BREAKING WAVES AND SOLITARY WAVES TO THE ROTATION-TWO-COMPONENT CAMASSA-HOLM SYSTEM

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ABSTRACT. In this paper, we consider two types of solutions of the rotation-two-component Camassa-Holm (R2CH) system, a model in the equatorial water waves with the effect of the Coriolis force. The first type of solutions exhibits finite time singularity in the sense of wave-breaking. We perform a refined analysis based on the local structure of the dynamics to provide some criteria that leads to the blow-up of solutions. The other type of solutions we study is the solitary waves. We classify various localized solitary wave solutions for the R2CH system. In addition to those smooth solitary wave solutions, we show that there are solitary waves with singularities, like peakons and cuspons, depending on the values of the rotating parameter Ω and the balance index σ. We also prove that horizontally symmetric weak solutions of this model must be traveling waves.

Keywords: Rotation-two-component Camassa-Holm system, Solitary Waves, Peakons, wave-breaking.

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1. INTRODUCTION

In this paper we study the following rotation-two-component Camassa-Holm (R2CH) system (see [20] for the derivation of the model)

$$\begin{cases} u_t - u_{xxt} - Au_x + 3uu_x = \sigma(2u_xu_{xx} + uu_{xxx}) - (1-2\Omega A)\rho\rho_x + 2\Omega\rho(\rho u)_x, \\ \rho_t + (\rho u)_x = 0, \end{cases} \tag{1.1}$$

where $u(t,x)$ is the horizontal fluid velocity, $\rho(t,x)$ is related to the free surface elevation from equilibrium, the parameter A characterizes a linear underlying shear flow, the dimensionless constant σ is a parameter which provides the competition/balance in fluid convection between nonlinear steepening and amplification due to stretching, and Ω characterizes the angular velocity of the Earth's rotation. In practice, Ω is small ($\approx 73\cdot 10^{-6}$ rad/s). We will always assume that $0 < \Omega < \frac{1}{4}$ and $1-2\Omega A > 0$ throughout this article.

System (1.1) is strongly related to several models describing the motion of waves at the free surface of a shallow water under the influence of gravity. In absence of the Earth's rotation, i.e., $\Omega = 0$, system (1.1) becomes the generalized two-component Camassa-Holm system [7]

$$\begin{cases} u_t - u_{xxt} - Au_x + 3uu_x = \sigma(2u_xu_{xx} + uu_{xxx}) - \rho\rho_x, \\ \rho_t + (\rho u)_x = 0. \end{cases} \tag{1.2}$$

Further taking $\sigma = 1$, (1.2) recovers the standard two-component integrable Camassa-Holm system [16, 31]

$$\begin{cases} u_t - u_{xxt} - Au_x + 3uu_x + \rho\rho_x = 2u_x u_{xx} + uu_{xxx}, \\ \rho_t + (\rho u)_x = 0. \end{cases} \tag{1.3}$$

Moreover, in the case $\rho = 0$, (1.2) is reduced to the Camassa-Holm (CH) equation [4, 5, 17, 21]:

$$u_t + 2\omega u_x - u_{txx} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \tag{1.4}$$

where $\omega = A/2$ is a constant related to the critical shallow water wave speed.

One of the motivations for the discovery of the CH equation is the quest for model equations that can exhibit wave-breaking phenomenon that the well-known KdV equations does not have. In fact, the H^1 norm of the CH solution remains finite, and hence classical solutions can only develop singularities in finite time in the form of wave-breaking (i.e., a solution that remains bounded while its slope becomes unbounded in finite time) [12, 13, 14, 15]. Another remarkable property the CH equation possesses is the presence of multi-solitons consisting of a train of peaked solitary waves (called "peakons") [4, 5, 6]. Peakons interact in a similar way to that of the KdV (smooth) solitons, but wave-breaking may occur during head-on collision of a peakon-antipeakon pair (cf. [15, 32]).

The above two distinctive features are also captured by the two-component system (1.2). One can refer to [7, 16, 19, 22, 23, 24, 33] and the references therein for details. The goal of this paper is to understand whether the two properties persist under the influence of the Coriolis force.

The wave-breaking phenomenon of system (1.1) has already been investigated in [20]. Utilizing the transport structure of (1.1) it is shown in [20] that the solution blows up at time T if and only if $\lim_{t\uparrow T^-} \{\inf_{x\in\mathbb{R}} u_x(t,x)\} = -\infty$. Unlike many other quasi-linear model equations, a notable difference in the blow-up analysis for (1.1) stems from the cubic term $\Omega\rho(\rho u)_x$, which fails to be controlled by the conservation laws. Using the transport equation of ρ, such a term can be rewritten as $-\Omega(\rho^2)_t$. This suggests that instead of considering solely the evolution of u, one can keep track of the dynamics of $K = u + \Omega(1-\partial_x^2)^{-1}(\rho^2)$. Note that from the conservation law (2.2), u_x blows up if and only if K_x blows up. But the advantage of considering K is that in the equation for K and K_x, the cubic terms can be bounded by the conservation laws, which enables one to carry out a standard procedure to reach a Riccati type inequality for K_x

$$\frac{d}{dt}K_x \lesssim -K_x^2 + C,$$

and thus by choosing K_x sufficiently negative initially, the corresponding solution blows up in finite time, cf. [20, Theorem 3.4]. A crucial ingredient in this argument is the use of the "global" information of solutions (like the conservation laws) in deriving various estimates. However the "local" structure of solutions is underappreciated. On the other hand, the non-diffusive nature of the system indicates that the local structure of data may strongly affect the the evolution of the solutions, in particular, the blow-ups. This has recently been evidenced in a class of CH-type equations in a series of works of Brandolese and Cortez [1, 2, 3], and later extended to some other quasilinear model equations with higher order nonlinearities [9, 10]. One of the main ideas lies in understanding of the interplay between the solution and its gradient. For (1.1), this amounts to tracking the dynamics of $K \pm K_x$ along the characteristics. Due to the nonlocal character involved in K, the conservation law is still needed to establish the convolution estimate. However it is now much apparent to see how rotation affects the wave-breaking. In particular, when the Coriolis effect is turned off our wave-breaking criterion recovers the one for the classical CH equation in [2], cf. Theorem 2.1.

Another issue we want to address here is concerned with the solitary wave solutions of (1.1), i.e. solutions of the form

$$(u(x,t), \rho(x,t)) = (\varphi(x-ct), \tilde{\rho}(x-ct)), \quad c \in \mathbb{R}$$

for functions φ and $\tilde{\rho} : \mathbb{R} \to \mathbb{R}$ such that $\varphi \to 0$ and $\tilde{\rho} \to 1$ as $|x| \to \infty$. In the study of the CH traveling waves it was observed that both peaked and cusped traveling waves exist [29]. Later Lenells [26, 27] used a suitable framework for weak solutions to classify all weak traveling waves of the CH equation. For the two-component CH system (1.3), it was shown in [16, 30, 33] that all solitary waves are smooth, symmetric and monotonic away from the crest, and decays exponentially far out. In [25], the authors considered a modified two-component CH equation which allows dependence on average density as well as pointwise density and a linear dispersion is added to the first equation of the system. They showed that the modified system admits peaked solitary wave solution in both u and ρ. For the generalized two-component CH system (1.2), the balance parameter σ leads to the possibility of existence of singular solitary waves (see [8]). Moreover, it is shown that when $\sigma \leq 1$, all smooth solitary waves are orbitally stable.

However it is unclear whether the R2CH system (1.1) supports solitary waves with singularities. Using a natural weak formulation of the R2CH system (1.1), see (3.10) when $\sigma = 0$ and (3.16) when $\sigma \neq 0$, we can define exactly in what sense the peaked and cusped solitary waves are solutions. In fact, it turns out that the equation for φ takes the form

$$\varphi_x^2 = R(\varphi)$$

where R is a rational function. A standard phase-plane analysis determines the behavior of solution near the zeros and poles of R. In fact, peaked solitary waves exist when R has a removable pole and cusped solitary waves correspond to when R has a non-removable pole. Due to the added rotational term, the numerator of R contains a quadratic polynomial $f(\varphi)$ whose root distribution is quite complicated. By analyzing each possible case carefully, we show here peaked and cusped solitary waves do exist for (1.1) and provide an implicit formula for the peaked solitary waves.

From the classification of the solitary waves for the R2CH system, we find that the solutions include very exotic shapes. But when restricted to smooth solutions, the situation is clearer. In particular, the smooth solutions are all symmetric around the crest. This raises the intriguing question whether the classes of traveling and symmetric waves are identical. Adapting the idea of [18], we are able to give an affirmative answer for the two-component system (1.1).

The remainder of this paper is organized as follows. In Section 2 we give a wave-breaking criterion (Theorem 2.1) which addresses the local structure of the solutions and also indicates explicitly how rotation is involved. We further provide an upper bound of u_x along each characteristics emanating from a vanishing point of ρ_0. In Section 3 we introduce the weak formulation of system (1.1) and define the class of solitary waves with singularities. In Section 4, we classify various solitary wave solutions. In Section 5, we demonstrate that an x-symmetric weak solution of system (1.1) must be a traveling wave.

2. Blow up for rotational 2CH

When $\sigma = 1$, the R2CH system reads

$$\begin{cases} u_t - u_{xxt} - Au_x + 3uu_x = (2u_x u_{xx} + uu_{xxx}) - (1-2\Omega A)\rho\rho_x + 2\Omega\rho(\rho u)_x, \\ \rho_t + (\rho u)_x = 0. \end{cases} \tag{2.1}$$

The above system admits an $H^1 \times L^2$ conservation law

$$E(t) = \frac{1}{2}\int_{\mathbb{R}} \left(u^2 + u_x^2 + (1-2\Omega A)(\rho-1)^2\right). \tag{2.2}$$

The blow-up criterion for the R2CH system (2.1) can be formulated as

Lemma 2.1. [20] *Assume that $1 - 2\Omega A > 0$. Let $(u_0, \rho_0 - 1) \in H^s \times H^{s-1}$ with $s > 3/2$. Then the corresponding solution (u, ρ) to system (2.1) with initial data (u_0, ρ_0) blows up in finite time $T < \infty$ if and only if*

$$\lim_{t\uparrow T^-} \left\{ \inf_{x\in\mathbb{R}} u_x(t,x) \right\} = -\infty. \tag{2.3}$$

Now, we give a condition which can guarantees wave-breaking in finite time.

