot-Carathéodory spaces include: [B], [FS], [H], [G], [GN], and [Lu].

2.1. **Calculus in $\mathcal{H}$.** Given a function $u : \mathcal{H} \mapsto \mathbb{R}$ we consider

$$Du = (X_1 u, X_2 u, X_3 u) \in \mathbb{R}^3,$$

the (full) gradient of u. As a vector field, this is written

$$Du = (X_1 u)\, X_1 + (X_2 u)\, X_2 + (X_3 u)\, X_3.$$

The *horizontal* gradient of u is

$$D_0 u = (X_1 u, X_2 u) \in \mathbb{R}^2,$$

or as a vector field $D_0 u = (X_1 u)\, X_1 + (X_2 u)\, X_2$.

Theorem 6. Ball-Box Theorem (simple version): *(See* [B]*,* [G]*,* [NSW]*)* *Set $|p| = d_{cc}(p, 0)$, the Carnot-Carathéodory gauge and*

$$|p|_{\mathcal{H}} = ((x^2 + y^2)^2 + z^2)^{1/4}$$

the Heisenberg gauge. We have

$$d_{cc}(p, 0) \approx |p|_{\mathcal{H}} \approx |x| + |y| + |z|^{1/2}$$

and

$$Vol(B(0, r)) \approx r^4,$$

where $B(0, r)$ is the Carnot-Carathèdory ball centered at 0 of radius r.

Remark 1. *With our choice of vector fields we have*

$$|(0, 0, z)|_{cc} = \sqrt{4\pi}\ |z|^{1/2}$$

2.2. **Taylor Formula.** Suppose that $u : \mathcal{H} \mapsto \mathbb{R}$ is a smooth function. Euclidean Taylor at 0 says:

$$\begin{aligned} u(x, y, z) = u(0, 0, 0) & \\ & + u_x \cdot x + u_y \cdot y + u_z \cdot z \\ & + \frac{1}{2} \left\{ u_{xx} \cdot x^2 + u_{yy} \cdot y^2 + u_{zz} \cdot z^2 \right. \\ & \qquad \left. + 2u_{xy} xy + 2u_{xz} xz + 2u_{yz} yz \right\} \\ & + o(x^2 + y^2 + z^2) \end{aligned}$$

Using the fact that $|p|^2 \approx x^2 + y^2 + |z|$, we obtain the horizontal Taylor expansion

$$\begin{aligned} u(x,y,z) = {} & u(0,0,0) \\ & + u_x \cdot x + u_y \cdot y + u_z \cdot z \\ & + \frac{1}{2}\left\{ u_{xx}x^2 + u_{yy}\cdot y^2 + 2u_{xy}xy \right\} \\ & + o(|p|^2). \end{aligned}$$

At another point $p_0 = (x_0, y_0, z_0)$, we get the horizontal Taylor formula by left-translation:

$$u(p) = u(p_0) + \langle Du(p_0), p_0^{-1}\cdot p\rangle + \frac{1}{2}\langle (D_0^2 u(p_0))^* h, h\rangle + o(|p_0^{-1}\cdot p|^2),$$

where $Du(p_0)$ is the full gradient of u at p_0, the matrix

$$(D_0^2 u)^* = \begin{pmatrix} X_1^2 u & \frac{1}{2}(X_1X_2u + X_2X_1u) \\ \frac{1}{2}(X_1X_2u + X_2X_1u) & X_2^2 u \end{pmatrix}$$

is the horizontal symmetrized second derivative and the vector and $h = (x - x_0, y - y_0)$ is the horizontal projection of $p_0^{-1}\cdot p$. We could also have used the non-symmetrized second derivative

$$D_0^2 u = \begin{pmatrix} X_1^2 u & X_1X_2u \\ X_2X_1u & X_2^2 u \end{pmatrix}.$$

We prefer to use the symmetrized version since the quadratic form associated to a matrix A is determined by its symmetric part $\frac{1}{2}\left(A + A^t\right)$. Here A^t denotes the transpose of A.

2.3. **Subelliptic Jets.** Let u be an upper-semicontinuous real function in $\mathcal{H}$. The second order superjet of u at p_0 is defined as

$$\begin{aligned} J^{2,+}(u,p_0) = \Big\{ & (\eta, \mathcal{X}) \in \mathbb{R}^3 \times \mathcal{S}^2(\mathbb{R}) \text{ such that} \\ & u(p) \leqq u(p_0) + \langle \eta, p_0^{-1}\cdot p\rangle + \frac{1}{2}\langle \mathcal{X}h, h\rangle + o(|p_0^{-1}p|^2) \Big\} \end{aligned}$$

Similarly, for lower-semincontinuous u, we define the second order subjet

$$\begin{aligned} J^{2,-}(u,p_0) = \Big\{ & (\eta, \mathcal{Y}) \in \mathbb{R}^3 \times \mathcal{S}^2(\mathbb{R}) \text{ such that} \\ & u(p) \geq u(p_0) + \langle \eta, p_0^{-1}p\rangle + \frac{1}{2}\langle \mathcal{Y}h, h\rangle + o(|p_0^{-1}p|^2) \Big\}. \end{aligned}$$

One easy way to get jets is by using smooth functions that touch u from above or below.

$$K^{2,+}(u,p_0)=\Big\{(D\varphi(p_0),(D^2\varphi(p_0))^*):\varphi\in C^2 \text{ in } X_1,X_2,\varphi\in C^1 \text{ in } X_3,$$
$$\varphi(p_0)=u(p_0)$$
$$\varphi(p)\geq u(p),\ p\neq p_0 \text{ in } a \text{ neighborhood of } p_0\Big\}$$

As in the Euclidean case every jet can be obtained by this method.

Lemma 1. *(*[C] *for the Euclidean case and* [Bi] *for the subelliptic case) We always have*

$$K^{2,+}(u,p_0)=J^{2,+}(u,p_0)$$

and

$$K^{2,-}(u,p_0)=J^{2,-}(u,p_0)$$

We also define the closure of the second order superjet of an upper-semicontinuous function u at p_0, denoted by $\bar{J}^{2,+}(u,p_0)$, as the set of pairs $(\eta,\mathcal{X})\in\mathbb{R}^3\times\mathcal{S}^2(\mathbb{R})$ such that there exist sequences of points p_m and pairs $(\eta_m,\mathcal{X}_m)\in J^{2,+}(u,p_m)$ such that

$$(p_m,u(p_m),\eta_m,\mathcal{X}_m)\to(p_0,u(p_0),\eta,\mathcal{X})$$

as $m\to\infty$. The closure of the second order subjet of a lower-semicontinuous function v at p_0, denoted by $\bar{J}^{2,-}(u,p_0)$ is defined in an analogous manner.

2.4. **Fully Non-Linear Equations.** Consider a continuous function

$$F:\mathcal{H}\times\mathbb{R}\times h\times S(\mathbb{R}^2)\longrightarrow\mathbb{R}$$
$$(p,u,\eta,\mathcal{X})\longrightarrow F(p,u,\eta,\mathcal{X}).$$

We will assume that F is proper; that is, F is increasing in u and F is decreasing in $\mathcal{X}$.

Definition 3. *A lower semicontinuous function u is a viscosity supersolution of the equation*

$$F(p,u(p),Du(p),(D^2u(p))^*)=0$$

if whenever $(\eta,\mathcal{X})\in J^{2,-}(u,p_0)$ we have

$$F(p_0,u(p_0),\eta,\mathcal{X})\geq 0.$$

Equivalently, if φ touches u from below, is C^2 in X_1,X_2 and C^1 in X_3, then we must have

$$F(p_0,u(p_0),D\varphi(p_0),\ (D^2\varphi(p_0))^*)\geq 0.$$

Definition 4. *An upper semicontinuous function u is a viscosity subsolution of the equation*
$$F(p, u(p), Du(p), (D^2u(p))^*) = 0$$
if whenever $(\eta, \mathcal{X}) \in J^{2,+}(u, p_0)$ we have
$$F(p_0, u(p_0), \eta, \mathcal{X}) \leq 0.$$
Equivalently, if φ touches u from above, is C^2 in X_1, X_2 and C^1 in X_3, then we must have
$$F(p_0, u(p_0), D\varphi(p_0),\ (D^2\varphi(p_0))^*) \leqq 0.$$
Note that if u is a viscosity subsolution and $(\eta, \mathcal{X}) \in \bar{J}^{2,+}(u, p_0)$ then, by the continuity of F, we still have
$$F(p_0, u(p_0), \eta, \mathcal{X}) \leq 0.$$
A similar remark applies to viscosity supersolutions and the closure of second order subjets.

A viscosity solution is defined as being both a viscosity subsolution and a viscosity supersolution. Observe that since F is proper, it follows easily that if u is a smooth classical solution then u is a viscosity solution.

Examples of F:

- Subelliptic Laplace equation (the Hörmander-Kohn operator):
$$-\Delta_0 u = -\left(X_1^2 u + X_2^2 u\right) = 0$$
- Subelliptic p-Laplace equation, $2 \leq p < \infty$:
$$\begin{aligned} -\Delta_p u &= -\left[|D_0 u|^{p-2}\Delta_0 u + (p-2)|D_0 u|^{p-4}\Delta_{0,\infty} u\right] \\ &= -\mathrm{div}\ (|D_0 u|^{p-2} D_0 u) = 0 \end{aligned}$$

Strictly speaking we need $p \geq 2$ for the continuity of the corresponding F. In the Euclidean case it is possible to extend the definition to the full range $p > 1$. This is a non-trivial matter not yet studied in the case of the Heisenberg group (to the best of my knowledge.) See [JLM] for the Euclidean case.

- Subelliptic ∞-Laplace equation ([Bi]):
$$-\Delta_{0,\infty} u = -\left[\sum_{i,j}^{2} (X_i u)(X_j u) X_i X_j u\right] = -\langle (D_0^2 u)^* D_0 u, D_0 u\rangle$$
- "Naive" subelliptic Monge-Ampère
$$-\det(D_0^2 u)^* = f$$

Here the corresponding $F(\mathcal{X}) = -\det \mathcal{X}$ is only proper in the cone of positive semidefinite matrices. As mentioned in the introduction, Gutiérrez and Montanari have considered the Monge-Ampère operator
$$-\left\{\det(D_0^2 u)^* + \frac{3}{4}(Tu)^2\right\}.$$

They show, among other things, that it can always be defined in the sense of measures for a convex function. See [GM1] and [GM2].

In order to have a reasonable theory we need a comparison principle, since it implies uniqueness. It also implies existence (under some additional hypothesis) via *Perron's method* (See [C] and [IL]).

The comparison principle for viscosity solutions is based on the "maximum principle for semicontinuous functions"(Crandall-Ishii-Lions, [CIL], in $\mathbb{R}^n$). This principle gives a substitute for the "maximum principle" for smooth functions easily obtained from the subelliptic Taylor formula.

If $u, v \in C^2(\Omega)$ and $u - v$ has a local maximum at $p \in \Omega$, we have

$$Du(p) = Dv(p)$$

and

$$(D_0^2 u(p))^* \leqq (D_0^2 v(p))^*$$

2.5. **The Subelliptic Maximum Principle.** The subelliptic version of the maximum principle for semicontinuous functions was proved by Bieske [Bi] for the case of the Heisenberg group.