Theorem 2.1. *Assume $1-2\Omega A > 0$. Let (u, ρ) be the solution of (2.1) with initial data $(u_0, \rho_0 - 1) \in H^s \times H^{s-1}$ with $s > 3/2$ and T be the maximal time of existence. Assume there exists a x_0 such that*

$$\rho_0(x_0) = 0, \tag{2.4}$$

and

$$u_{0,x}(x_0) < -\left|u_0(x_0) - \frac{A}{2}\right| - 4\Omega C_1 - 4\sqrt{\Omega C_1\sqrt{E(0)}}, \tag{2.5}$$

where

$$C_1 = \frac{3E(0)}{4(1-2\Omega A)} + \frac{3}{2}. \tag{2.6}$$

Then the corresponding solution (u, ρ) to system (2.1) blows up in finite time. Moreover, the blow-up time T^* satisfies

$$T^* \leq \frac{8}{\sqrt{K_{0,x}^2(x_0) - \left(K_0(x_0) - \frac{A}{2}\right)^2}}, \tag{2.7}$$

where

$$K_0(x) = u_0(x) + \Omega(1-\partial_x^2)^{-1}(\rho^2)(0,x).$$

Remark 2.1. *Note that in the case when $\Omega = 0$, the condition (2.5) on the velocity u reduces to the same one as for the classical Camassa-Holm equation with linear dispersion (see [2, Corollary 2.4]). Here the appearance of the the Coriolis effect brings up delicate interaction between the surface and the velocity. To control the additional terms in the blow-up analysis we are forced to use the conservation law of $E(t)$, as can be seen from the following proof.*

Proof of Theorem 2.1. By a simple density argument, we need only to prove this theorem for $s \geq 3$. We follow the characteristics of the R2CH system to generate finite-time blow-up. Hence we define the characteristics $q(t,x)$ as

$$\begin{cases} q_t(t,x) = u(t, q(t,x)), \\ q(0,x) = x, \end{cases} \qquad x \in \mathbb{R}, \quad t \in [0,T). \tag{2.8}$$

One can easily check that $q \in C^1([0,T)\times\mathbb{R},\mathbb{R})$ with $q_x(t,x)>0$ for all $(t,x)\in[0,T)\times\mathbb{R}$.

Denote $p(x)=\frac{1}{2}e^{-|x|}$ the fundamental solution of $1-\partial_x^2$ on $\mathbb{R}$, and define the two convolution operators p_+, p_- as

$$\begin{aligned} p_+ * f(x) &= \frac{e^{-x}}{2}\int_{-\infty}^{x} e^{y} f(y)dy,\\ p_- * f(x) &= \frac{e^{x}}{2}\int_{x}^{\infty} e^{-y} f(y)dy. \end{aligned} \tag{2.9}$$

Then we have the relation

$$p = p_+ + p_-, \qquad p_x = p_- - p_+. \tag{2.10}$$

It is easily checked that the derivatives of u and u_x along the characteristics can be obtained from the following computation

$$u_t + uu_x = -\, p_x * \left(-Au + u^2 + \frac{1}{2}u_x^2 + \frac{1-2\Omega A}{2}\rho^2 - \Omega\rho^2 u\right) + \Omega p * (\rho^2 u_x), \tag{2.11}$$

$$\begin{aligned} u_{xt} + uu_{xx} = &-\frac{1}{2}u_x^2 + u^2 + \frac{1-2\Omega A}{2}\rho^2 - \Omega u\rho^2 + Ap_{xx} * u\\ &- p * \left(u^2 + \frac{1}{2}u_x^2 + \frac{1-2\Omega A}{2}\rho^2 - \Omega\rho^2 u\right) + \Omega p_x * (\rho^2 u_x). \end{aligned} \tag{2.12}$$

For wave-breaking, one would like to choose some initial data such that u_x approaches $-\infty$ in finite time. The difficulty in the analysis of the dynamics of u_x sources from the last term $p_x * (\rho^2 u_x)$, which fails to be controlled by the conservation laws. Our idea is to absorb this term by considering the dynamics of the quantity

$$K := u + \Omega p * \rho^2 \tag{2.13}$$

together with its derivative K_x. A direct computation shows that the dynamics of K and K_x are given by [20]

$$K_t + uK_x = -\, p_x * \left(u^2 + \frac{1}{2}u_x^2\right) - \frac{1-2\Omega A}{2} p_x * \rho^2 + Ap_x * u + \Omega u p_x * \rho^2, \tag{2.14}$$

$$\begin{aligned} K_{xt} + uK_{xx} = &-\frac{1}{2}u_x^2 + u^2 - p * \left(u^2 + \frac{1}{2}u_x^2\right) + \frac{1-2\Omega A - 2\Omega u}{2}\rho^2 + \Omega u p * \rho^2\\ &+ Ap * u - Au - \frac{1-2\Omega A}{2} p * \rho^2. \end{aligned} \tag{2.15}$$

For $x_0 \in \mathbb{R}$ given in the theorem, let

$$\gamma(t) = \rho(t, q(t,x_0)), \quad t \in [0,T), \tag{2.16}$$

where $q(t,x_0)$ is defined by (2.8). Along with the trajectory of $q(t,x)$, we have

$$\gamma'(t) = -\gamma u_x, \quad t \in [0,T). \tag{2.17}$$

From assumption (2.4) we know that $\gamma(0)=\rho_0(x_0)=0$ and hence equation (2.17) implies

$$\gamma(t) \equiv 0, \quad \text{for } t \in [0,T). \tag{2.18}$$

From now on we make an abuse of notation by denoting

$$u(t) = u(t,q(t,x_0)),\;\; u_x(t) = u_x(t,q(t,x_0)),\;\; K(t) = K(t,q(t,x_0)),\;\; K_x(t) = K_x(t,q(t,x_0)).$$

We further denote $'$ the material derivative $\partial_t + u\partial_x$ along the characteristics $q(t, x_0)$. Then from (2.14), (2.15) and (2.18) we see that

$$(K+K_x)' = -2p_- * \left(u^2 - Au + \frac{1}{2}u_x^2\right) - \frac{1}{2}u_x^2 + u^2 - Au - (1 - 2\Omega A - 2\Omega u)p_- * \rho^2,$$

$$(K-K_x)' = 2p_+ * \left(u^2 - Au + \frac{1}{2}u_x^2\right) + \frac{1}{2}u_x^2 - u^2 + Au + (1 - 2\Omega A - 2\Omega u)p_+ * \rho^2.$$

Applying [2, Lemma 3.1 (1)] with $m = -A^2/4$ and $K = 1$ we have the following convolution estimates

$$p_\pm * \left(u^2 - Au + \frac{1}{2}u_x^2\right) \geq \frac{1}{4}\left(u^2 - Au - \frac{A^2}{4}\right). \tag{2.19}$$

This in turn provides the bounds for $(K \pm K_x)'$ as

$$(K+K_x)' \leq -\frac{1}{2}\left[u_x^2 - \left(u - \frac{A}{2}\right)^2\right] - (1 - 2\Omega A - 2\Omega u)p_- * \rho^2,$$

$$(K-K_x)' \geq \frac{1}{2}\left[u_x^2 - \left(u - \frac{A}{2}\right)^2\right] + (1 - 2\Omega A - 2\Omega u)p_+ * \rho^2.$$

Using the fact that

$$(K \pm K_x)' = \left[\left(K - \frac{A}{2}\right) \pm K_x\right]', \quad (1 - 2\Omega A)p_\pm * \rho^2 \geq 0$$

we can further deduce that

$$\left[\left(K - \frac{A}{2}\right) + K_x\right]' \leq -\frac{1}{2}\left[u_x^2 - \left(u - \frac{A}{2}\right)^2\right] + 2\Omega u p_- * \rho^2,$$

$$\left[\left(K - \frac{A}{2}\right) - K_x\right]' \geq \frac{1}{2}\left[u_x^2 - \left(u - \frac{A}{2}\right)^2\right] - 2\Omega u p_+ * \rho^2.$$

The convolution terms in the above estimates can be bounded by

$$\begin{aligned} 0 \leq p_\pm * \rho^2 &= p_\pm * (\rho - 1)^2 + 2p_\pm * (\rho - 1) + p_\pm * 1 \\ &\leq \|p_\pm\|_{L^\infty}\|\rho - 1\|_{L^2}^2 + 2\|p_\pm\|_{L^2}\|\rho - 1\|_{L^2} + 1 \\ &\leq \frac{3}{4}\|\rho - 1\|_{L^2}^2 + \frac{3}{2} \leq \frac{3E(0)}{4(1 - 2\Omega A)} + \frac{3}{2} = C_1, \end{aligned} \tag{2.20}$$

where we have used the definition (2.6) and the fact that

$$\|p_\pm\|_{L^\infty} = \frac{1}{2}, \quad \|p_\pm\|_{L^2} = \frac{1}{2\sqrt{2}}, \quad \|\rho - 1\|_{L^2}^2 \leq \frac{E(0)}{1 - 2\Omega A}.$$

From (2.20) we can also bound

$$|up_\pm * \rho^2| \leq \|u\|_{L^\infty}\|p_\pm * \rho^2\|_{L^\infty} \leq \sqrt{E(0)}C_1. \tag{2.21}$$