Theorem 7. Maximum principle for semicontinuous functions: *Let u be upper semi-continuous in a bounded domain $\Omega \subset \mathcal{H}$. Let v be lower semi-continuous in Ω. Suppose that for $x \in \partial\Omega$ we have*

$$\limsup_{y\to x} u(y) \leq \liminf_{y\to x} v(y),$$

where both sides are not $+\infty$ or $-\infty$ simultaneously. If $u-v$ has an interior local maximum

$$\sup_{\Omega}(u - v) > 0$$

then we have:

For $\tau > 0$ we can find points $p^\tau, q^\tau \in \mathcal{H}$ such that

i)

$$\lim_{\tau\to\infty} \tau\psi(p_\tau \cdot q_\tau^{-1}) = 0,$$

where

$$\psi(x, y, z) = x^4 + y^4 + z^2,$$

ii) *There exists a point $\hat{p} \in \Omega$ such that $p_\tau \to \hat{p}$ (and so does q_τ by (i)) and*

$$\sup_{\Omega}(u - v) = u(\hat{p}) - v(\hat{p}) > 0,$$

iii) *there exist symmetric matrices*

$$\mathcal{X}_\tau, \mathcal{Y}_\tau \in S(\mathbb{R}^2)$$

and vectors

$$\eta_\tau \in \mathbb{R}^3$$

so that

iv)
$$(\eta_\tau, \mathcal{X}_\tau) \in \overline{J}^{2,+}(u, p_\tau),$$
v)
$$(\eta_\tau, \mathcal{Y}_\tau) \in \overline{J}^{2,-}(v, q_\tau),$$
and

vi)
$$\mathcal{X}_\tau \leqq \mathcal{Y}_\tau + o(1)$$
as $\tau \to \infty$.

The last statement means that if $\xi \in \mathbb{R}^2$ we have
$$\langle \mathcal{X}_\tau \xi, \xi \rangle - \langle \mathcal{Y}_\tau, \xi, \xi \rangle \ \leq a(\tau)|\xi|^2,$$
where $a(\tau) \to 0$ as $\tau \to \infty$.

Proof. For $\tau > 0$ set
$$M_\tau = \sup_{\Omega \times \Omega} \{u(p) - v(q) - \tau \psi(p \cdot q^{-1})\}.$$
By semi-continuity $M_\tau < \infty$ and M_τ is attained at a point (p_τ, q_τ). Note that M_τ is decreasing in τ and uniform bounded. We have
$$M_{\tau/2} \geq u(p_\tau) - v(q_\tau) - \frac{\tau}{2}\psi(p_\tau \cdot q_\tau^{-1})$$
$$M_{\tau/2} - \frac{\tau}{2}\psi(p_\tau \cdot q_\tau^{-1}) \geq M_\tau$$
$$M_{\tau/2} - M_\tau \geq \frac{1}{2}\tau\psi(p_\tau \cdot q_\tau^{-1})$$
Conclude first that
$$\lim_{\tau \to \infty} \tau\psi(p_\tau \cdot q_\tau^{-1}) = 0.$$
Next we observe,
$$\sup_\Omega (u(p) - v(p)) \leq M_\tau = u(p_\tau) - v(q_\tau) - \tau\psi(p_\tau \cdot q_\tau^{-1})$$
Since $p_\tau \to \hat{p}$ as well as $q_\tau \to \hat{p}$ (for a subsequence of τ's tending to ∞) we get
$$\sup_\Omega (u(p) - v(p)) = \lim_{\tau \to \infty} M_\tau = u(\hat{p}) - v(\hat{p}).$$
We apply now the **Euclidean maximum principle** for semicontinuous functions of Crandall-Ishii-Lions [CIL]. There exist 3×3 symmetric matrices X_τ, Y_τ so that
$$\big(\tau D_p(\psi(p \cdot q^{-1})), X_\tau\big) \in \overline{J}^{2,+}_{\text{eucl.}}\ (u, p_\tau)$$
and
$$\big(-\tau D_q(\psi(p \cdot q^{-1})), Y_\tau\big) \in \overline{J}^{2,-}_{\text{eucl.}}\ (v, q_\tau)$$
with the property
$$\langle X_\tau \gamma, \gamma \rangle - \langle Y_\tau \chi, \chi \rangle \leqq \langle C\gamma \oplus \chi,\ \gamma \oplus \chi \rangle$$

where the vectors $\gamma, \chi \in \mathbb{R}^3$, and

$$C = \tau(A^2 + A)$$

and

$$A = D^2_{p,q}(\psi(p \cdot q^{-1}))$$

are 6×6 matrices.

We need now a way to get subelliptic jets from euclidean jets.

Lemma 2. Subelliptic jets from Euclidean jets: *Let* $(\eta, X) \in J^{2,+}_{eucl.}(u,p)$ *be a second order Euclidean superjet. Then, we have*

$$(DL_p \cdot \eta, (DL_p \cdot X \cdot (DL_p)^t)_{2\times 2}) \in J^{2,+}(u,p)$$

In the lemma the subindex 2×2 indicates the principal 2×2 minor of a 3×3 matrix. The mapping L_p is just left multiplication by p in $\mathcal{H}$. One can easily see that its differential is given by

$$DL_p = \begin{pmatrix} 1 & 0 & -y/2 \\ 0 & 1 & x/2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Proof. Easy when $p = 0$. Left translate for general p. □

Using this lemma we conclude that

$$\left(\tau DL_{p_\tau} \cdot D_p\psi(p \cdot q^{-1}), (DL_{p_\tau} \cdot X_\tau \cdot (DL_{p_\tau})^t)_{2\times 2}\right) \in \overline{J}^{2,+}(u, p_\tau)$$

and

$$\left(-\tau DL_{q_\tau} \cdot D_q\psi(p \cdot q^{-1}), (DL_{q_\tau} \cdot Y_\tau \cdot (DL_{q_\tau})^t)_{2\times 2}\right) \in \overline{J}^{2,-}(v, q_\tau).$$

Our choice of $\psi(p \cdot q^{-1})$ implies that

$$DL_{p_\tau} \cdot D_p\psi(p \cdot q^{-1}) = -DL_{q_\tau} \cdot D_q\psi(p \cdot q^{-1}).$$

We call this common value η_τ. Let $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$. Write

$$\xi_{p_\tau} = (\xi_1, \xi_2, \ \frac{1}{2}(\xi_2 x_1^\tau - \xi_1 y_1^\tau)),$$

where we have set $p_\tau = (x_1^\tau, y_1^\tau, z_1^\tau)$.

The vector ξ_{p_τ} is chosen so that

$$((DL_{p_\tau})^t)^{-1} \ \xi_{p_\tau} = (\xi_1, \xi_2, 0)$$

Similarly set

$$\xi_{q_\tau} = (\xi_1, \xi_2, \ \frac{1}{2}(\xi_2 x_2^\tau - \xi_1 y_2^\tau)),$$

where we have set $q_\tau = (x_2^\tau, y_2^\tau, z_2^\tau)$. It satisfies

$$((DL_{q_\tau})^t)^{-1} \ \xi_{q_\tau} = (\xi_1, \xi_2, 0).$$

Set

$$\mathcal{X}_\tau = (DL_{p_\tau} \cdot X_\tau \cdot (DL_{p_\tau})^t)_{2\times 2}$$

and

$$\mathcal{Y}_\tau = (DL_{q_\tau} \cdot Y_\tau \cdot (DL_{q_\tau})^t)_{2\times 2}.$$

With these choices we have

$$\begin{aligned}
\langle \mathcal{X}_\tau \xi, \xi\rangle - \langle \mathcal{Y}_\tau \xi, \xi\rangle = & \\
&= \langle X_\tau \xi_{p_\tau}, \xi_{p_\tau}\rangle - \langle Y_\tau \xi_{q_\tau}, \xi_{q_\tau}\rangle \\
&\leqq \langle C(\xi_{p_\tau} \oplus \xi_{q_\tau}), \xi_{p_\tau} \oplus \xi_{q_\tau}\rangle = \\
&= |\xi|^2 \left(\frac{x_2^\tau y_1^\tau - x_1^\tau y_2^\tau}{2} + z_1^\tau - z_2^\tau \right)^2 \tau \\
&\leqq o(1)\ |\xi|^2. \text{ by (i).}
\end{aligned}$$

□

Remark 2. *Bieske* [Bi] *has considered more general functions ψ of the form*

$$\psi(x, y, z) = x^{2n} + y^{2n} + z^{2m}$$

Estimate (vi) in this case is more complicated. It takes the form

$$\begin{aligned}
\langle \mathcal{X}_\tau \xi, \xi\rangle - \mathcal{Y}_\tau \xi, \xi\rangle &\leqq 2m^2 |\xi|^2\ \tau (z_1^\tau - z_2^\tau + \frac{1}{2}(z_2^\tau y_1^\tau - z_1^\tau y_1^\tau))^{-2+4m} \\
&\leqq 2m^2 |\xi|^2\ \tau \psi^{2-\frac{1}{m}}(p_\tau \cdot q_\tau^{-1}) \\
&\leqq 2m^2 |\xi|^2\ (\tau \psi(p_\tau \cdot q_\tau^{-1}))\ \psi^{1-\frac{1}{m}}(p_\tau \cdot q_\tau^{-1}).
\end{aligned}$$

This improvement is necessary to prove uniqueness for the subelliptic ∞-Laplacian, [Bi].

2.6. **Examples of Comparison Principles.** We have included a couple of examples where the comparison principle holds. As explained in [CIL], once we have the subelliptic version of the maximum principle, many more cases can be analyzed. See also [IL] for more examples.

2.7. **Degenerate Elliptic Equations, Constant Coefficients.**

Theorem 8. *Let F be continuous and independent of p. Suppose that F satisfies*

$$\frac{\partial F}{\partial u} \geq \gamma > 0,$$

F is decreasing in $\mathcal{X}$, and

$$|F(u, \eta, \mathcal{X}) - F(u, \eta, \mathcal{Y})| \leqq w(\mathcal{X} - \mathcal{Y}),$$

where $w \to 0$ as $\mathcal{X} - \mathcal{Y} \to 0$. Let u be an upper semicontinuous subsolution and v a lower semicontinuous supersolution of

$$F(u, Du, (D_0^2 u)^*) = 0$$

in a domain Ω such that

$$\limsup_{y\to x} u(y) \le \liminf_{y\to x} v(y), \quad x \in \partial\Omega,$$

where both sides are not $+\infty$ or $-\infty$ simultaneously. Then

$$u(x) \leqq v(x) \quad \text{for all } x \in \Omega.$$

Proof. Let us proceed by contradiction. Suppose that $\sup_\Omega(u-v) > 0$ and apply the subelliptic maximum principle from §2.5. For τ large enough we have:

$$\begin{aligned} 0 < \gamma(u(p_\tau) - v(q_\tau)) &\le F(u(p_\tau), \eta_\tau, \mathcal{X}_\tau) - F(v(q_\tau), \eta_\tau, \mathcal{X}_\tau) \\ &\quad + F(v(q_\tau), \eta_\tau, \mathcal{Y}_\tau) - F(v(q_\tau), \eta_\tau, \mathcal{Y}_\tau) \\ &\leqq F(v(q_\tau), \eta_\tau, \mathcal{Y}_\tau) - F(v(q_\tau), \eta_\tau, \mathcal{X}_\tau) \\ &\leqq F(v(q_\tau), \eta_\tau, \mathcal{Y}_\tau) - F(v(q_\tau), \eta_\tau, \mathcal{Y}_\tau + \mathcal{R}_\tau) \\ &\le w(\mathcal{R}_\tau) \to 0 \quad \text{as } \tau \to \infty, \end{aligned}$$

where we have used the facts that u is a subsolution, that v is a supersolution, the inequality

$$\mathcal{X}_\tau \le \mathcal{Y}_\tau + \mathcal{R}_\tau,$$

the fact that F is proper and the continuity of F.