Putting together, we can further conclude that

$$\begin{aligned} \left[\left(K - \frac{A}{2}\right) + K_x\right]' &\leq -\frac{1}{2}\left[u_x^2 - \left(u - \frac{A}{2}\right)^2\right] + 2\Omega C_1\sqrt{E(0)}, \\ \left[\left(K - \frac{A}{2}\right) - K_x\right]' &\geq \frac{1}{2}\left[u_x^2 - \left(u - \frac{A}{2}\right)^2\right] - 2\Omega C_1\sqrt{E(0)}. \end{aligned} \tag{2.22}$$

In addition, it follows from (2.10), (2.13) and (2.20) that

$$\begin{aligned} &\left(u-\frac{A}{2}\right)+u_x \leq \left(K-\frac{A}{2}\right)+K_x \leq \left(u-\frac{A}{2}\right)+u_x+2\Omega C_1, \\ &\left(u-\frac{A}{2}\right)-u_x \leq \left(K-\frac{A}{2}\right)-K_x \leq \left(u-\frac{A}{2}\right)-u_x+2\Omega C_1. \end{aligned} \tag{2.23}$$

Now if the assumption (2.5) holds, we have that

$$\frac{1}{2}\left[u_x^2-\left(u-\frac{A}{2}\right)^2\right]-2\Omega C_1\sqrt{E(0)}>0,$$

which implies, from (2.22), that

$$\left[\left(K-\frac{A}{2}\right)+K_x\right]'(0)<0, \quad \left[\left(K-\frac{A}{2}\right)-K_x\right]'(0)>0. \tag{2.24}$$

Hence at least for a short time t, $K(t)+K_x(t)$ is non-increasing and $K(t)-K_x(t)$ is non-decreasing. From (2.5), the definition (2.13) of K and (2.23) we know that

$$\begin{aligned} &\left(K(0)-\frac{A}{2}\right)+K_x(0)<-4\sqrt{\Omega C_1\sqrt{E(0)}}-2\Omega C_1, \\ &\left(K(0)-\frac{A}{2}\right)-K_x(0)>4\sqrt{\Omega C_1\sqrt{E(0)}}+4\Omega C_1. \end{aligned} \tag{2.25}$$

The short time monotonicity (2.24) indicates that the above bounds continue to hold, at least for a short time. Therefore going back to u and u_x using (2.23) again we have that

$$\begin{aligned} &\left(u(t)-\frac{A}{2}\right)+u_x(t)<-4\sqrt{\Omega C_1\sqrt{E(0)}}-2\Omega C_1, \\ &\left(u(t)-\frac{A}{2}\right)-u_x(t)>4\sqrt{\Omega C_1\sqrt{E(0)}}+2\Omega C_1, \end{aligned} \tag{2.26}$$

which, when plugging in to (2.22), shows that the monotonicity of $\left(K-\frac{A}{2}\right)\pm K_x$ persists and thus the bounds of the form in (2.25) continue to hold for later time. Therefore the estimates (2.26) still hold true, pushing the monotonicity even further in time. Hence, we always have $K(t)-\frac{A}{2}+K_x(t)<0$ is non-increasing, and $K(t)-\frac{A}{2}-K_x(t)>0$ is non-decreasing, which allows us to define the function

$$h(t)=\sqrt{K_x^2(t)-[K(t)-A/2]^2}>0.$$

Computing the derivative of h leads to

$$\begin{aligned} h'(t)&=\frac{-(K-A/2+K_x)'(K-A/2-K_x)-(K-A/2+K_x)(K-A/2-K_x)'}{2\sqrt{K_x^2(t)-[K(t)-A/2]^2}} \\ &\geq\left\{\frac{1}{2}\left[u_x^2-\left(u-\frac{A}{2}\right)^2\right]-2\Omega C_1\sqrt{E(0)}\right\}\frac{(K-A/2-K_x)-(K-A/2+K_x)}{2\sqrt{K_x^2(t)-[K(t)-A/2]}} \\ &\geq\frac{1}{2}\left[u_x^2-\left(u-\frac{A}{2}\right)^2\right]-2\Omega C_1\sqrt{E(0)}>0, \end{aligned}$$

where we have used the fact that

$$\frac{(K-A/2-K_x)-(K-A/2+K_x)}{2}\geq h.$$

From (2.23) and (2.26) it follows that

$$0 < \left(K - \frac{A}{2}\right) - K_x \le \left(u - \frac{A}{2}\right) - u_x + 2\Omega C_1 < 2\left[\left(u - \frac{A}{2}\right) - u_x\right],$$
$$0 < -\left(K - \frac{A}{2}\right) - K_x \le -\left(u - \frac{A}{2}\right) - u_x.$$

Therefore

$$K_x^2 - \left(K - \frac{A}{2}\right)^2 \le 2\left[u_x^2 - \left(u - \frac{A}{2}\right)^2\right],$$

and hence

$$h' \ge \frac{1}{4}h^2 - 2\Omega C_1\sqrt{E(0)}. \tag{2.27}$$

Evaluating (2.23) at initial time we have

$$\left(K(0) - \frac{A}{2}\right) + K_x(0) \le \left(u_0(x_0) - \frac{A}{2}\right) + u_{0,x}(x_0) + 2\Omega C_1 < -4\sqrt{\Omega C_1\sqrt{E(0)}},$$
$$\left(K(0) - \frac{A}{2}\right) - K_x(0) \ge \left(u_0(x_0) - \frac{A}{2}\right) - u_{0,x}(x_0) > 4\sqrt{\Omega C_1\sqrt{E(0)}}.$$

Therefore we know that

$$h^2(0) > 16\Omega C_1\sqrt{E(0)}.$$

Therefore from (2.27) we see that h is increasing and in fact we have

$$h' \ge \frac{1}{8}h^2.$$

This is enough to show that h blows up in finite time. Indeed, we can solve to get

$$h(t) \ge \frac{8h(0)}{8 - th(0)}.$$

Therefore we see that

$$h(t) \to +\infty \quad \text{as} \quad t \to \frac{8}{h(0)}.$$

On the other hand, since

$$h(t) \le -K_x = -u_x - \Omega p_x * \rho^2,$$

and from (2.20) we know that

$$h(t) \le -u_x + 2C_1.$$

Therefore $-u_x$ must blow up at time T^* which satisfies

$$T^* \le \frac{8}{h(0)}, \tag{2.28}$$

completing the proof. □

It is known from Lemma 2.1 that the solution of system (2.1) breaks down in finite time T if and only if

$$\lim_{t\to T^-} \inf_{x\in\mathbb{R}} u_x(t,x) = -\infty.$$

An interesting question is whether u_x has an upper bound. The investigation on this issue gives the following result.

Proposition 2.1. *Assume that* $1-2\Omega A>0$. *Let* $(u_0,\rho_0-1)\in H^s\times H^{s-1}$ *with* $s>3/2$, *and* $T>0$ *be the maximal time of existence of the solution* (u,ρ) *to system* (2.1) *with initial data* (u_0,ρ_0). *Then for* $x\in\Lambda:=\{x\in\mathbb{R}:\ \rho_0(x)=0\}$, *we have that* $u_x(t,q(t,x))$ *is bounded from above for* $t\in[0,T)$.

Proof. Similar to the arguments in the beginning of the proof of Theorem 2.1, we need only to prove this theorem for $s\geq 3$. Given $x\in\mathbb{R}$, let

$$M_1(t)=K_x(t,q(t,x)),\quad \gamma(t)=\rho(t,q(t,x)),\quad t\in[0,T), \tag{2.29}$$

where $q(t,x)$ is defined by (2.8). Along the trajectory of $q(t,x)$, we have

$$\gamma'(t)=-\gamma u_x,\quad t\in[0,T). \tag{2.30}$$

For any $x\in\Lambda$, equation (2.30) implies

$$\gamma(t)=\rho(t,q(t,x))=0,\quad \text{for } t\in[0,T). \tag{2.31}$$

Then (2.15) has the form

$$M_1'(t)=-\frac{1}{2}(M_1-\Omega\partial_x p*\rho^2)^2+f(t,q(t,x)) \tag{2.32}$$

with

$$f=\Omega up*\rho^2+A\partial_x^2p*u+u^2-p*\left(u^2+\frac{1}{2}u_x^2+\frac{1-2\Omega A}{2}\rho^2\right), \tag{2.33}$$

for $x\in\Lambda$, where "$'$" is the derivative with respect to t. And we can get the upper bound of f

$$f\leq C_{E(0)}^2, \tag{2.34}$$

where $C_{E(0)}$ denotes a constant that depends only on $E(0)$. Given any $x\in\mathbb{R}$, let us define

$$P(t)=M_1(t)-\|u_{0,x}\|_{L^\infty}-2\Omega C_1-2C_{E(0)},$$

where C_1 is defined by (2.6). Observing $P(t)$ is a C^1-differentiable function in $[0,t)$ and satisfies

$$\begin{aligned}P(0)&=M_1(0)-\|u_{0,x}\|_{L^\infty}-2\Omega C_1-2C_{E(0)}\\&\leq u_{0,x}(x)+\Omega p_x*\rho^2(0,x)-\|u_{0,x}\|_{L^\infty}-2\Omega C_1\leq 0,\end{aligned}$$

where we have used the estimate (2.20). We now claim

$$P(t)\leq 0,\quad \forall\, t\in[0,T). \tag{2.35}$$

Assume the contrary that there is $t_0\in[0,T)$ such that $P(t_0)>0$. Let

$$t_1=\max\{t<t_0;P(t)=0\}.$$

Then $P(t_1)=0$ and $P'(t_1)\geq 0$, or equivalently,

$$M_1(t_1)=\|u_{0,x}\|_{L^\infty}+2\Omega C_1+2C_{E(0)} \tag{2.36}$$

and

$$M_1'(t_1)\geq 0. \tag{2.37}$$

By (2.32), (2.34) and (2.36), it then follows that

$$\begin{aligned}{M_1}'(t_1) &= -\frac{1}{2}(M_1(t_1) - \Omega\partial_x p * \rho^2)^2 + f(t_1, q(t_1, x)) \\ &\leq -\frac{1}{2}\left(\|u_{0,x}\|_{L^\infty} + 2C_{E(0)}\right)^2 + C_{E(0)}^2 < 0,\end{aligned}$$

which is a contradiction to (2.37). This verifies the estimate in (2.35). Therefore, for any x such that $\rho(x) = 0$

$$\sup_{t\in[0,T)} \{u_x(t, q(t, x)) + \Omega\partial_x p * \rho^2(t, q(t, x))\} \leq 2\Omega C_1 + 2C_{E(0)} + \|u_{0,x}\|_{L^\infty},$$

which implies

$$\sup_{t\in[0,T)} u_x(t, q(t, x)) \leq \|u_{0,x}\|_{L^\infty} + 4\Omega C_1 + 2C_{E(0)}.$$

This completes the proof of Proposition 2.1. □

3. Weak formulations

In this section, we derive the weak formulations for system (1.1), introduce the notion of various types of solitary waves, and derive the ODEs for the solitary waves.

Since $\rho \to 1$ as $|x| \to \infty$ in (1.1), we define $\rho = 1 + \eta$ with $\eta \to 0$ as $|x| \to \infty$, and hence we can rewrite system (1.1) as

$$\begin{cases} u_t - u_{txx} - Au_x + 3uu_x - \sigma(2u_xu_{xx} + uu_{xxx}) \\ \quad +(1 - 2\Omega A)(1 + \eta)\eta_x - 2\Omega(1 + \eta)\left((1 + \eta)u\right)_x = 0, \\ \eta_t + ((1 + \eta)u)_x = 0. \end{cases} \tag{3.1}$$

Using the kernel p defined in the previous section, we can further rewrite system (3.1) in a weak form as

$$\begin{cases} u_t + \sigma uu_x = \\ \quad -p_x * \left[-Au + \frac{3-\sigma}{2}u^2 + \frac{\sigma}{2}u_x^2 + \frac{1-2\Omega A}{2}(1 + \eta)^2 - 2\Omega\partial_x^{-1}\left((1 + \eta)\left((1 + \eta)u\right)_x\right)\right], \\ \eta_t + ((1 + \eta)u)_x = 0. \end{cases} \tag{3.2}$$

This way we can define a weak solution to (1.1) as follows.

Definition 3.1. *Assume that $\vec{u} = (u, \eta) \in X(\mathbb{R})$ where $X(\mathbb{R}) = C(\mathbb{R}_+, H^1(\mathbb{R}) \times L^2(\mathbb{R}))$, and that $\vec{u}$ satisfies*

$$\begin{cases} \int\int_{\mathbb{R}_+\times\mathbb{R}}[u(1 - \partial_x^2)\psi_t + \Omega(1 + \eta)^2\psi_t \\ -(Au - \frac{3}{2}u^2 - \frac{\sigma}{2}u_x^2 - \frac{1-2\Omega A}{2}(1 + \eta)^2)\psi_x - \frac{\sigma}{2}u^2\psi_{xxx}]dtdx = 0, \\ \int\int_{\mathbb{R}_+\times\mathbb{R}}[\eta\psi_t + (1 + \eta)u\psi_x]dtdx = 0. \end{cases} \tag{3.3}$$

for all $\psi \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R})$. Then $\vec{u}(t, x)$ is a weak solution to the system (3.1).

Now we give the definitions of solitary waves, peakons and cuspons of (3.1).

Definition 3.2. *A solitary wave of* (3.1) *is a nontrivial traveling wave solution of* (3.1) *of the form $(\varphi(x - ct), \eta(x - ct)) \in H^1 \times H^1$with $c \in \mathbb{R}$ and φ, η vanishing at infinity.*

Remark 3.1. *Here we demand more regularity of η for a solitary wave than for a general solution due to the continuity requirement of η in the subsequent discussions, which is also important to obtain Proposition 3.1.*

Definition 3.3. [26] *We say that a continuous function φ has a peak at x if φ is smooth locally on either side of x and*

$$0 \neq \lim_{y\uparrow x} \varphi_x(y) = -\lim_{y\downarrow x} \varphi_x(y) \neq \pm\infty.$$

Wave profiles with peaks are called peaked waves or peakons.

Definition 3.4. [26] *We say that a continuous function φ has a cusp at x if φ is smooth locally on either side of x and*

$$\lim_{y\uparrow x} \varphi_x(y) = -\lim_{y\downarrow x} \varphi_x(y) = \pm\infty.$$

Wave profiles with cusps are called cusped waves or cuspons.

It is easily seen that a solitary wave (φ, η) with speed $c \in \mathbb{R}$ satisfies

$$\begin{cases} \left[-(c+A)\varphi + c\varphi_{xx} + \dfrac{3}{2}\varphi^2 - \sigma\varphi\varphi_{xx} - \dfrac{\sigma}{2}\varphi_x^2 + \dfrac{1-2\Omega A-2\Omega c}{2}(1+\eta)^2\right]_x = 0, \\ (-c\eta + (1+\eta)\varphi)_x = 0, \end{cases} \tag{3.4}$$

where we have used $((1+\eta)\varphi)_x = c\eta_x$ in the first equation.

Integrating the system we get

$$\begin{cases} -(c+A)\varphi + c\varphi_{xx} + \frac{3}{2}\varphi^2 \\ \quad = \sigma\varphi\varphi_{xx} + \frac{\sigma}{2}\varphi_x^2 - \frac{1-2\Omega A-2\Omega c}{2}(1+\eta)^2 + \frac{1-2\Omega A-2\Omega c}{2}, \quad \text{in } \mathcal{D}'(\mathbb{R}). \\ -c\eta + (1+\eta)\varphi = 0. \end{cases} \tag{3.5}$$

The fact that the second equation of the above system holds in a strong sense comes from the regularity of φ and η.

Our next goal is to decouple system (3.5) to derive a closed ODE for φ. From the first equation in (3.5) we see that if the coefficient of $(1+\eta)^2$ vanishes then the resulting system is decoupled. Hence we split the case into two.

3.1. **When** $1 - 2\Omega A - 2\Omega c = 0$**.** In this case system (3.5) becomes an ODE for φ solely and an algebraic equation for η. Moreover, the equation for φ is reminiscent of the case of compressible elastic rod equation [27] and the Camassa-Holm equation with a linear dispersion [26], where a very detailed classification of traveling waves has been given. Repeating the analysis performed in [26, 27] carefully one can recover the classification for solitary waves for the φ component. In particular, for the interest of the peaked waves we find that peaked φ-solitary wave exists only when

$$\sigma = 1, \quad A = 0, \quad 1 - 2\Omega c = 0, \tag{3.6}$$

which takes the form

$$\varphi(x) = ce^{-|x|}.$$

However when one turns to the η component the second equation in (3.5) leads to

$$\eta = \frac{\varphi}{c - \varphi},$$

and thus when φ exhibits a peak singularity then η given from the above formula leaves H^1, and therefore in this case peaked waves are also excluded.

Given that the classification of solitary waves in the case when $1 - 2\Omega A - 2\Omega c = 0$ can be done following [26, 27], our main effort will be to gain information about the solitary waves with an emphasis on peaked solitary waves

3.2. **when** $1-2\Omega A-2\Omega c\neq 0$**.** In this case one needs to solve the second equation in (3.5) for η and then plug the result into the first one. To do so, we need the following

Proposition 3.1. *If* (φ,η) *is a solitary wave of* (3.1) *for some* $c\in\mathbb{R}$*, then* $c\neq 0$ *and* $\varphi(x)\neq c$ *for any* $x\in\mathbb{R}$*.*

Proof. Since the constant $\frac{1-2\Omega A-2\Omega c}{2}\neq 0$, the proof follows closely to the one in [8, Proposition 2.4], and hence we omit the details. □

Using Proposition 3.1 we obtain from the second equation of (3.5) that

$$\eta=\frac{\varphi}{c-\varphi}. \tag{3.7}$$

Plugging this into the first equation of (3.5), we obtain a single equation for the unknown φ

$$\begin{aligned}&-(c+A)\varphi+c\varphi_{xx}+\frac{3}{2}\varphi^2\\&=\sigma\varphi\varphi_{xx}+\frac{\sigma}{2}\varphi_x^2-\frac{1-2\Omega A-2\Omega c}{2}\frac{c^2}{(c-\varphi)^2}+\frac{1-2\Omega A-2\Omega c}{2},\quad\text{in }\mathcal{D}'(\mathbb{R}).\end{aligned}\tag{3.8}$$

Now we discuss (3.8) in the cases $\sigma=0$ and $\sigma\neq 0$ separately.
Case A: When $\sigma=0$, (3.8) becomes

$$\varphi_{xx}=\frac{c+A}{c}\varphi-\frac{3}{2c}\varphi^2+\frac{1-2\Omega A-2\Omega c}{2c}-\frac{1-2\Omega A-2\Omega c}{2}\frac{c^2}{c(c-\varphi)^2}. \tag{3.9}$$

Since $\varphi\in H^1$ and from Proposition 3.1, $c\neq 0$, and $c-\varphi\neq 0$, we know that $|c-\varphi|$ is bounded away from 0. Hence from the standard local regularity theory to elliptic equations we see that $\varphi\in C^\infty$ and so is η. Therefore in this case all solitary waves are smooth. Multiplying (3.9) by φ_x and integrating on $(-\infty,x]$, we get

$$\begin{aligned}\varphi_x^2&=\frac{\varphi^2\left[(c-\varphi)^2+A(c-\varphi)-(1-2\Omega A-2\Omega c)\right]}{c(c-\varphi)}\\&=\frac{\varphi^2\left[\varphi^2-(2c+A)\varphi+c^2+Ac-(1-2\Omega A-2\Omega c)\right]}{c(c-\varphi)}\\&:=\frac{\varphi^2 f(\varphi)}{c(c-\varphi)}:=G(\varphi),\end{aligned}\tag{3.10}$$

where

$$f(\varphi)=\varphi^2-(2c+A)\varphi+c^2+Ac-(1-2\Omega A-2\Omega c). \tag{3.11}$$

Case B: When $\sigma\neq 0$, we can rewrite (3.8) as

$$\begin{aligned}\left(\left(\varphi-\frac{c}{\sigma}\right)^2\right)_{xx}=\varphi_x^2-\frac{2(c+A)}{\sigma}\varphi+\frac{3}{\sigma}\varphi^2-\frac{1-2\Omega A-2\Omega c}{\sigma}\\+\frac{1-2\Omega A-2\Omega c}{\sigma}\frac{c^2}{(c-\varphi)^2},\quad\text{in }\ \mathcal{D}'(\mathbb{R}).\end{aligned}\tag{3.12}$$

The following lemma concerns the regularity of the solitary waves when $\sigma\neq 0$. The idea is inspired by the study of the travelling waves of Camassa-Holm equation [26].