□

2.8. **Uniformly Elliptic Equations with no First Derivatives Dependence.**

Theorem 9. *Let F be continuous, proper, independent of η and satisfying*

$$|F(p, u, \mathcal{X}) - F(q, u, \mathcal{X})| \leqq w(p, q),$$

where $w(p,q) \to 0$ as $q^{-1} \cdot p \to 0$. Suppose that F uniformly elliptic in the following sense. There exists a constant $\gamma > 0$ such that

$$F(p, u, \mathcal{X}) - F(p, u, \mathcal{Y}) \leqq -\gamma \,\mathit{trace}\,(\mathcal{X} - \mathcal{Y})$$

whenever $\mathcal{X} \ge \mathcal{Y}$. Let u be an upper semicontinuous subsolution of

$$F(p, u, (D_0^2 u)^*) = 0$$

in a domain Ω, v a lower semicontinuous supersolution, $u \leqq v$ on $\partial\Omega$ as in 2.7. Then

$$u(x) \leqq v(x) \quad \text{for all } x \in \Omega.$$

Proof. Suppose this is not the case. Then $\sup_\Omega\,(u - v) > 0$. Let φ be a non-negative smooth function such that

$$(D_0^2 \varphi)^* < 0$$

everywhere in the bounded domain Ω (φ could be a quadratic polynomial). Note that for $\delta > 0$ small enough, the function $u - v_\delta$ has an interior maximum, where $v_\delta = v + \delta\varphi$.

Claim. *The function v_δ is a strict supersolution of*

$$F(p, u, (D_0^2 u)^*) = 0.$$

Proof. Let ψ be a smooth test function touching v_δ from below at p_0

$$\begin{aligned} \psi(p_0) &= v_\delta(p_0) \\ \psi(p) &< v_\delta(p), \quad p \neq p_0. \end{aligned}$$

Then $\psi(p) - \delta\varphi(p)$ is a smooth test function that touches v from below at p_0. Thus, we have

$$F(p_0, \psi(p_0) - \delta\varphi(p_0),\ (D_0^2\psi)^* - \delta(D_0^2\varphi)^*) \geq 0,$$

which implies

$$F(p_0, \psi(p_0),\ (D_0^2\psi)^* - \delta(D_0^2\varphi)^*) \geq 0.$$

Using the uniform ellipticity we get

$$\begin{aligned} F(p_0, \psi(p_0), (D_0^2\psi)^*) - F(p_0, \psi(p_0),\ (D_0^2\psi)^* {-} \delta(D_0^2\varphi)^*) \\ \geq \gamma \text{ trace } (-\delta(D_0^2\varphi)^*(p_0)). \end{aligned}$$

We conclude that

$$F(p_0, \psi(p_0),\ (D_0^2\psi)^*) \geq \delta\gamma \text{ trace } (-(D_0^2\varphi)^*)(p_0) > 0$$

thereby proving the claim. □

Apply the subelliptic maximum principle to $u - v_\delta$. Since F is increasing in u:

$$\begin{aligned} 0 < \ & F(p_\tau, u(p_\tau), \mathcal{X}_\tau) - F(p_\tau, v_\delta(q_\tau), \mathcal{X}_\tau) \\ & + F(q_\tau, v(q_\tau), \mathcal{X}_\tau) - F(q_\tau, v_\delta(q_\tau), \mathcal{X}_\tau) \\ & \leqq 0 + w(p_\tau, q_\tau)) \end{aligned}$$

getting a contradiction for τ large enough. □

3. Lecture III: The Maximum Principle for Riemannian Vector Fields

Consider a frame $\mathfrak{X} = \{X_1, X_2, \ldots, X_n\}$ in $\mathbb{R}^n$ consisting of n linearly independent smooth vector fields. Write

$$X_i(x) = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial}{\partial x_j}$$

for some smooth functions $a_{ij}(x)$. Denote by $\mathbb{A}(x)$ the matrix whose (i,j)-entry is $a_{ij}(x)$. We always assume that $\det(\mathbb{A}(x)) \neq 0$ in $\mathbb{R}^n$.

We need to write down an appropriate Taylor theorem. We will use exponential coordinates as done in [NSW]. Fix a point $p \in \mathbb{R}^n$ and let $t = (t_1, t_2, \ldots, t_n)$ denote a vector close to zero. We define the exponential based at p of t, denoted by $\Theta_p(t)$, as follows: Let γ be the unique solution to the system of ordinary differential equations

$$\gamma'(s) = \sum_{i=1}^{n} t_i X_i(\gamma(s))$$

satisfying the initial condition $\gamma(0) = p$. We set $\Theta_p(t) = \gamma(1)$ and note this is defined in a a neighborhood of zero. In fact we have the following Calculus lemma.

Lemma 3. *Write $\Theta_p(t) = \big(\Theta_p^1(t), \Theta_p^2(t), \dots, \Theta_p^n(t)\big)$. Note that we can think of $X_i(x)$ as the i-th row of $\mathbb{A}(x)$. Similarly $D\Theta_p^k(0)$ is the k-th column of $\mathbb{A}(p)$ so that*

$$D\Theta_p(0) = \mathbb{A}(p).$$

For the second derivative we get

$$\langle D^2\Theta_p^k(0)h, h\rangle = \langle \mathbb{A}^t(p)h, D(\mathbb{A}^t(p)h)_k\rangle$$

for all vectors $h \in \mathbb{R}^n$.

In particular, the mapping $t \mapsto \Theta_p(t)$ is a diffeomorphism taking a a neighborhood of 0 into a neighborhood of p.

The gradient of a function u relative to the frame $\mathfrak{X}$ is

$$D_{\mathfrak{X}}u = (X_1(u), X_2(u), \dots, X_n(u)).$$

The second derivative matrix $D^2_{\mathfrak{X}}u$ is an $n \times n$ - not necessarily symmetric matrix with entries $X_i(X_j(u))$. We shall be interested in the quadratic form determined by this matrix, which is the same as the quadratic form determined by the symmetrized second derivative

$$(D^2_{\mathfrak{X}}u)^* = \frac{1}{2}\left(D^2_{\mathfrak{X}}u + (D^2_{\mathfrak{X}}u)^t\right).$$

The Taylor expansion from [NSW] can be stated as follows:

Lemma 4. *Let u be a smooth function in a neighborhood of p. We have:*

$$u\left(\Theta_p(t)\right) = u(p) + \langle D_{\mathfrak{X}}u(p), t\rangle + \frac{1}{2}\langle \left(D^2_{\mathfrak{X}}u(p)\right)^* t, t\rangle + o(|t|^2)$$

as $t \to 0$.

A natural question is how $D_{\mathfrak{X}}u$ and $(D^2_{\mathfrak{X}}u)^*$ change if we change frames. We could write down a long (and uninteresting) formula for the change between two general frames. Here is how to go from the canonical frame $\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots \frac{\partial}{\partial x_n}\}$ to the frame $\mathfrak{X}$. The gradient relative to the canonical frame is denoted ∇.

Lemma 5. *For smooth functions u we have*

$$D_{\mathfrak{X}}u = \mathbb{A} \cdot \nabla u$$

and for all $t \in \mathbb{R}^n$

$$\langle \left(D^2_{\mathfrak{X}} u\right)^{*} \cdot t, t\rangle = \langle \mathbb{A} \cdot D^2 u \cdot \mathbb{A}^t \cdot t, t\rangle + \sum_{k=1}^{n} \langle \mathbb{A}^t \cdot t, \nabla \left(\mathbb{A}^t \cdot t\right)_k\rangle \frac{\partial u}{\partial x_k}.$$

The Taylor series gives the following counterpart of theorem 1.

Lemma 6. *Let* u *and* v *be smooth functions such that* $u - v$ *has an interior local maximum at* p. *Then we have*

$$D_{\mathfrak{X}} u(p) = D_{\mathfrak{X}} v(p) \tag{3.1}$$

and

$$\left(D^2_{\mathfrak{X}} u(p)\right)^{*} \leq \left(D^2_{\mathfrak{X}} v(p)\right)^{*}. \tag{3.2}$$

For the purposes of illustrations let us consider some examples:

Example 1. *The canonical frame*

This is just $\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots \frac{\partial}{\partial x_n}\}$. The first and second derivatives are just the usual ones and the exponential mapping is just addition

$$\Theta_p(t) = p + t.$$

Example 2. *The Heisenberg group*

The frame is given by the left invariant vector fields $\{X_1, X_2, X_3\}$. For $p = (x, y, z)$ the matrix $\mathbb{A}$ is just

$$\mathbb{A}(p) = \begin{pmatrix} 1 & 0 & -y/2 \\ 0 & 1 & x/2 \\ 0 & 0 & 1 \end{pmatrix}.$$

A simple calculation shows that

$$\langle \mathbb{A}^t \cdot t, D\left(\mathbb{A}^t \cdot t\right)_k\rangle = 0$$

not only for $k = 1$ and $k = 2$ but also for $k = 3$. That is, although $\mathbb{A}$ is not constant, we have that Lemma 5 simplifies to

$$\langle \left(D^2_{\mathfrak{X}} u\right)^{*} \cdot t, t\rangle = \langle \mathbb{A} \cdot D^2 u \cdot \mathbb{A}^t \cdot t, t\rangle. \tag{3.3}$$

The exponential mapping is just the group multiplication

$$\Theta_p(t) = p \cdot \Theta_0(t) = (x + t_1, y + t_2, z + t_3 + (1/2)(x t_2 - y t_1)).$$

From Lemma 5 we see that the additional simplification of (3.3) occurs whenever $D^2 \Theta^k_p(0) = 0$. This is true for all step 2 groups as it can be seen from the Campbell-Hausdorff formula.

Example 3. *The Engel group*

This is a step 3 group for which the analogue of (3.3) does not work. Denote by $p=(x,y,z,w)$ a point in $\mathbb{R}^4$. The frame is given by the vector fields:

$$\begin{cases} X_1 &= \frac{\partial}{\partial x} - \frac{y}{2}\frac{\partial}{\partial z} + \left(\frac{-xy}{12} - \frac{z}{2}\right)\frac{\partial}{\partial w} \\ X_2 &= \frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial z} + \frac{x^2}{12}\frac{\partial}{\partial w} \\ X_3 &= \frac{\partial}{\partial z} + \frac{x}{2}\frac{\partial}{\partial w} \\ X_4 &= \frac{\partial}{\partial w} \end{cases}$$

So that the matrix $\mathbb{A}$ is

$$\mathbb{A} = \begin{pmatrix} 1 & 0 & -\frac{y}{2} & \frac{-xy}{12} - \frac{z}{2} \\ 0 & 1 & \frac{x}{2} & \frac{x^2}{12} \\ 0 & 0 & 1 & \frac{x}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Let $t=(t_1,t_2,t_3,t_4)$. Direct calculations shows that

$$\begin{aligned} \langle \mathbb{A}^t \cdot t, D\left(\mathbb{A}^t \cdot t\right)_1\rangle &= 0 \\ \langle \mathbb{A}^t \cdot t, D\left(\mathbb{A}^t \cdot t\right)_2\rangle &= 0 \\ \langle \mathbb{A}^t \cdot t, D\left(\mathbb{A}^t \cdot t\right)_3\rangle &= 0 \end{aligned}$$

but for $k=4$ we get

$$\langle \mathbb{A}^t \cdot t, D\left(\mathbb{A}^t \cdot t\right)_k\rangle = \frac{x}{3}t_1t_2 - \frac{y}{3}t_1^2 + \frac{1}{2}t_1t_3.$$

Therefore, for a smooth function u we have

$$D_{\mathfrak{X}}u = \mathbb{A}\cdot Du$$

and for all $t\in\mathbb{R}^n$

$$\langle \left(D^2_{\mathfrak{X}}u\right)^* \cdot t, t\rangle = \langle \mathbb{A}\cdot D^2u \cdot \mathbb{A}^t \cdot t, t\rangle + \left(\frac{x}{3}t_1t_2 - \frac{y}{3}t_1^2 + \frac{1}{2}t_1t_3\right)\frac{\partial u}{\partial w}.$$

3.1. **Jets.** To define second order superjets of an upper-semicontinuous function u, let us consider smooth functions φ touching u from above a a point p.

$$K^{2,+}(u,p) = \Big\{\varphi \in C^2 \text{in a neighborhood of } p, \varphi(p) = u(p),$$
$$\varphi(q) \geq u(q),\ q \neq p \text{ in } a \text{ neighborhood of } p\Big\}$$

Each function $\varphi \in K^{2,+}(u,p)$ determines a pair (η, A) by

$$\begin{aligned} \eta &= \big(X_1\varphi(p), X_2\varphi(p), \ldots, X_n\varphi(p)\big) \\ A_{ij} &= \tfrac{1}{2}\big(X_i(X_j(\varphi))(p) + X_j(X_i(\varphi))(p)\big). \end{aligned} \tag{3.4}$$

This representation clearly depends on the frame $\mathfrak{X}$. Using the Taylor theorem for φ and the fact that φ touches u from above at p we get

$$u\left(\Theta_p(t)\right) \leq u(p) + \langle \eta, t\rangle + \frac{1}{2}\langle At, t\rangle + o(|t|^2) \tag{3.5}$$

We may also consider $J^{2,+}_{\mathfrak{X}}(u,p)$ defined as the collections of pairs (η, X) such that (3.5) holds. Using the identification given by (3.4) it is clear that

$$K^{2,+}(u,p) \subset J^{2,+}_{\mathfrak{X}}(u,p).$$

In fact, we have equality. This is the analogue of the Crandall-Ishii Lemma of [C].

Lemma 7.

$$K^{2,+}(u,p) = J^{2,+}_{\mathfrak{X}}(u,p).$$

Proof. Given a pair $(\eta, X) \in J^{2,+}_{\mathfrak{X}}(u,p)$ we must find a C^2 function φ so that (3.4) holds. Given any pair (ξ, Y) the version of the lemma for the canonical frame in [C] gives a C^2 function φ touching u from above at p such that $D\varphi(p) = \xi$ and $D^2\varphi(p) = Y$. Using Lemma 5 we get

$$D_{\mathfrak{X}}\varphi(p) = \mathbb{A}(p) \cdot \xi$$

and

$$\langle \left(D^2_{\mathfrak{X}}\varphi\right)^{*} \cdot t, t\rangle = \langle \mathbb{A} \cdot Y \cdot \mathbb{A}^{t} \cdot t, t\rangle + \sum_{k=1}^{n} \langle \mathbb{A}^{t} \cdot t, D\left(\mathbb{A}^{t} \cdot t\right)_k \rangle \xi_k.$$

Thus, it suffices to solve for (ξ, Y) the equations

$$\eta = \mathbb{A}(p) \cdot \xi$$

and

$$\langle X \cdot t, t\rangle = \langle \mathbb{A} \cdot Y \cdot \mathbb{A}^{t} \cdot t, t\rangle + \sum_{k=1}^{n} \langle \mathbb{A}^{t} \cdot t, D\left(\mathbb{A}^{t} \cdot t\right)_k \rangle \xi_k.$$

□

Theorem 10. [BBM] The maximum principle for semicontinuous functions: *Let u be upper semi-continuous in a bounded domain $\Omega \subset \mathbb{R}^n$. Let v be lower semi-continuous in Ω. Suppose that for $x \in \partial\Omega$ we have*

$$\limsup_{y \to x} u(y) \le \liminf_{y \to x} v(y),$$

where both sides are not $+\infty$ or $-\infty$ simultaneously. If $u - v$ has a positive interior local maximum

$$\sup_{\Omega}(u - v) > 0$$

then we have:

For $\tau > 0$ we can find points $p_\tau, q_\tau \in \mathbb{R}^n$ such that

i)
$$\lim_{\tau \to \infty} \tau |p_\tau - q_\tau|^2 = 0,$$

ii) *there exists a point $\hat{p} \in \Omega$ such that $p_\tau \to \hat{p}$ (and so does q_τ by (i)) and*
$$\sup_{\Omega}(u - v) = u(\hat{p}) - v(\hat{p}) > 0,$$

iii) *there exist symmetric matrices $\mathcal{X}_\tau, \mathcal{Y}_\tau$ and vectors η^+_τ, η^-_τ so that*

iv)
$$(\eta_\tau^+, \mathcal{X}_\tau) \in \overline{J}^{2,+}_{\mathfrak{X}}(u, p_\tau),$$
v)
$$(\eta_\tau^-, \mathcal{Y}_\tau) \in \overline{J}^{2,-}_{\mathfrak{X}}(v, q_\tau),$$
vi)
$$\eta_\tau^+ - \eta_\tau^- = o(1)$$
and

vi)
$$\mathcal{X}_\tau \leqq \mathcal{Y}_\tau + o(1)$$
as $\tau \to \infty$.

Proof. The idea of the proof as in the case of theorem 7 is to use the Euclidean theorem to get the jets and then twist them into position. As in the proof of theorem 7, for $\tau > 0$ we get points p_τ and q_τ so that (i) and (ii) hold. We apply now the **Euclidean maximum principle** for semicontinuous functions of Crandall-Ishii-Lions [CIL]. There exist $n \times n$ symmetric matrices X_τ, Y_τ so that
$$(\tau D_p(\psi(p_\tau, q_\tau)), X_\tau) \in \overline{J}^{2,+}_{\text{eucl.}}(u, p_\tau)$$
and
$$(-\tau D_q(\psi(p_\tau, q_\tau)), Y_\tau) \in \overline{J}^{2,-}_{\text{eucl.}}(v, q_\tau)$$
with the property
$$\langle X_\tau \gamma, \gamma\rangle - \langle Y_\tau \chi, \chi\rangle \leqq \langle C\gamma \oplus \chi,\ \gamma \oplus \chi\rangle \tag{3.6}$$
where the vectors $\gamma, \chi \in \mathbb{R}^n$, and
$$C = \tau(A^2 + A)$$
and
$$A = D^2_{p,q}(\psi(p_\tau, q_\tau))$$
are $2n \times 2n$ matrices.

Let us now twist the jets according to Lemma 5. Call $\xi_\tau^+ = \tau D_p(\psi(p_\tau, q_\tau))$ and $\xi_\tau^- = -\tau D_q(\psi(p_\tau, q_\tau))$. By our choice of ψ we get $\xi_\tau^+ = \xi_\tau^-$. Set
$$\eta_\tau^+ = \mathbb{A}(p_\tau) \cdot \xi_\tau^+$$
and
$$\eta_\tau^- = \mathbb{A}(q_\tau) \cdot \xi_\tau^-.$$
We see that
$$\begin{aligned} |\eta_\tau^+ - \eta_\tau^-| &= |\mathbb{A}(p_\tau) - \mathbb{A}(q_\tau)||\xi_\tau^+| \\ &\leq C\tau|p_\tau - q_\tau||D_p(\psi(p_\tau, q_\tau))| \\ &\leq C\tau\psi(p_\tau, q_\tau) \\ &= o(1), \end{aligned}$$
where we have used the fact that $|p-q||D_p\psi(p,q)| \leq C\psi(p,q)$, property (i) and the smoothness, in the form of a Lispchitz condition, of $\mathbb{A}(p)$.

The second order parts of the jets are given by

$$\langle \mathcal{X}_\tau \cdot t, t\rangle = \langle \mathbb{A}(p_\tau) X_\tau \mathbb{A}^t(p_\tau)\cdot t, t\rangle + \sum_{k=1}^n \langle \mathbb{A}^t(p_\tau)\cdot t, D(\mathbb{A}^t(p)\cdot t)_k[p_\tau]\rangle (\xi_\tau^+)_k$$

and

$$\langle \mathcal{Y}_\tau \cdot t, t\rangle = \langle \mathbb{A}(q_\tau) Y_\tau \mathbb{A}^t(q_\tau)\cdot t, t\rangle + \sum_{k=1}^n \langle \mathbb{A}^t(q_\tau)\cdot t, D(\mathbb{A}^t(p)\cdot t)_k[q_\tau]\rangle (\xi_\tau^-)_k.$$

In order to estimate their difference we write

$$\begin{aligned} \langle \mathcal{X}_\tau \cdot t, t\rangle - \langle \mathcal{Y}_\tau \cdot t, t\rangle &= \langle X_\tau \mathbb{A}^t(p_\tau)\cdot t, \mathbb{A}^t(p_\tau)\cdot t\rangle - \langle Y_\tau \mathbb{A}^t(q_\tau)\cdot t, \mathbb{A}^t(q_\tau)\cdot t\rangle \\ &\quad + \textstyle\sum_{k=1}^n \langle \mathbb{A}^t(p_\tau)\cdot t, D(\mathbb{A}^t(p)\cdot t)_k[p_\tau]\rangle (\xi_\tau^+)_k \\ &\quad - \textstyle\sum_{k=1}^n \langle \mathbb{A}^t(q_\tau)\cdot t, D(\mathbb{A}^t(p)\cdot t)_k[q_\tau]\rangle (\xi_\tau^-)_k. \end{aligned}$$

Using inequality 3.6, we get

$$\begin{aligned} \langle \mathcal{X}_\tau \cdot t, t\rangle - \langle \mathcal{Y}_\tau \cdot t, t\rangle &\le \langle C\left(\mathbb{A}(p_\tau)\cdot t \oplus \mathbb{A}(q_\tau)\cdot t\right), \mathbb{A}(p_\tau)\cdot t \oplus \mathbb{A}(q_\tau)\cdot t\rangle \\ &\quad + \tau\left\{\textstyle\sum_{k=1}^n \langle \mathbb{A}^t(p_\tau)\cdot t, D(\mathbb{A}^t(p)\cdot t)_k[p_\tau]\rangle \frac{\partial\psi}{\partial p_k}(p_\tau, q_\tau)\right\} \\ &\quad - \tau\left\{\textstyle\sum_{k=1}^n \langle \mathbb{A}^t(q_\tau)\cdot t, D(\mathbb{A}^t(p)\cdot t)_k[q_\tau]\rangle \frac{\partial\psi}{\partial p_k}(p_\tau, q_\tau)\right\} \end{aligned}$$

To estimate the first term in the right hand side we note that symmetries of ψ give a block structure to $D^2_{p,q}\psi$ so that we have

$$\langle C(\gamma\oplus\delta), (\gamma\oplus\delta)\rangle \le C\tau|\gamma-\delta|^2.$$

Replacing γ by $\mathbb{A}(p_\tau)\cdot t$ and δ by $\mathbb{A}(q_\tau)\cdot t$, using the smoothness of $\mathbb{A}$, and property (i) we get that this first term is $o(1)$. The second and third term together are also $o(1)$ since their difference is estimated by $\tau|p_\tau - q_\tau||D_p\psi(p_\tau, q_\tau)|$.