Lemma 3.1. *Let $\sigma \neq 0$ and (φ, η) is a solitary wave of* (3.1). *Then*

$$\left(\varphi - \frac{c}{\sigma}\right)^k \in C^j\left(\mathbb{R} \setminus \varphi^{-1}\left(\frac{c}{\sigma}\right)\right), \ for\ k \geq 2^j. \tag{3.13}$$

Therefore

$$\varphi \in C^\infty\left(\mathbb{R} \setminus \varphi^{-1}\left(\frac{c}{\sigma}\right)\right).$$

Proof. From Proposition 3.1 we know that $c \neq 0$ and $\varphi \neq c$ and thus φ satisfies (3.12). Let $v = \varphi - \frac{c}{\sigma}$ and denote

$$r(v) = \frac{3}{\sigma}\left(v + \frac{c}{\sigma}\right)^2 - \frac{2(c+A)}{\sigma}\left(v + \frac{c}{\sigma}\right) - \frac{1 - 2\Omega A - 2\Omega c}{\sigma}.$$

Then $r(v)$ is a polynomial in v. By the fact that $\varphi - c \neq 0$, we get that

$$\frac{\sigma - 1}{\sigma} c - v = c - \varphi \neq 0. \tag{3.14}$$

Then v satisfies

$$(v^2)_{xx} = v_x^2 + r(v) + \frac{(1 - 2\Omega A - 2\Omega c)c^2}{\sigma}\left(\frac{\sigma - 1}{\sigma} c - v\right)^{-2}.$$

Using the assumption, one have $(v^2)_{xx} \in L^1_{loc}(\mathbb{R})$. Hence $(v^2)_x$ is absolutely continuous and hence

$$v^2 \in C^1(\mathbb{R}), \quad \text{and then } \ v \in C^1\left(\mathbb{R} \backslash v^{-1}(0)\right).$$

Hence from (3.14) and that $v + \frac{c}{\sigma} \in H^1(\mathbb{R}) \subset C(\mathbb{R})$ we know

$$\left(\frac{\sigma - 1}{\sigma} c - v\right)^{-2} \in C(\mathbb{R}) \cap C^1\left(\mathbb{R} \backslash v^{-1}(0)\right).$$

Moreover,

$$\begin{aligned}
(v^k)_{xx} &= (kv^{k-1}v_x)_x = \frac{k}{2}\left(v^{k-2}(v^2)_x\right)_x \\
&= k(k-2)v^{k-2}v_x^2 + \frac{k}{2}v^{k-2}(v^2)_{xx} \\
&= k(k-2)v^{k-2}v_x^2 + \frac{k}{2}v^{k-2}\left[v_x^2 + r(v) + \frac{(1 - 2\Omega A - 2\Omega c)c^2}{\sigma}\left(\frac{\sigma - 1}{\sigma} c - v\right)^{-2}\right] \\
&= k\left(k - \frac{3}{2}\right)v^{k-2}v_x^2 + \frac{k}{2}v^{k-2}r(v) + \frac{k(1 - 2\Omega A - 2\Omega c)c^2}{2\sigma}v^{k-2}\left(\frac{\sigma - 1}{\sigma} c - v\right)^{-2}.
\end{aligned} \tag{3.15}$$

For $k = 3$, the right-hand side of (3.15) is in $L^1_{loc}(\mathbb{R})$, which implies that

$$v^3 \in C^1(\mathbb{R}).$$

For $k \geq 4$, we infer from (3.15) that

$$\begin{aligned}
(v^k)_{xx} =& \frac{k}{4}\left(k - \frac{3}{2}\right)v^{k-4}\left[(v^2)_x\right]^2 + \frac{k}{2}v^{k-2}r(v) \\
&+ \frac{k(1 - 2\Omega A - 2\Omega c)c^2}{2\sigma}v^{k-2}\left(\frac{\sigma - 1}{\sigma} c - v\right)^{-2} \in C(\mathbb{R}).
\end{aligned}$$

Therefore $v^k \in C^2(\mathbb{R})$ for $k \geq 4$.

For $k \geq 8$ we know from the above that

$$v^4, v^{k-4}, v^{k-2}, v^{k-2}r(v) \in C^2(\mathbb{R}), \text{ and } v^{k-2}\left(\frac{\sigma-1}{\sigma}c - v\right)^{-2} \in C^2\left(\mathbb{R}\backslash v^{-1}(0)\right).$$

Moreover we have

$$v^{k-2}v_x^2 = \frac{1}{4}(v^4)_x\frac{1}{k-4}(v^{k-4})_x \in C^1(\mathbb{R}).$$

Hence from (3.15) we conclude that

$$v^k \in C^3\left(\mathbb{R}\backslash v^{-1}(0)\right), \quad k \geq 8.$$

Applying the same argument to higher values of k we prove that $v^k \in C^j\left(\mathbb{R}\backslash v^{-1}(0)\right)$ for $k \geq 2^j$, and hence (3.13). □

Denote $\overline{x} = \min\{x : \varphi(x) = \frac{c}{\sigma}\}$ (if $\varphi \neq \frac{c}{\sigma}$ for all x then let $\overline{x} = +\infty$), then $\overline{x} \leq +\infty$. By Lemma 3.1, a solitary wave φ is smooth on $(-\infty, \overline{x})$ and (3.8) holds pointwise on $(-\infty, \overline{x})$. Multiplying (3.12) by φ_x and integrating on $(-\infty, x]$ for $x < \overline{x}$ to get

$$\begin{aligned}\varphi_x^2 &= \frac{\varphi^2\left[(c-\varphi)^2 + A(c-\varphi) - (1-2\Omega A - 2\Omega c)\right]}{(c-\varphi)(c-\sigma\varphi)}\\ &= \frac{\varphi^2\left[\varphi^2 - (2c+A)\varphi + c^2 + Ac - (1-2\Omega A - 2\Omega c)\right]}{(c-\varphi)(c-\sigma\varphi)}\\ &= \frac{\varphi^2 f(\varphi)}{(c-\varphi)(c-\sigma\varphi)} := F(\varphi),\end{aligned} \tag{3.16}$$

where $f(\varphi)$ is defined by (3.11).

Putting together, we obtain the ODEs for φ as follows.

$$\varphi_x^2 = \begin{cases} G(\varphi), & \text{when } \sigma = 0, \\ F(\varphi), & \text{when } \sigma \neq 0. \end{cases} \tag{3.17}$$

Since both G and F are rational functions of φ, a simple phase-plane analysis determines the behavior of solutions near the zeros and poles of G and F. We will first look at the case when $\sigma \neq 0$.

Case 1. When φ approaches a simple zero m of $F(\varphi)$, it follows that $F(m) = 0$ and $F'(m) \neq 0$. Then the solution φ of (3.16) satisfies

$$\varphi_x^2 = (\varphi - m)F'(m) + O((\varphi-m)^2) \quad \text{as } \varphi \to m,$$

Hence

$$\varphi(x) = m + \frac{1}{4}(x-x_0)^2F'(m) + O((x-x_0)^4) \quad \text{as } x \to x_0, \tag{3.18}$$

where $\varphi(x_0) = m$.

Case 2. If $F(\varphi)$ has a double zero at $\varphi = 0$, so that $F'(0) = 0$ and $F''(0) > 0$, then

$$\varphi_x^2 = \varphi^2 F''(0) + O(\varphi^3) \quad \text{as } \varphi \to 0,$$

and we get

$$\varphi(x) \backsim \alpha\text{exp}(-|x|\sqrt{F''(0)}) \quad \text{as } |x| \to +\infty, \tag{3.19}$$

for some constant α. Thus $\varphi \to 0$ exponentially as $|x| \to \infty$.

Case 3. If φ approaches a simple pole $\varphi(x_0) = \frac{c}{\sigma}$ (when $\sigma \neq 1$). Then

$$\varphi(x) - \frac{c}{\sigma} = \beta_1 |x - x_0|^{2/3} + O((x - x_0)^{4/3}) \quad \text{as } x \to x_0, \tag{3.20}$$

and

$$\varphi_x = \begin{cases} \frac{2}{3}\beta_1 |x - x_0|^{-1/3} + O((x - x_0)^{1/3}) & \text{as } x \downarrow x_0, \\ -\frac{2}{3}\beta_1 |x - x_0|^{-1/3} + O((x - x_0)^{1/3}) & \text{as } x \uparrow x_0, \end{cases} \tag{3.21}$$

for some constant $\beta_1 > 0$. In particular, whenever $F(\varphi)$ has a pole, the solution φ has a cusp.

Case 4. Peaked solitary waves occur when φ suddenly changes direction: $\varphi_x \to -\varphi_x$ according to (3.16).

When $\sigma = 0$, similar conclusions in Case 1 and Case 2 are also valid for $G(\varphi)$.