□

Once we have the maximum principle (Theorem 10) we get easily comparison theorems for viscosity solutions for fully nonlinear equations of the general form

$$F(x, u(x), D_{\mathfrak{X}}u(x), (D^2_{\mathfrak{X}}u(x))^*) = 0$$

where F is continuous and proper (increasing in u and decreasing in $(D^2_{\mathfrak{X}}u(x))^*$).) as it is done in [CIL]. Here is an example:

Corollary 1. *Suppose $F(p, z, \eta, X)$ satisfies*

$$\begin{aligned} \sigma(r-s) &\le F(p,r,\eta,X) - F(p,s,\eta,X), \\ |F(p,r,\eta,X) - F(q,r,\eta,X)| &\le \omega_1(|p-q|), \\ |F(p,r,\eta,X) - F(p,r,\eta,Y)| &\le \omega_2(|X-Y|) \text{ an} \\ |F(p,r,\eta,X) - F(p,r,\xi,X)| &\le \omega_3(|\eta-\xi|), \end{aligned}$$

where the constant $\sigma > 0$ and the functions $\omega_i \colon [0,\infty) \mapsto [0,\infty)$ satisfy $\omega_i(0^+) = 0$ for $i = 1, 2, 3$. Let u be an upper-continuous viscosity solution and v a lower semi-continuous viscosity supersolution to

$$F(x, u(x), D_{\mathfrak{X}}u(x), (D^2_{\mathfrak{X}}u(x))^*) = 0$$

in a domain Ω *so that for all* $p \in \partial\Omega$ *we have*

$$\limsup_{q\in\Omega, q\to p} u(q) \le \liminf_{q\in\Omega, q\to p} v(q)$$

and both sides are not ∞ *or* $-\infty$ *simultaneously. Then*

$$u(p) \le v(p)$$

for all $p \in \Omega$.

4. Lecture IV: The Maximum Principle in Carnot Groups

A Carnot group $\mathcal{G}$ of step $r \ge 1$ is a simply connected nilpotent Lie group whose Lie algebra $\mathfrak{g}$ is stratified. This means that $\mathfrak{g}$ admits a decomposition as a vector space sum

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_r$$

such that

$$[\mathfrak{g}_1, \mathfrak{g}_j] = \mathfrak{g}_{j+1}$$

for $j = 1, \ldots, r$ with $\mathfrak{g}_k = \{0\}$ for $k > j$. Note that $\mathfrak{g}$ is generated as a Lie algebra by $\mathfrak{g}_1$.

Let $m_j = \dim(\mathfrak{g}_j)$ and choose a basis of $\mathfrak{g}_j$ formed by left-invariant vector fields $X_{i,j}$, $i = 1, \ldots, m_j$. The dimension of $\mathcal{G}$ as a manifold is $m = m_1 + m_2 + \ldots + m_r$. The horizontal tangent space at a point $p \in \mathcal{G}$ is

$$T_{\mathrm{h}}(p) = \text{linear span}\{X_{1,1}(p), X_{2,1}(p), \ldots, X_{m_1,1}(p)\}.$$

As in the Heisenberg group case, we say that a piecewise smooth curve $t \mapsto \gamma(t)$ is horizontal if $\gamma'(t) \in T_h(\gamma(t))$ whenever $\gamma'(t)$ exists. Given two points $p, q \in \mathcal{G}$ denote by

$$\Gamma(p, q) = \{\text{horizontal curves joining } p \text{ and } \ q\}.$$

Chow's theorem states that $\Gamma(p, q) \neq \phi$.

For convenience, fix an ambient Riemannian metric in $\mathcal{G}$ so that $\mathfrak{X} = \{X_{i,j}\}_{1\le m_j, j=1\le r}$ is a left invariant orthonormal frame and

$$\text{Riemannian vol. element} = \text{Haar measure of } \mathcal{G} = \text{ Lebesgue meas. in } \mathbb{R}^m.$$

The Carnot-Carathèodory metric is then defined by

$$d_{cc}(p, q) = \inf\{\text{length}(\gamma) : \gamma \in \Gamma(p, q)\}.$$

It depends only on the restriction of the ambient Riemannian metric to the horizontal distribution generated by the horizontal tangent spaces. The exponential mapping $\exp\colon \mathfrak{g} \mapsto \mathcal{G}$ is a global diffeomorphism.

A point $p \in \mathcal{G}$ has exponential coordinates $(p_{i,j})_{1\le i\le m_j, 1\le j\le r}$ if

$$p = \exp\left(\sum_{j=1}^{r}\sum_{i=1}^{m_j} p_{i,j} X_{i,j}\right).$$

Denoting by $\cdot$ the group operation in $\mathcal{G}$, the mapping $(p, q) \mapsto p \cdot q$ has polynomial entries when written in exponential coordinates.

The non-isotropic dilations are the group homorphisms given by

$$\delta_t\left(\sum_{j=1}^{r}\sum_{i=1}^{m_j} p_{i,j}X_{i,j}\right) = \sum_{j=1}^{r}\sum_{i=1}^{m_j} t^j p_{i,j}X_{i,j},$$

where $t > 0$.

Recall that references for Carnot groups and Carnot-Carathéodory spaces include: [B], [FS], [H], [G], [GN], and [Lu].

4.1. **Calculus in $\mathcal{G}$.** Given a function $u : \mathcal{G} \mapsto \mathbb{R}$ we consider

$$D_{\mathfrak{X}}u = (X_{i,j}u)_{1\le i\le m_j, 1\le j\le r} \in \mathbb{R}^m,$$

the (full) gradient of u. As a vector field, this is written

$$D_{\mathfrak{X}}u = \sum_{j=1}^{r}\sum_{i=1}^{m_j} (X_{i,j}u)X_{i,j}$$

The *horizontal* gradient of u is

$$D_0 u = (X_{i,1}u)_{1\le i\le m_1} \in \mathbb{R}^{m_1},$$

or as a vector field $D_0 u = \sum_{i=1}^{m_1}(X_{i,1}u)X_{i,1}$.

Theorem 11. Ball-Box Theorem: *(See* [B], [G], [NSW]*) Set* $|p| = d_{cc}(p,0)$*, the Carnot-Carathéodory gauge and let*

$$|p|_{\mathcal{G}} = \left(\sum_{j=1}^{r}\left(\sum_{i=1}^{m_j}|p_{i,j}|^2\right)^{\frac{r!}{j}}\right)^{\frac{1}{2r!}}$$

be a smooth gauge. Then:

$$d_{cc}(p,0) \approx |p|_{\mathcal{G}} \approx \sum_{j=1}^{r}\sum_{i=1}^{m_j}|p_{i,j}|^{\frac{1}{j}}$$

and

$$Vol(B(0,r)) \approx r^Q,$$

where $B(0,r)$ *is the Carnot-Carathéodory ball centered at* 0 *of radius* r *and* $Q = \sum_{j=1}^{r} j m_j$ *is the homogeneous dimension of* $\mathcal{G}$.

4.2. **Taylor Formula.** Suppose that $u : \mathcal{G} \mapsto \mathbb{R}$ is a smooth function. Let us write down the Taylor expansion with respect to the frame $\mathfrak{X}$ at $p = 0$.

Write $P = \sum_{j=1}^{r}\sum_{i=1}^{m_j} p_{i,j}X_{i,j}$ so that $p = (p_{i,j}) = \exp(P)$.

$$\begin{aligned} u(\exp(P)) &= u(0) \\ &+ \sum_{j=1}^{r}\sum_{i=1}^{m_j}(X_{i,j}u)(0)p_{i,j} \\ &+ \frac{1}{2}\left(\sum_{i,j,k,l}(X_{k,l}X_{i,j}u)(0)p_{k,l}p_{i,j}\right) \\ &+ o\left(\sum_{j=1}^{r}\sum_{i=1}^{m_j}|p_{i,j}|^2\right) \end{aligned}$$

as $p \to 0$. We want to replace the quadratic norm in the error term by

$$|p|_{\mathcal{G}}^2 \approx \sum_{j=1}^{r}\sum_{i=1}^{m_j}|p_{i,j}|^{\frac{2}{j}}.$$

For $j \geq 3$ it is clear that $p_{i,j} = o(|p_{i,j}|^{\frac{2}{j}})$ and for $j, k \geq 2$ it is also clear that $p_{i,j}p_{l,k} = o(|p_{i,j}|^{\frac{2}{j}})$. Throwing all these terms into the reminder we obtain the subriemannian Taylor formula

$$\begin{aligned} u(\exp(P)) &= u(0) \\ &+ \sum_{j=1}^{2}\sum_{i=1}^{m_j}(X_{i,j}u)(0)p_{i,j} \\ &+ \frac{1}{2}\left(\sum_{k,i=1}^{m_1}(X_{k,1}X_{i,1}u)(0)p_{k,1}p_{i,1}\right) \\ &+ o|p|_{\mathcal{G}}^2 \end{aligned}$$

We already had defined the horizontal gradient of u as

$$D_0u = (X_{i,1}u)_{1\leq i\leq m_1}.$$

This is the part of the gradient corresponding to $\mathfrak{g}_1$. It is clear from the second order Taylor formula that the part corresponding to $\mathfrak{g}_2$, which we call second order horizontal gradient for lack of a better name,

$$D_1u = (X_{i,2}u)_{1\leq i\leq m_2}$$

will also play a role in theory of second order subelliptic equations. We also write

$$D_0^2u = (X_{i,1}X_{j,1}u)_{i,j=1}^{m_1}$$

for the second order derivatives corresponding to $\mathfrak{g}_1$ and $(D_0^2u)^*$ for its symmetric part $\frac{1}{2}(D_0^2u + (D_0^2u)^t)$. With these notations the Taylor formula at 0 reads

$$\begin{aligned} u(p) = u(0) \\ &+ \langle D_0u(0), p_1\rangle + \langle D_1u(0), p_2\rangle \\ &+ \frac{1}{2}\langle (D_0^2u(0))^* \cdot p_1, p_1\rangle \\ &+ o|p|^2_{\mathcal{G}}, \end{aligned}$$

where we have written $p_1 = (p_{i,1})_{1\leq i\leq m_1}$ and $p_2 = (p_{i,2})_{1\leq i\leq m_2}$.

At another point p_0, we get the horizontal Taylor formula by left-translation.

Lemma 8. *If $u\colon \mathcal{G} \mapsto \mathbb{R}$ is a smooth function near p_0 we have*

$$\begin{aligned} u(p) = u(p_0) &+ \langle D_0u(p_0), (p_0^{-1}\cdot p)_1\rangle + \langle D_1u(p_0), (p_0^{-1}\cdot p)_2\rangle \\ &+ \frac{1}{2}\langle (D_0^2u(p_0))^*(p_0^{-1}\cdot p)_1, (p_0^{-1}\cdot p)_1\rangle + o(|p_0^{-1}\cdot p|^2_{\mathcal{G}}) \end{aligned}$$

as $p \longrightarrow p_0$.

We continue the development of the theory as we did in the Heisenberg group case.

4.3. **Subelliptic Jets.** Let u be an upper-semicontinuous real function in $\mathcal{G}$. The second order superjet of u at p_0 is defined as

$$\begin{aligned} J^{2,+}(u,p_0) = \Big\{ &(\eta,\xi,\mathcal{X}) \in \mathbb{R}^{m_1}\times\mathbb{R}^{m_2}\times\mathcal{S}^{m_1}(\mathbb{R}) \text{ such that} \\ &u(p) \leqq u(p_0) + \langle \eta, (p_0^{-1}\cdot p)_1\rangle + \langle \xi, (p_0^{-1}\cdot p)_2\rangle \\ &\quad + \frac{1}{2}\langle \mathcal{X}(p_0^{-1}\cdot p)_1, (p_0^{-1}\cdot p)_1\rangle + o(|p_0^{-1}p|^2_{\mathcal{G}}) \Big\} \end{aligned}$$

Similarly, for lower-semincontinuous v, we define the second order subjet

$$\begin{aligned} J^{2,-}(v,p_0) = \Big\{ &(\eta,\xi,\mathcal{Y}) \in \mathbb{R}^{m_1}\times\mathbb{R}^{m_2}\times\mathcal{S}^{m_1}(\mathbb{R}) \text{ such that} \\ &v(p) \geq v(p_0) + \langle \eta, (p_0^{-1}\cdot p)_1\rangle + \langle \xi, (p_0^{-1}\cdot p)_2\rangle \\ &\quad + \frac{1}{2}\langle \mathcal{Y}(p_0^{-1}\cdot p)_1, (p_0^{-1}\cdot p)_1\rangle + o(|p_0^{-1}p|^2_{\mathcal{G}}) \Big\} \end{aligned}$$

As before, one way to get jets is by using smooth functions that touch u from above or below. Let Γ^2 denote the class of function ϕ such that $D_0\phi, D_1\phi$ and $D_0^2\phi$ are continuous. We define

$$\begin{aligned} K^{2,+}(u,p_0) = \Big\{ &(D_0\varphi(p_0), D_1\varphi(p_0), (D^2\varphi(p_0))^*) : \varphi \in \Gamma^2 \\ &\varphi(p_0) = u(p_0) \\ &\varphi(p) \geq u(p),\ p \neq p_0 \text{ in } a \text{ neighborhood of } p_0 \Big\}. \end{aligned}$$

The set $K^{2,+}(u,p_0)$ is defined analogously.

Lemma 9. *(See* [C] *for the Euclidean case,* [Bi] *for the Heisenberg group case, and* [BM] *for the Carnot case) We always have*

$$K^{2,+}(u,p_0) = J^{2,+}(u,p_0)$$

and

$$K^{2,-}(u,p_0) = J^{2,-}(u,p_0)$$

We also define the closure of the second order superjet of an upper-semicontinuous function u at p_0, denoted by $\bar{J}^{2,+}(u,p_0)$, as the set of triples $(\eta,\xi,\mathcal{X}) \in \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times \mathcal{S}^2(\mathbb{R})$ such that there exist sequences of points p_m and triples $(\eta_m, \xi_m \mathcal{X}_m) \in J^{2,+}(u,p_m)$ such that

$$(p_m, u(p_m), \eta_m, \xi_m, \mathcal{X}_m) \to (p_0, u(p_0), \eta, \xi, \mathcal{X})$$

as $m \to \infty$. The closure of the second order subjet of a lower-semicontinuous function v at p_0, denoted by $\bar{J}^{2,-}(u,p_0)$ is defined in an analogous manner.

4.4. **Fully Non-Linear Equations.** Consider a continuous function

$$\begin{aligned} F :& \mathcal{G} \times \mathbb{R} \times \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times S(\mathbb{R}^{m_1}) \longrightarrow \mathbb{R} \\ &(p,u,\eta,\xi,\mathcal{X}) \longrightarrow F(p,u,\eta,\xi,\mathcal{X}). \end{aligned}$$

We will always assume that F is proper; that is, F is increasing in u and F is decreasing in $\mathcal{X}$.

Definition 5. *A lower semicontinuous function v is a viscosity supersolution of the equation*

$$F(p, v(p), D_0 v(p), D_1 v(p), (D^2 v(p))^*) = 0$$

if whenever $(\eta,\xi,\mathcal{Y}) \in J^{2,-}(v,p_0)$ we have

$$F(p_0, v(p_0), \eta, \xi, \mathcal{Y}) \geq 0.$$

Equivalently, if φ touches v from below and is in Γ^2, then we must have

$$F(p_0, v(p_0), D_0\varphi(p_0), D_1\varphi(p_0),\ (D^2\varphi(p_0))^*) \geq 0.$$

Definition 6. *An upper semicontinuous function u is a viscosity subsolution of the equation*

$$F(p, v(p), D_0 v(p), D_0 v(p), (D^2 v(p))^*) = 0$$

if whenever $(\eta,\xi,\mathcal{X}) \in J^{2,+}(u,p_0)$ we have

$$F(p_0, u(p_0), \eta, \xi, \mathcal{X}) \leq 0.$$

Equivalently, if φ touches u from above, is in Γ^2, then we must have

$$F(p_0, u(p_0), D_0\varphi(p_0), D_0\varphi(p_0),\ (D^2\varphi(p_0))^*) \leqq 0.$$

Note that if u is a viscosity subsolution and $(\eta, \xi, \mathcal{X}) \in \bar{J}^{2,+}(u, p_0)$ then, by the continuity of F, we still have

$$F(p_0, u(p_0), \eta, \xi, \mathcal{X}) \leq 0.$$

A similar remark applies to viscosity supersolutions and the closure of second order subjets.

A viscosity solution is defined as being both a viscosity subsolution and a viscosity supersolution. Observe that since F is proper, it follows easily that if u is a smooth classical solution then u is a viscosity solution.

Examples of F:

- Subelliptic Laplace equation (the Hörmander-Kohn operator):

$$-\Delta_0 u = -\left(\sum_{1 \leq j \leq m_1} X_{j,1} X_{j,1} u \right) = 0$$

- Subelliptic ∞-Laplace equation:

$$-\Delta_{0,\infty} u = -\left[\sum_{i,j=1}^{m_1} (X_{1,i} u)(X_{1,j} u) X_{1,i} X_{1,j} u \right] = -\langle (D_0^2 u)^* D_0 u, D_0 u \rangle$$

- Subelliptic p-Laplace equation, $2 \leq p < \infty$:

$$\begin{aligned} -\Delta_p u &= -\left[|D_0 u|^{p-2} \Delta_0 u + (p-2) |D_0 u|^{p-4} \Delta_{0,\infty} u \right] \\ &= -\text{div}\, (|D_0 u|^{p-2} D_0 u) = 0 \end{aligned}$$

Strictly speaking we need $p \geq 2$ for the continuity of the corresponding F. In the Euclidean case it is possible to extend the definition to the full range $p > 1$. This is a non-trivial matter not yet studied in the case of the Heisenberg group (to the best of my knowledge.) See [JLM] for the Euclidean case.

- "Naive" subelliptic Monge-Ampère

$$-\det(D_0^2 u)^* = f$$

Here the corresponding $F(\mathcal{X}) = -\det \mathcal{X}$ is only proper in the cone of positive semidefinite matrices.

The next step is to generalize to semi-continuous functions the "maximum principle" for smooth functions easily obtained from the subelliptic Taylor formula. If $u, v \in \Gamma^2(\Omega)$ and $u - v$ has a local maximum at $p \in \Omega$, we have

$$D_0 u(p) = D_0 v(p),$$

$$D_1 u(p) = D_1 v(p),$$

and

$$(D_0^2 u(p))^* \leqq (D_0^2 v(p))^*$$

Theorem 12. THE MAXIMUM PRINCIPLE FOR SEMICONTINUOUS FUNCTIONS: *Let u be upper semi-continuous in a bounded domain $\Omega \subset \mathcal{G}$. Let v be lower semi-continuous in Ω. Suppose that for $x \in \partial\Omega$ we have*

$$\limsup_{y \to x} u(y) \leq \liminf_{y \to x} v(y),$$

where both sides are not $+\infty$ or $-\infty$ simultaneously. If $u - v$ has a positive interior local maximum

$$\sup_{\Omega}(u - v) > 0$$

then we have:

For $\tau > 0$ we can find points $p_\tau, q_\tau \in \mathcal{G}$ such that

i)
$$\lim_{\tau \to \infty} \tau |p_\tau - q_\tau|^2 = 0,$$

ii) *there exists a point $\hat{p} \in \Omega$ such that $p_\tau \to \hat{p}$ (and so does q_τ by (i)) and*
$$\sup_{\Omega}(u - v) = u(\hat{p}) - v(\hat{p}) > 0,$$

iii) *there exist $m_1 \times m_1$ symmetric matrices $\mathcal{X}_\tau, \mathcal{Y}_\tau$ and vectors η_τ^+, ξ_τ^+, η_τ^- and ξ_τ^- so that*

iv)
$$(\eta_\tau^+, \xi_\tau^+, \mathcal{X}_\tau) \in \overline{J}^{2,+}(u, p_\tau),$$

v)
$$(\eta_\tau^-, \xi_\tau^-, \mathcal{Y}_\tau) \in \overline{J}^{2,-}(v, q_\tau),$$

vi)
$$\eta_\tau^+ - \eta_\tau^- = o(1),$$
$$\xi_\tau^+ - \xi_\tau^- = o(1)$$

and

vi)
$$\mathcal{X}_\tau \leqq \mathcal{Y}_\tau + o(1)$$

as $\tau \to \infty$.

Proof. Let us apply Theorem 10 to the Carnot group $\mathcal{G}$ endowed with the left-invariant frame $\mathfrak{X}$. We get riemannian jets

$$(\beta_\tau^+, X_\tau) \in \overline{J}_{\mathfrak{X}}^{2,+}(u, p_\tau)$$

and

$$(\beta_\tau^-, Y_\tau) \in \overline{J}_{\mathfrak{X}}^{2,-}(v, q_\tau),$$

satisfying

$$\beta_\tau^+ - \beta_\tau^- = o(1)$$

and

$$X_\tau \leq Y_\tau + o(1).$$

All we need to check is that by keeping the parts of β_τ^+ and β_τ^- in $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ and restricting X_τ and Y_τ to $\mathfrak{g}_1$ we get subelliptic jets.

The following lemma follows from the Taylor theorem arguing as in section 4.2 regarding higher order derivatives as part of the error term.

Lemma 10. Subelliptic jets from Riemannian jets: *Let* $(\beta, X) \in J^{2,+}_{\mathfrak{X}}(u,p)$ *be a second order superjet. Then, we have*

$$(\beta^1, \beta^2, X_{m_1 \times m_1}) \in J^{2,+}(u,p)$$

□

4.5. **Absolutely Minimizing Lipschitz Extensions.** Let $\Omega \subset \mathbb{R}^n$ be a domain. Consider m linearly independent vector fields

$$\{X_1, X_2, \dots, X_m\},$$

where $m \leq n$. If there is an integer $r \geq 1$ such that at any point $x \in \Omega$, the linear span of $\{X_1, X_2, \dots, X_m\}$ and all their commutators up to order r has dimension n, the systems of vector fields $\{X_1, X_2, \dots, X_m\}$ is said to satisfy Hörmander's condition. In [Ho] Hörmander proved that if this condition is satisfied the second order operator $X_1^2 + X_2^2 + \dots + X_m^2$ is hypoelliptic.

The control distance associated to $\{X_1, X_2, \dots, X_m\}$ is defined using horizontal curves as we did in Section §4. This control distance is a genuine metric by the corresponding version of Chow's theorem, see [NSW] and [B], and it satisfies

$$\frac{1}{c_K}|x-y| \leq d_{cc}(x,y) \leq c_K|x-y|^{\frac{1}{r}}$$

for some constant $c_K > 0$ for $x, y \in K$ compact subset of Ω.

For a function $u \colon \Omega \to \mathbb{R}$ the horizontal gradient of u is

$$Xu = (X_1u, X_2u, \dots, X_mu).$$

The Sobolev space $HW^{1,p}(\Omega)$, $1 \leq p \leq \infty$, consists of functions $u \in L^p(\Omega)$ whose distributional horizontal gradient is also in $L^p(\Omega)$. Endowed with the norm

$$\|u\|_p + \|Xu\|_p$$

the space $HW^{1,p}(\Omega)$ is a Banach space. As it is the case in the Euclidean case, Lipschitz functions in a bounded domain Ω with respect to the Carnot-Carathèodory metric d_{cc} are precisely functions is $HW^{1,\infty}(\Omega)$. See [FSS] and [GN].

Definition 7. *A function* $u \in HW^{1,\infty}(\Omega)$ *is an absolute minimizing Lipschitz extension (AMLE) if whenever* $D \subset \Omega$ *is open,* $v \in HW^{1,\infty}(D)$ *and* $u = v$ *on* ∂D *we have*

$$\|Xu\|_{L^\infty(D)} \leq \|Xv\|_{L^\infty(D)}.$$

Given any Lipschitz function $f : \partial\Omega \to \mathbb{R}$ it can always be extended to an AMLE in Ω. This existence result holds in very general metric spaces as shown by Juutinen [Ju2].