From looking at the forms of G and F, cf. (3.10) and (3.16), we see that the only term that remains complicated is $f(\varphi)$ in the numerator. The following discussion enlists all possible distribution of the roots of f.

(a) $f(\varphi)$ has no zeros: If

$$\begin{cases} c > \frac{1}{2\Omega} - 2\Omega > 0, \\ 4\Omega - 2\sqrt{4\Omega^2 + 2\Omega c - 1} < A < 4\Omega + 2\sqrt{4\Omega^2 + 2\Omega c - 1}, \end{cases} \tag{3.22}$$

where we have used the fact that $0 < \Omega < \frac{1}{4}$, then

$$f(\varphi) > 0.$$

And a simple calculation shows that

$$\frac{2c + A}{2} > c + 2\Omega - \sqrt{4\Omega^2 + 2\Omega c - 1} > 0. \tag{3.23}$$

(b) $f(\varphi)$ has a double zero: If

$$\begin{cases} c \geq \frac{1}{2\Omega} - 2\Omega > 0, \\ A = 4\Omega \pm 2\sqrt{4\Omega^2 + 2\Omega c - 1}, \end{cases} \tag{3.24}$$

then

$$A^2 - 8\Omega A + 4(1 - 2\Omega c) = 0.$$

Hence

$$f(\varphi) = \left(\varphi - \frac{2c + A}{2}\right)^2, \tag{3.25}$$

with

$$\frac{2c + A}{2} > 0. \tag{3.26}$$

(c) $f(\varphi)$ has two simple zeros: If

$$c < \frac{1}{2\Omega} - 2\Omega, \tag{3.27}$$

or

$$\begin{cases} c > \frac{1}{2\Omega} - 2\Omega, \\ A < 4\Omega - 2\sqrt{4\Omega^2 + 2\Omega c - 1}, \quad \text{or} \quad A > 4\Omega + 2\sqrt{4\Omega^2 + 2\Omega c - 1}, \end{cases} \tag{3.28}$$

then

$$A^2 - 8\Omega A + 4(1 - 2\Omega c) > 0.$$

Hence

$$f(\varphi) = (\varphi - M_1)(\varphi - M_2), \tag{3.29}$$

where

$$M_1 = \frac{(2c + A) - \sqrt{A^2 - 8\Omega A + 4(1 - 2\Omega c)}}{2}, \tag{3.30}$$

$$M_2 = \frac{(2c + A) + \sqrt{A^2 - 8\Omega A + 4(1 - 2\Omega c)}}{2}, \tag{3.31}$$

and $M_1 < M_2$.

4. Classification of solitary waves when $1 - 2\Omega A - 2\Omega c \neq 0$

With the results established in the previous section, we are in position to classify all solitary waves of system (1.1) for various σ, under the assumption that $1 - 2\Omega A - 2\Omega c \neq 0$.

4.1. The case $\sigma = 0$.

From the discussion in Section 3 Case A, we know that all solitary waves are smooth in this case. As $\varphi = 0$ is a double zero of $G(\varphi)$, we know from the Section 3 Case 2 that the solitary waves decay exponentially to 0 as $|x| \to \infty$. Hence there must exist

$$m_\varphi = \min_{x\in\mathbb{R}} \varphi(x), \qquad M_\varphi = \max_{x\in\mathbb{R}} \varphi(x),$$

with $m_\varphi M_\varphi < 0$, and we have $\varphi_x \to 0$ as $\varphi \to m_\varphi$ or M_φ. Hence (3.10) shows that $G(m_\varphi) = G(M_\varphi) = 0$. Thus $f(\varphi)$ must have two simple zeros or a double zero.

On the other hand, combining (3.10) with the decay property of φ at infinity and $\varphi < c$, we know that necessarily

$$M_1M_2 = c^2 + Ac - (1 - 2\Omega A - 2\Omega c) = (c - K_1)(c - K_2) \geq 0, \tag{4.1}$$

where M_1, M_2 are defined by (3.30) and (3.31) and

$$\begin{aligned} K_1 &= \frac{-(A + 2\Omega) + \sqrt{(A - 2\Omega)^2 + 4}}{2} = \frac{-(A + 2\Omega) + \sqrt{(A + 2\Omega)^2 + 4(1 - 2\Omega A)}}{2}, \\ K_2 &= \frac{-(A + 2\Omega) - \sqrt{(A - 2\Omega)^2 + 4}}{2} = \frac{-(A + 2\Omega) - \sqrt{(A + 2\Omega)^2 + 4(1 - 2\Omega A)}}{2} \end{aligned} \tag{4.2}$$

are the two roots of the equation $c^2 + Ac - (1 - 2\Omega A - 2\Omega c) = 0$. Since $1 - 2\Omega A > 0$, we know $K_1 > 0 > K_2$.

Combining (4.1) with $\varphi \neq c$ and $M_1 < M_2$, the fact that 0 is the double zero of $G(\varphi)$, and the conclusion in Section 2 Case 2, we have that

(i) if $c > 0$, then $0 \leq M_1 < c$ or $M_2 \leq 0$;

(ii) if $c < 0$, then $0 \geq M_2 > c$ or $M_1 \geq 0$.

Furthermore, one can prove that:

Theorem 4.1. *Suppose that* $1 - 2\Omega A - 2\Omega c \neq 0$. *When* $\sigma = 0$*, and*

(I) if

$$\begin{cases} c \geq \frac{1}{2\Omega} - 2\Omega > 0, \\ A = 4\Omega \pm 2\sqrt{4\Omega^2 + 2\Omega c - 1}, \end{cases}$$

i.e., from Section 3 case (b), $f(\varphi)$ *has a double zero, then* (3.1) *does not admit a solitary solution* $\varphi < 0$. *Besides,* (3.1) *also does not admit a solitary solution* $\varphi > 0$ *for* $A > 0$.

(II) if

$$c < \frac{1}{2\Omega} - 2\Omega, \tag{4.3}$$

or

$$\begin{cases} c > \frac{1}{2\Omega} - 2\Omega, \\ A < 4\Omega - 2\sqrt{4\Omega^2 + 2\Omega c - 1}, \quad \textit{or} \quad A > 4\Omega + 2\sqrt{4\Omega^2 + 2\Omega c - 1}, \end{cases} \tag{4.4}$$

i.e., from Section 3 case (c), $f(\varphi)$ *has two simple zeros, then* (3.1) *admits a solitary solution if and only if*

$$\begin{aligned} &(i) \quad c > M_1 > 0 \quad \textit{or} \quad M_2 < 0, \quad \textit{when } c > 0, \;\; \textit{or} \\ &(ii) \quad 0 > M_2 > c \quad \textit{or} \quad M_1 > 0, \quad \textit{when } c < 0. \end{aligned}$$

Moreover, all solitary waves are smooth.

Proof. The regularity has been discussed in Section 3 Case A. So we will just focus on the existence part.

First, we consider the case (I). In this case, we have

$$\varphi_x^2 = \frac{\varphi^2(\varphi - \frac{2c+A}{2})^2}{c(c-\varphi)} = G(\varphi) \tag{4.5}$$

and

$$G'(\varphi) = \frac{c\varphi(\frac{2c+A}{2} - \varphi)\left(3\varphi^2 - (4c + \frac{2c+A}{2})\varphi + c(2c+A)\right)}{c^2(c-\varphi)^2}. \tag{4.6}$$

From (4.5), we know that φ cannot oscillate around zero near infinity. Consider the following two cases:

Case 1: $\varphi(x) < 0$ near $-\infty$. Then there is some x_0 sufficiently negative so that $\varphi(x_0) = -\varepsilon < 0$, with $\varepsilon > 0$ sufficiently small, and $\varphi_x(x_0) < 0$. From standard ODE theory, we can generate a unique local solution $\varphi(x)$ on $[x_0 - L, x_0 + L]$ for some $L > 0$. By (4.6), we have $G'(\varphi) < 0$ for $\varphi < 0$. Therefore $G(\varphi)$ decreases for $\varphi < 0$. Because $\varphi_x(x_0) < 0$, φ decreases near x_0, so $G(\varphi)$ increases near x_0. Hence by (4.6), φ_x decreases near x_0, and then φ and φ_x both decrease on $[x_0 - L, x_0 + L]$. Since $\sqrt{G(\varphi)}$ is local Lipschitz in φ for $\varphi < 0$, we can continue the local solution to all of $\mathbb{R}$ and obtain that $\varphi(x) \to -\infty$ as $x \to \infty$, which fails to be in H^1. Thus there is no solitary wave in this case.

Case 2: $\varphi(x) > 0$ near $-\infty$. Then there is some x_0 sufficiently negative so that $\varphi(x_0) = \varepsilon > 0$, with $\varepsilon > 0$ sufficiently small, and $\varphi_x(x_0) > 0$. From standard ODE theory, we can generate a unique local solution $\varphi(x)$ on $[x_0 - L, x_0 + L]$ for some $L > 0$. If $A > 0$, then $\frac{2c+A}{2} > c$. Thus $G(\varphi) = \varphi_x^2 > 0$ for $0 < \varphi < c$, which contradicts the fact that for the smooth solitary waves there must exist some points such that $G(m_\varphi) = G(M_\varphi) = 0$. Thus there are no solitary waves in this case.

Now we turn our attention to the case (II). Then we need to show that

$$M_1 M_2 = (c - K_1)(c - K_2) > 0, \tag{4.7}$$

where $f(\varphi)$ and K_1, K_2 are defined by (3.11) and (4.2) respectively.