Jensen proved in [J2] that AMLE in Euclidean space are viscosity solutions of the ∞-Laplace equation

$$\sum_{i,j=1}^{n} \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} = 0,$$

and also proved a uniqueness theorem for solutions of the Dirichlet problem. As a corollary, he therefore obtained that AMLEs are determined by their boundary values.

For general Carnot groups Bieske and Capogna [BC] proved that AMLEs are viscosity solutions of the corresponding ∞-Laplace equation

$$\Delta_{\infty,0} u = \sum_{i,j=1}^{m} X_i u X_i u X_i X_j u = 0. \tag{4.1}$$

Changyou Wang has recently extended Bieske and Capogna theorem to general Hörmander vector fields:

Theorem 13. [W1] *Let $\{X_1, X_2, \ldots, X_m\}$ be system of Hörmander vector fields and let u be an AMLE. Then u is a viscosity solution of the equation 4.1.*

The natural question to consider is now the uniqueness of viscosity solutions of 4.1. This was proven by Bieske [Bi] in the case of the Heisenberg group by using the extension of the maximum principle mentioned in Remark 2 of Section 2. Bieske has also considered the Grušin plane case in [Bi2], [Bi3]. The case of general Carnot group has recently being settled by Changyou Wang [W1], who introduced subelliptic sup-convolutions. To the best of my knowledge, the case of general Hörmander vector fields is open.

5. Lecture V: Convex functions on Carnot groups

Let us recall the definition of convexity in the viscosity sense:

Definition 8. *Let $\Omega \subset \mathcal{G}$ be an open set and $u\colon \Omega \to \mathbb{R}$ be an upper-semicontinuous function. We say that u is convex in Ω if*

$$(D_0^2 u)^* \geq 0$$

in the viscosity sense. That is, if $p \in \Omega$ and $\phi \in C^2$ touches u from above at p ($\phi(p) = u(p)$ and $\phi(q) \geq u(q)$ for q near p) we have $(D_0^2 \phi)^*(p) \geq 0$.

This definition is compatible with the stratified group structure since convexity is preserved by left-translations and by dilations. Also, uniform limits of convex functions are convex and the supremum of a family of convex functions is convex, since these results hold for viscosity subsolutions in general.

Note that we require the a-priori assumption of upper-semicontinuity as it is done in the definition of sub-harmonic functions. This is not needed when *horizontally convex* functions are considered. These are defined by requiring that whenever $p \in \Omega$ and the horizontal vector $h \in \mathcal{G}_1$ are such that $\{p \cdot \delta_t(h) \colon t \in (-1,1)\} \subset \Omega$ the function of one real variable

$$t \mapsto u(p \cdot \delta_t(h)) \tag{5.1}$$

is convex for $-1 < t < 1$. A development of the theory based on the notion of horizontal convexity can be found in [DGN].

It was established in [LMS] that upper-semicontinuous horizontally convex functions are indeed convex in the sense of Definition 8. In this section we will show that the reciprocal is also true (see Theorem 15 below.) This equivalence has also been established independently by Wang [W2] and Magnani [M].

In the Heisenberg group Balogh and Rickley [BR] proved that condition (5.1) by itself, without requiring upper-semicontinuity, suffices to guarantee that u is continuous - and therefore Lipschitz continuous - and also showed that horizontally convex functions are convex in the sense of Definition 8.

5.1. **Convexity in the Viscosity Sense in $\mathbb{R}^n$.**

In order to illustrate our approach in the case of general Carnot groups, we present here the Euclidean version of Theorem 15 below.

Theorem 14. *Let $\Omega \subset \mathbb{R}^n$ be an open set and $u \colon \Omega \to \mathbb{R}$ be an upper-semicontinuous function. The following statements are equivalent:*

i) *whenever $x, y \in \Omega$ and the segment joining x and y is also in Ω we have*

$$u(\lambda x + (1-\lambda)y) \leq \lambda u(x) + (1-\lambda)u(y) \tag{5.2}$$

for all $0 \leq \lambda \leq 1$.

ii) *u is a viscosity subsolution of all equations*

$$F(x, u(x), Du(x), D^2u(x)) = 0,$$

where $F(x, z, p, M)$ is a continuous function in $\Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n$ satisfying the conditions in Section 4.4 and homogeneous; that is $F(x, z, p, 0) = 0$.

iii) *u is a viscosity subsolution of all linear equations with constant coefficients*

$$F(x, u, Du, D^2u) = -\operatorname{trace}\left(A \cdot D^2u\right) = 0,$$

where $A \in \mathcal{S}^n$ is positive definite.

iv) *$x \to u(Ax)$ is subharmonic for all $A \in \mathcal{S}^n$ positive definite;*

v) *u satisfies the inequality*

$$D^2u \geq 0$$

in the viscosity sense;

vi) *u satisfies $-\operatorname{trace}(A \cdot D^2u) \leq 0$ in the sense of distributions for all $A \in \mathcal{S}^n$ positive definite.*

A function u is convex if one of the above equivalent statements holds.

Proof. For the equivalence between i), ii), iii), and v) we refer to [LMS]. The equivalence between vi) and iii) for any given matrix A is part of viscosity folklore. This is just the simplest case of the theory of Hessian measures of Trudinger and Wang [TW1]. To prove the equivalence between iv) and iii) observe that $x \to u(Ax)$ is subharmonic for all $A > 0$ if and only if $\mathrm{trace}(A \cdot D^2u(x)A^t) \geq 0$ in the sense of distributions for all $A > 0$. This occurs precisely when $\mathrm{trace}(A^t \cdot A \cdot D^2u(x)) \geq 0$ in the sense of distribution for all $A > 0$. Since every positive definite matrix B has a positive definite square root $B = A^2 = A^t \cdot A$, we see that $x \to u(Ax)$ is subharmonic for all $A > 0$ if and only if $\mathrm{trace}(B \cdot D^2u(x)) \geq 0$ in the sense of distribution for all $B > 0$. □

5.2. **Convexity in Carnot Groups.** A key observation is that the notion of convexity depends only on the horizontal distribution and not on the particular choice of a basis of $\mathfrak{g}_1$. More precisely, let us consider two linearly independent horizontal frames

$$\mathfrak{X}_{\mathrm{h}} = \{X_1, \ldots, X_{m_1}\}, \quad \mathfrak{Y}_{\mathrm{h}} = \{Y_1, \ldots, Y_{m_1}\}$$

and write $X_i = \sum_{j=1}^{m_1} a_{ij} Y_j$, for some constants a_{ij}. Let A be the matrix with entries a_{ij}. The matrix A is not singular and the following formula holds for any smooth function ϕ

$$(D^2_{0,\mathfrak{X}}\phi(p))^* = A(D^2_{0,\mathfrak{Y}}\phi(p))^* A^t.$$

Thus the matrix $(D^2_{\mathrm{h},\mathfrak{X}}\phi(p))^*$ is positive definite if and only if $(D^2_{\mathrm{h},\mathfrak{Y}}\phi(p))^*$ is positive definite.

Given a frame $\mathfrak{X}$ we denote by

$$\Delta_{\mathfrak{X}} u = \sum_{i=1}^{m_1} X_i^2 u$$

the corresponding Hörmander-Kohn Laplacian.

The main result of this lecture is the analogue to Theorem 14.

Theorem 15. *Let $\Omega \subset \mathcal{G}$ be an open set and $u \colon \Omega \to \mathbb{R}$ be an upper-semicontinuous function. The following statements are equivalent:*

i) *whenever $p \in \Omega$ and $h \in \mathcal{G}_1$ are such that $\{p \cdot \delta_t(h) \colon t \in (-1, 1)\} \subset \Omega$ the function of one real variable*

$$t \mapsto u(p \cdot \delta_t(h))$$

is convex for $-1 < t < 1$.

ii) *u is a viscosity subsolution of all equations*

$$F(p, u(p), D_0u(p), (D_0^2u(p))^*) = 0,$$

where $F(x, z, p, M)$ is proper and homogeneous.

iii) *u is a viscosity subsolution of all linear equations with constant coefficients*

$$F(p, u, D_0u, (D_0^2u)^*) = -\,\mathrm{trace}\left(A \cdot (D_0^2u)^*\right) = 0,$$

where $A \in \mathcal{S}^{m_1}$ is positive definite.

iv) *u satisfies the inequality $-\Delta_{\mathfrak{Y}} u \leq 0$ in the viscosity sense for all frames $\mathfrak{Y}$ such that $\mathfrak{Y}_{\mathrm{h}} = A\mathfrak{X}_{\mathrm{h}}$, where $A \in \mathcal{S}^{m_1}$ is positive definite.*

v) *u satisfies the inequality*

$$(D_0^2 u)^* \geq 0$$

in the viscosity sense.

vi) *u satisfies $-\operatorname{trace}(A \cdot (D_0^2 u)^*) \leq 0$ in the sense of distributions for all $A \in \mathcal{S}^{m_1}$ positive definite.*

A few remarks are in order. Condition i) is called horizontal convexity in [LMS] and H-convexity in [DGN]. Note that iv) is indeed the analogue of iv) in Theorem 14. Condition v) is called v-convexity in [LMS] and in [W2].

The equivalence of the four viscosity related conditions ii), iii), iv), and v) follows easily from elementary linear algebra facts as in theorem 14. Moreover if one of these conditions holds, then u is locally bounded. This is the case because u is always a subsolution of the corresponding ∞-Laplacian (see 5.3 below). The details in the case of the Heisenberg group are contained in the proof of Lemma 3.1 in [LMS].

To show that iv) implies vi) we may do it one matrix $A \in \mathcal{S}^{m_1}$ at a time. Thus, we may assume that A is the identity matrix. If u is a bounded viscosity subsolution of the Hörmander-Kohn Laplacian, it follows using the same proofs as in Lemma 2.2 and Lemma 2.3 from [LMS] that u is weak-subsolution with first horizontal derivatives locally square integrable. If u is not bounded below, we use the truncation $u_M(x) = \max\{-M, u(x)\}$ and standard limit theorems (see [TW2].)

To prove the equivalence of (i) with the other conditions we need to establish that convex functions can be approximated by smooth convex functions. Note that this is relatively easy to do for horizontally convex functions since the inequality (5.1) is preserved by convolution with a smooth mollifier (see the proof of Theorem 4.2 in [LMS].) Fortunately, Bonfiglioli and Lanconelli [BL] have characterized subharmonic functions by a sub-mean value property and proved that subharmonic functions can be approximated by smooth subharmonic functions. Moreover these approximations are frame independent. If follows from these results that vi) implies iv) since the implication holds for smooth functions, and viscosity subsolutions are preserved by locally uniform limits. The complete details are in [JLMS].

5.3. **Regularity of Convex Functions.** Convex functions are subsolutions of all homogeneous ellliptic equations. In particular we consider the Hörmander-Kohn Laplace equation

$$-\Delta_{\mathrm{h}} u = -(X_1^2 u + \cdots X_m^2 u) = 0,$$

and the subelliptic ∞-Laplace equation

$$-\Delta_{\infty,\mathrm{h}} u = -\sum_{i,j=1}^{m} (X_i u)(X_j u)(X_i X_j u) = 0. \tag{5.3}$$

These equations can certainly be written in the form

$$F(p, u(p), D_0 u(p), (D_0^2 u)^*(p)) = 0.$$

The proof that a convex function u is bounded is done in two steps. To control u^+ we use the subelliptic version of the De Giorgi-Moser estimate for subsolutions of linear elliptic equations

$$\sup_{B_R}(u^+) \leq C \fint_{B_{4R}} u^+ \, dx.$$

To control u^- we use comparison with cones defined using ∞-harmonic functions. The details are in [LMS].

To prove that a convex funtion is Lipschitz, the key ingredient is that subsolutions of (5.3) are Lipschtiz continuous. This was established by Jensen [J2] in the Euclidean case and Bieske [B] for the Heisenberg group. The case of general Carnot groups follows from Wang [W1].

We collect these results in the following:

Theorem 16. *Let $\Omega \subset G$ be an open set and $u\colon \Omega \to \mathbb{R}$ be a convex function. Let B_R be a ball such that $B_{4R} \subset \Omega$. Then u is locally bounded and we have.*

$$\|u\|_{L^\infty(B_R)} \leq C \fint_{B_{4R}} |u| \, dx. \tag{5.4}$$

Moreover, u is locally Lipschitz and we have the bound

$$\|D_{\mathrm{h}} u\|_{L^\infty(B_R)} \leq \frac{C}{R} \|u\|_{L^\infty(B_{2R})}. \tag{5.5}$$

Here C is a constant independent of u and R. If, in addition, u is C^2, then the symmetrized horizontal second derivatives are nonnegative

$$(D_{\mathrm{h}}^2 u)^* \geq 0. \tag{5.6}$$

A different proof of this theorem for horizontally convex functions is in [DGN].

6. Lecture VI: Subelliptic Cordes Estimates

The goal of this lecture is to present some estimates of Cordes type for linear subelliptic partial differential operators in non-divergence form with measurable coefficients in the Heisenberg group, including the linearized p-Laplacian. The following regularity theorems follow from these techniques.

Theorem 17. *Let $\frac{\sqrt{17}-1}{2} \leq p < \frac{5+\sqrt{5}}{2}$. Then any p-harmonic function in the Heisenberg group $\mathcal{H}$ initially in $HW^{1,p}_{\mathrm{loc}}$ is in $HW^{2,2}_{\mathrm{loc}}$.*

Theorem 18. *Given $0 < \alpha < 1$ there exists $\epsilon = \epsilon(\alpha)$ such for $|p-2| < \epsilon$, p-harmonic functions in the Heisenberg group $\mathcal{H}$ have horizontal derivatives that are Hölder continuous with exponent α.*

The results from this section are basend on the work of Marchi [Ma, Ma2] extended by Domokos [D1], which give non-uniform bounds of the $HW^{2,2}$ (or $HW^{2,p}$) norm of approximate p-harmonic functions. Using the Cordes condition [Co, T] and Strichartz's spectral analysis [S] we establish $HW^{2,2}$ estimates for linear subelliptic partial differential operators with measurable coefficients.

Let $\Omega \subset \mathcal{H}$ be a domain in the Heisenberg group. Consider the following Sobolev space with respect to the horizontal vector fields X_i as

$$HW^{2,2}(\Omega) = \{u \in L^2(\Omega) \ : \ X_i X_j u \in L^2(\Omega)\,, \text{ for all } i, j \in \{1, 2\}\}$$

endowed with the inner-product

$$(u, v)_{HW^{2,2}(\Omega)} = \int_\Omega \Big(u(x)v(x) + \sum_{i,j=1}^{2} X_i X_j u(x) \cdot X_i X_j v(x) \Big) dx\,.$$

$HW^{2,2}(\Omega)$ is a Hilbert space and let $HW_0^{2,2}(\Omega)$ be the closure of $C_0^\infty(\Omega)$ in this Hilbert space.

Recall that $\mathcal{X}^2 u$ is the matrix of second order horizontal derivatives whose entries are $(\mathcal{X}^2 u)_{ij} = X_j(X_i u)$, and $\Delta_0 u = \sum_{i=1}^{2} X_i X_i u$ is the subelliptic Laplacian associated to the horizontal vector fields X_i. The first step in proving Cordes estimates is to control the L^2-norm of all the second derivatives by the L^2-norm of the sublaplacian. For the symmetric part of the second derivative, this follows easily by integration by parts but to control Tu we need a more refined argument based on Strichartz [S] spectral analysis.

Lemma 11. *For all $u \in HW_0^{2,2}(\Omega)$ we have*

$$\|\mathcal{X}^2 u\|_{L^2(\Omega)} \leq \sqrt{3}\ \|\Delta_0 u\|_{L^2(\Omega)}.$$

The constant $\sqrt{3}$ is sharp when $\Omega = \mathcal{H}$.

Proof. Since $-\Delta_H$ and iT commute they share the same system of eigenvectors. Strichartz [S] computed them explicitly as well as the corresponding eigenvalues. Computing L^2-norms is now a matter of adding eigenvalues. Details can be found in [DM].

□

6.1. **Cordes conditions.** Let us consider now

$$\mathcal{A}u = \sum_{i,j=1}^{2} a_{ij}(x) X_i X_j u$$

where the functions $a_{ij} \in L^\infty(\Omega)$. Let us denote by $A = (a_{ij})$ the 2×2 matrix of coefficients.

Definition 9. [Co, T] *We say that A satisfies the Cordes condition $K_{\varepsilon,\sigma}$ if there exists $\varepsilon \in (0,1]$ and $\sigma > 0$ such that*

$$0 < \frac{1}{\sigma} \leq \sum_{i,j=1}^{2} a_{ij}^2(x) \leq \frac{1}{1+\varepsilon}\left(\sum_{i=1}^{2} a_{ii}(x)\right)^2, \text{ a.e. } x \in \Omega\,. \tag{6.1}$$

Theorem 19. *Let $0 < \varepsilon \leq 1$, $\sigma > 0$ such that $\gamma = \sqrt{3}\sqrt{1-\varepsilon} < 1$ and A satisfies the Cordes condition $K_{\varepsilon,\sigma}$. Then for all $u \in HW_0^{2,2}(\Omega)$ we have*

$$\|X^2 u\|_{L^2} \leq \sqrt{3}\,\frac{1}{1-\gamma}\,\|\alpha\|_{L^\infty}\|\mathcal{A}u\|_{L^2}\,, \tag{6.2}$$

where

$$\alpha(x) = \frac{\langle A(x), I\rangle}{\|A(x)\|^2}\,.$$

Proof. This is just Linear Algebra. See [DM] for the details. □

6.2. **$HW^{2,2}$-interior regularity for p-harmonic functions in $\mathcal{H}$.** Let $\Omega \subset \mathcal{H}$ be a domain, $h \in HW^{1,p}(\Omega)$ and $p > 1$. Consider the problem of minimizing the functional

$$\Phi(u) = \int_\Omega |\mathcal{X}u(x)|^p\,dx$$

over all $u \in HW^{1,p}(\Omega)$ such that $u - h \in HW_0^{1,p}(\Omega)$. The Euler equation for this problem is the p-Laplace equation

$$\sum_{i=1}^{2} X_i\left(|\mathcal{X}u|^{p-2}\,X_i u\right) = 0\,, \text{ in } \Omega\,. \tag{6.3}$$

A function $u \in HW^{1,p}(\Omega)$ is called a weak solution of (6.3) if

$$\sum_{i=1}^{2}\int_\Omega |\mathcal{X}u(x)|^{p-2} X_i u(x)\cdot X_i\varphi(x)dx = 0\,, \quad \text{for all } \varphi \in HW_0^{1,p}(\Omega)\,. \tag{6.4}$$

Φ is a convex functional on $HW^{1,p}$, therefore weak solutions are minimizers for Φ. For $m \in \mathbb{N}$ define the approximating problems of minimizing functionals

$$\Phi_m(u) = \int_\Omega \left(\frac{1}{m} + |\mathcal{X}u(x)|^2\right)^{\frac{p}{2}}$$

and the corresponding Euler equations

$$\sum_{i=1}^{2} X_i\left(\left(\frac{1}{m} + |\mathcal{X}u|^2\right)^{\frac{p-2}{2}} X_i u\right) = 0\,, \text{ in } \Omega\,. \tag{6.5}$$

The differentiated version of this equation has the form

$$\sum_{i,j=1}^{2} a_{ij}^m\,X_i X_j u = 0\,, \text{ in } \Omega \tag{6.6}$$

where

$$a_{ij}^m(x) = \delta_{ij} + (p-2)\frac{X_i u(x)\, X_j u(x)}{\frac{1}{m} + |\mathcal{X}u(x)|^2}\,.$$

Consider a weak solution $u_m \in HW^{1,p}(\Omega)$ of equation (6.5). Then coefficients a_{ij}^m are bounded. Define the mapping $L_m : HW_0^{2,2}(\Omega) \to L^2(\Omega)$ by

$$L_m(v)(x) \;=\; \sum_{i,j=1}^{2} a_{ij}^m(x) X_i X_j v(x)\,. \tag{6.7}$$

The Cordes condition is satisfied precisely when

$$p-2 \in \left(\frac{1-\sqrt{5}}{2}\,,\,\frac{1+\sqrt{5}}{2}\right)$$

independently of m.

Theorem 20. The case $p > 2$: *For $2 \le p < 2 + \frac{1+\sqrt{5}}{2}$ if $u \in W^{1,p}(\Omega)$ is a minimizer for the functional Φ, then $u \in HW^{2,2}_{loc}(\Omega)$.*

Proof. The case $p = 2$ it is well known, so let us suppose $p \neq 2$. Let $u \in HW^{1,p}(\Omega)$ be a minimizer for Φ. Consider $x_0 \in \Omega$ and $r > 0$ such that $B_{4r} = B(x_0, 4r) \subset\subset \Omega$. Choose a cut-off function $\eta \in C_0^\infty(B_{2r})$ such that $\eta = 1$ on B_r. Also consider minimizers u_m for Φ_m on $HW^{1,p}(B_{2r})$ subject to $u_m - u \in HW_0^{1,p}(B_{2r})$. Then $u_m \to u$ in $HW^{1,p}(B_{2r})$ as $m \to \infty$.

By [D1, Ma] for $2 \le p < 4$ we have $u_m \in HW^{2,2}_{loc}(\Omega)$, but with bounds depending on m. Also that u_m satisfies equation $L_m(u_m) = 0$ a.e. in B_{2r}. So, in B_{2r} we have a.e.

$$X_iX_j(\eta^2 u_m) = X_iX_j(\eta^2)u_m + X_j(\eta^2)X_i u_m + X_i(\eta^2)X_j u_m + \eta^2 X_iX_j u_m$$

and hence

$$L_m(\eta^2 u_m) = u_m\, L_m(\eta^2) + \sum_{i,j=1}^{2} a_{ij}^m(x)\Big(X_j(\eta^2)X_i u_m + X_i(\eta^2)X_j u_m\Big)\,.$$

From the Cordes estimate it follows that

$$\begin{aligned}\|\mathcal{X}^2 u_m\|_{L^2(B_r)} &\le \|\mathcal{X}^2(\eta^2 u_m)\|_{L^2(B_{2r})} \le c\|L_m(\eta^2 u_m)\|_{L^2(B_{2r})}\\ &\le c\|u_m\|_{W^{1,p}(B_{2r})} \le c\|u\|_{HW^{1,p}(B_{2r})}\end{aligned}$$

where c is independent of m. Therefore, $u \in HW^{2,2}(B_r)$. □

For $p < 2$ a different argument is needed since we only get $u \in HW^{2,p}_{loc}(\Omega)$. We finish by quoting the result obtained in [DM]:

Theorem 21. The case $p < 2$: *For the range $\frac{\sqrt{17}-1}{2} \le p \le 2$ if $u \in HW^{1,p}(\Omega)$ is a minimizer for the functional Φ, then $u \in W^{2,2}_{loc}(\Omega)$.*

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Department of Mathematics, University of Pittsburgh, Pittsburgh, PA 15260, USA

E-mail address: `manfredi@pitt.edu`