If $c = K_1$, then (3.10) becomes

$$\varphi_x^2 = \frac{-\varphi^3(2K_1 + A - \varphi)}{K_1(K_1 - \varphi)} = \frac{-\varphi^3(K_1 - K_2 - 2\Omega - \varphi)}{K_1(K_1 - \varphi)} := G_1(\varphi), \tag{4.8}$$

where

$$2K_1 + A = \sqrt{(A-2\Omega)^2+4} - 2\Omega \geq 2 - 2\Omega > 0.$$

Hence we see that $\varphi(x) < 0$ near $-\infty$. Because $\varphi(x) \to 0$ as $x \to -\infty$, there is some x_0 sufficiently negative so that $\varphi(x_0) = -\varepsilon < 0$, with $\varepsilon > 0$ sufficiently small, and $\varphi_x(x_0) < 0$. From standard ODE theory, we can generate a unique local solution $\varphi(x)$ on $[x_0 - L, x_0 + L]$ for some $L > 0$. Since $K_1 > 0 > K_2$, we have

$$\left[\frac{-\varphi^3(2K_1 + A - \varphi)}{(K_1 - \varphi)}\right]' = \frac{\varphi^2[-3\varphi^2 + 2(4K_1 + A)\varphi - 3K_1(2K_1 + A)]}{(K_1 - \varphi)^2} < 0, \tag{4.9}$$

for $\varphi < 0$. Therefore $G_1(\varphi)$ decreases for $\varphi < 0$. A similar argument as Case 1 shows there is no solitary wave in this case.

Similarly we can prove that when $c = K_2$ there is no solitary wave. The proof of this theorem is thus completed. □

4.2. **The case $\sigma \neq 0$.** Now we give the following theorem on the existence of solitary waves of (3.1) for $\sigma \neq 0$.

Theorem 4.2. *Suppose* $1 - 2\Omega A - 2\Omega c \neq 0$. *For* $\sigma \neq 0$ *we have*
(I) If

$$\begin{cases} c > \frac{1}{2\Omega} - 2\Omega > 0, \\ 4\Omega - 2\sqrt{4\Omega^2 + 2\Omega c - 1} < A < 4\Omega + 2\sqrt{4\Omega^2 + 2\Omega c - 1}, \end{cases}$$

i.e., from Section 3 Case (a), $f(\varphi)$ has no zeros, then we have:

I-1. If $\sigma > 1$, then there is cusped solitary wave $\varphi > 0$ with $\max_{x\in\mathbb{R}} \varphi(x) = \frac{c}{\sigma}$.

I-2. If $\sigma < 0$, then there is anticusped (the solution profile has a cusp pointing downward) solitary wave $\varphi < 0$ with $\min_{x\in\mathbb{R}} \varphi(x) = \frac{c}{\sigma}$.

(II) If

$$\begin{cases} c \geq \frac{1}{2\Omega} - 2\Omega > 0, \\ A = 4\Omega \pm 2\sqrt{4\Omega^2 + 2\Omega c - 1}, \end{cases}$$

i.e., from Section 3 Case (b), $f(\varphi)$ has a double zero, we have

II-1. If $\sigma > 1$, then there is a smooth solitary wave with $\max_{x\in\mathbb{R}} \varphi(x) = \frac{c}{\sigma} = \frac{2c+A}{2}$ *and a cusped solitary wave $\varphi > 0$ with* $\max_{x\in\mathbb{R}} \varphi(x) = \frac{c}{\sigma} < \frac{2c+A}{2}$.

II-2. If $0 < \sigma \leq 1$, there is no solitary wave $\varphi < 0$, and no solitary wave $\varphi > 0$ when $A \geq 0$.

II-3. If $\sigma < 0$, then there is an anticusped solitary wave $\varphi < 0$ with $\min_{x\in\mathbb{R}} \varphi(x) = \frac{c}{\sigma}$ *and there is no solitary wave $\varphi > 0$ when $A \geq 0$.*

(III) If

$$c < \frac{1}{2\Omega} - 2\Omega, \quad \text{or} \quad \begin{cases} c > \frac{1}{2\Omega} - 2\Omega, \\ A < 4\Omega - 2\sqrt{4\Omega^2 + 2\Omega c - 1} \quad \text{or} \quad A > 4\Omega + 2\sqrt{4\Omega^2 + 2\Omega c - 1} \end{cases}$$

i.e., from Section 3 Case (c), $f(\varphi)$ has two simple zeros, then we have

III-1. When $c = K_1$ *and* $\sigma < 0$*, there is an anticusped solitary wave* $\varphi < 0$ *with* $\min_{x\in\mathbb{R}} \varphi(x) = \frac{K_1}{\sigma}$.

III-2. When $c = K_2$ *and* $\sigma < 0$*, there is a cusped solitary wave* $\varphi > 0$ *with* $\max_{x\in\mathbb{R}} \varphi(x) = \frac{K_2}{\sigma}$.

III-3. When $M_1 < M_2 < 0 < c$*,*
(1) if $\sigma < 0$*, then* $\varphi < 0$*. Moreover, we have the following.*
If $M_2 > \frac{c}{\sigma}$*, then there is a smooth solitary wave with* $\min_{x\in\mathbb{R}} \varphi(x) = M_2$*;*
if $M_2 = \frac{c}{\sigma}$*, then there is a peaked solitary wave with* $\min_{x\in\mathbb{R}} \varphi(x) = M_2 = \frac{c}{\sigma}$*;*
if $M_2 < \frac{c}{\sigma}$*, then there is an anticusped solitary wave with* $\min_{x\in\mathbb{R}} \varphi(x) = \frac{c}{\sigma}$*;*
(2) if $0 < \sigma \leq 1$*, then there is a smooth wave* $\varphi < 0$ *with* $\min_{x\in\mathbb{R}} \varphi(x) = M_2$*;*
(3) if $\sigma > 1$*, then there is a smooth wave* $\varphi < 0$ *with* $\min_{x\in\mathbb{R}} \varphi(x) = M_2$ *and a cusped solitary wave* $\varphi > 0$ *with* $\max_{x\in\mathbb{R}} \varphi(x) = \frac{c}{\sigma}$.

III-4. When $0 < M_1 < M_2 < c$ *or* $0 < M_1 < c < M_2$*,*
(1) if $\sigma < 0$*, then there is a smooth wave* $\varphi > 0$ *with* $\max_{x\in\mathbb{R}} \varphi(x) = M_1$ *and an anticusped solitary wave* $\varphi < 0$ *with* $\min_{x\in\mathbb{R}} \varphi(x) = \frac{c}{\sigma}$*;*
(2) if $0 < \sigma \leq 1$*, then there is a smooth wave* $\varphi > 0$ *with* $\max_{x\in\mathbb{R}} \varphi(x) = M_1$*;*
(3) if $\sigma > 1$*, then* $\varphi > 0$*. Moreover, we have the following.*
If $M_1 < \frac{c}{\sigma}$*, then there is a smooth solitary wave with* $\max_{x\in\mathbb{R}} \varphi(x) = M_1$*;*
if $M_1 = \frac{c}{\sigma}$*, then there is a peaked solitary wave with* $\max_{x\in\mathbb{R}} \varphi(x) = M_1 = \frac{c}{\sigma}$*;*
if $M_1 > \frac{c}{\sigma}$*, then there is a cusped solitary wave with* $\max_{x\in\mathbb{R}} \varphi(x) = \frac{c}{\sigma}$.

III-5. When $0 < c < M_1 < M_2$*,*
(1) if $\sigma > 1$*, then there is a cusped solitary wave* $\varphi > 0$ *with* $\max_{x\in\mathbb{R}} \varphi(x) = \frac{c}{\sigma}$*;*
(2) if $\sigma < 0$*, then there is an anticusped solitary wave* $\varphi < 0$ *with* $\min_{x\in\mathbb{R}} \varphi(x) = \frac{c}{\sigma}$.

III-6. When $M_1 < M_2 < c < 0$*,*
(1) if $\sigma > 1$*, then there is an anticusped solitary wave* $\varphi < 0$ *with* $\min_{x\in\mathbb{R}} \varphi(x) = \frac{c}{\sigma}$*;*
(2) if $\sigma < 0$*, then there is a cusped solitary wave* $\varphi < 0$ *with* $\min_{x\in\mathbb{R}} \varphi(x) = \frac{c}{\sigma}$.

III-7. When $M_1 < c < M_2 < 0$ *or* $c < M_1 < M_2 < 0$*,*
(1) if $\sigma < 0$*, then there is a smooth wave* $\varphi < 0$ *with* $\min_{x\in\mathbb{R}} \varphi(x) = M_2$ *and a cusped solitary wave* $\varphi > 0$ *with* $\max_{x\in\mathbb{R}} \varphi(x) = \frac{c}{\sigma}$*;*
(2) if $0 < \sigma \leq 1$*, then there is a smooth wave* $\varphi < 0$ *with* $\min_{x\in\mathbb{R}} \varphi(x) = M_2$*;*
(3) if $\sigma > 1$*, then* $\varphi < 0$*. Moreover, we have the following.*
If $M_2 > \frac{c}{\sigma}$*, then there is a smooth solitary wave with* $\min_{x\in\mathbb{R}} \varphi(x) = M_2$*;*
if $M_1 = \frac{c}{\sigma}$*, then there is a peaked solitary wave with* $\min_{x\in\mathbb{R}} \varphi(x) = M_2 = \frac{c}{\sigma}$*;*
if $M_1 < \frac{c}{\sigma}$*, then there is a cusped solitary wave with* $\min_{x\in\mathbb{R}} \varphi(x) = \frac{c}{\sigma}$.

III-8. When $c < 0 < M_1 < M_2$*,*
(1) if $\sigma < 0$*, then* $\varphi > 0$*. Moreover, we have the following.*
If $M_1 < \frac{c}{\sigma}$*, then there is a smooth solitary wave with* $\max_{x\in\mathbb{R}} \varphi(x) = M_1$*;*
if $M_1 = \frac{c}{\sigma}$*, then there is a peaked solitary wave with* $\max_{x\in\mathbb{R}} \varphi(x) = M_1 = \frac{c}{\sigma}$*;*
if $M_1 > \frac{c}{\sigma}$*, then there is a cusped solitary wave with* $\max_{x\in\mathbb{R}} \varphi(x) = \frac{c}{\sigma}$*;*

(2) if $0 < \sigma \leq 1$*, then there is a smooth wave* $\varphi > 0$ *with* $\max_{x\in\mathbb{R}} \varphi(x) = M_1$*;*

(3) if $\sigma > 1$*, then there is a smooth wave* $\varphi > 0$ *with* $\max_{x\in\mathbb{R}} \varphi(x) = M_1$ *and an anticusped solitary wave* $\varphi < 0$ *with* $\min_{x\in\mathbb{R}} \varphi(x) = \frac{c}{\sigma}$*.*

Note that $f(\varphi)$*,* M_1, M_2 *and* K_1, K_2 *are defined by* (3.11)*,* (3.30)*,* (3.31) *and* (4.2) *respectively. Each kind of the above solitary waves is unique and even up to translation and all solitary waves decay exponentially to zero at infinity.*

Proof. We will discuss the cases (I), (II) and (III) respectively.
(I) First, we deal with the case (I), i.e., $f(\varphi) > 0$ *for all* $\varphi(x) \in \mathbb{R}$. In this case

$$\varphi_x^2 = \frac{\varphi^2 f(\varphi)}{(c-\varphi)(c-\sigma\varphi)} = F(\varphi), \tag{4.10}$$

The discussion in the beginning of Section 4.1 shows that there are no smooth solitary waves in this case and since $f(\varphi)$ has no zeros for $\varphi \in \mathbb{R}$, there can only exist cuspons or anticuspons. From (4.10) we see that φ cannot oscillate around zero near infinity.

(i) If $\varphi(x) > 0$ near $-\infty$. Then there is some x_0 sufficiently negative so that $\varphi(x_0) = \varepsilon > 0$, with ε sufficiently small, and $\varphi_x(x_0) > 0$. If $\sigma > 1$, $\sqrt{F(\varphi)}$ is locally Lipschitz in φ for $0 \leq \varphi \leq \frac{c}{\sigma} < c$, then $\frac{c}{\sigma}$ becomes a pole of $F(\varphi)$. And for $\sigma \leq 1$, $\frac{c}{\sigma}$ is not a pole of $F(\varphi)$. Thus we obtain we obtain a solitary wave with a cusp at $\varphi = \frac{c}{\sigma}$ for $\sigma > 1$ by using (3.20) and (3.21).

(ii) For the case $\varphi(x) < 0$ near $-\infty$, we obtain a solitary wave with an anticusp at $\varphi = \frac{c}{\sigma}$ for $\sigma < 0$ by using (3.20) and (3.21).

(II) Next, we deal with the case (II), i.e., $f(\varphi)$ *has a double zero*. In this case

$$\varphi_x^2 = \frac{\varphi^2(\varphi - \frac{2c+A}{2})^2}{(c-\varphi)(c-\sigma\varphi)} = F(\varphi). \tag{4.11}$$

From (4.11) we see that φ cannot oscillate around zero near infinity. We will only consider the case $\varphi(x) > 0$ near $-\infty$, the case $\varphi(x) < 0$ near $-\infty$ can be handled in a similar way

If $\varphi(x) > 0$ near $-\infty$. Then there is some x_0 sufficiently negative so that $\varphi(x_0) = \varepsilon > 0$, with ε sufficiently small, and $\varphi_x(x_0) > 0$.

(i) If $\sigma > 1$, $\sqrt{F(\varphi)}$ is locally Lipschitz in φ for $0 \leq \varphi \leq \frac{c}{\sigma} < c$. If $\frac{c}{\sigma} = \frac{2c+A}{2}$, then $\varphi_x^2 = \frac{\varphi^2(c-\sigma\varphi)}{\sigma^2(c-\varphi)}$. Hence the smooth solution can be constructed with the maximum height $\varphi = \frac{2c+A}{2} = \frac{c}{\sigma}$. Since $\varphi = 0$ is still a double zero of $F(\varphi)$, we still have the exponential decay here.

If $\frac{c}{\sigma} < \frac{2c+A}{2}$, then $\frac{c}{\sigma}$ becomes a pole of $F(\varphi)$. Using (3.20) and (3.21), we obtain a solitary wave with a cusp at $\varphi = \frac{c}{\sigma}$ and decays exponentially.

(ii) In the case $\sigma \leq 1$, one can use the arguments similar as the proof of Theorem 4.1 to get there would be no solitary wave in the case $A \geq 0$.

(III) Finally, we deal with the case (III), i.e., $f(\varphi)$ *has two simple zero*. In this case

$$\varphi_x^2 = \frac{\varphi^2(\varphi - M_1)(\varphi - M_2)}{(c-\varphi)(c-\sigma\varphi)} = F(\varphi). \tag{4.12}$$

From (4.12) and the decay of $\varphi(x)$ at infinity, we know that solitary waves exist if condition (4.1) holds. First we deal with the case $M_1 M_2 = (c - K_1)(c - K_2) = 0$, where M_1, M_2 and K_1, K_2 are defined by (3.30), (3.31) and (4.2) respectively.

If $c = K_1$, then (4.12) becomes

$$\varphi_x^2 = \frac{-\varphi^3(2K_1 + A - \varphi)}{(K_1 - \varphi)(K_1 - \sigma\varphi)} := F_1(\varphi). \tag{4.13}$$

Then we find that $\varphi(x) < 0$ near $-\infty$. Hence we can find some x_0 sufficiently negative with $\varphi(x_0) = -\varepsilon < 0$ and $\varphi_x(x_0) < 0$, and we can construct unique local solution $\varphi(x)$ on $[x_0 - L, x_0 + L]$ for some $L > 0$.

If $\sigma < 0$, we see that $\frac{1}{K_1 - \sigma\varphi}$ is decreasing when $K_1/\sigma < \varphi \leq 0$. Combining this with (4.9) we know that $F_1(\varphi)$ decreases for $\varphi < 0$. Because $\varphi_x(x_0) < 0$, $\varphi(x)$ decreases near x_0, so that $F_1(\varphi)$ increases near x_0. Hence from (4.13), $\varphi_x(x)$ decreases near x_0. Then φ and φ_x both decreases on $[x_0 - L, x_0 + L]$. Since $\sqrt{F_1(\varphi)}$ is locally Lipschitz in φ for $K_1/\sigma < \varphi \leq 0$, we can easily continue the local solution to $(-\infty, x_0 - L]$ with $\varphi(x) \to 0$ as $x \to -\infty$. As for $x \geq x_0 + L$, we can solve the initial valued problem

$$\begin{cases} \psi_x = -\sqrt{F_1(\varphi)}, \\ \psi(x_0 + L) = \varphi(x_0 + L) \end{cases}$$

all the way until $\psi = K_1/\sigma$, which is a simple pole of $F_1(\varphi)$. By (3.20) and (3.21), we deduce that we can construct an anticusped solution with a cusp singularity at $\varphi = K_1/\sigma$.

If $\sigma > 0$, then a direct computation shows

$$\begin{aligned} F_1'(\varphi) =& \frac{(2K_1 + A - \varphi)\left(-\sigma\varphi^2 + 2K_1(1+\sigma)\varphi - 3K_1^2\right)}{(K_1 - \varphi)^2(K_1 - \sigma\varphi)^2} \\ &+ \frac{\varphi(K_1 - \varphi)(K_1 - \sigma\varphi)}{(K_1 - \varphi)^2(K_1 - \sigma\varphi)^2} < 0 \end{aligned} \tag{4.14}$$

for $\varphi < 0$. A similar argument as Theorem 4.1 shows that there is no solitary wave in this case.

Similarly, we conclude that when $c = K_2$, there is no solitary wave when $\sigma > 0$. When $\sigma < 0$, there is an cusped solution with a cusp singularity at K_2/σ.

Now we deal with the case $M_1 M_2 = (c - K_1)(c - K_2) > 0$. Recalling that $M_1 < M_2$, 8 cases are there we will consider. We will only look at $0 < M_1 < M_2 < c$. The other cases can be handled in a very similar way. Applying (4.12), we know that φ can not oscillate around zero near infinity. Let us consider the following two cases.

Case 1: $\varphi(x) > 0$ near $-\infty$. Then there is some x_0 sufficiently negative so that $\varphi(x_0) = \varepsilon > 0$, with ε sufficiently small, and $\varphi_x(x_0) > 0$.

(i) When $\sigma \leq 1$, $\sqrt{F(\varphi)}$ is locally Lipschitz in φ for $0 \leq \varphi < M_1 < c$. Hence there is a local solution to

$$\begin{cases} \varphi_x = \sqrt{F(\varphi)}, \\ \varphi(x_0) = \varepsilon \end{cases}$$

on $[x_0 - L, x_0 + L]$ for some $L > 0$. Therefore by (3.18), we obtain a smooth solitary wave with maximum height $\varphi = M_1$ and an exponential decay to zero at infinity

$$\varphi(x) = O\left(\exp\left(-\frac{\sqrt{c^2 + Ac - (1 - 2\Omega A - 2\Omega c)}}{c}|x|\right)\right) \quad \text{as} \quad |x| \to \infty. \tag{4.15}$$

(ii) When $\sigma > 1$, $\sqrt{F(\varphi)}$ is locally Lipschitz in φ for $0 \leq \varphi < \frac{c}{\sigma}$. Thus if $M_1 < \frac{c}{\sigma}$, it becomes the same as (i) and we can obtain smooth solitary waves with exponential decay.

If $M_1 = \frac{c}{\sigma}$, then the smooth solution can be constructed until $\varphi = M_1 = \frac{c}{\sigma}$. However at $\varphi = M_1 = \frac{c}{\sigma}$ it can make a sudden turn and so gives rise to a peak. Since $\varphi = 0$ is still a double zero of $F(\varphi)$, we still have the exponential decay here.

If $M_1 > \frac{c}{\sigma}$, then $\varphi = \frac{c}{\sigma}$ becomes a pole of $F(\varphi)$. Using (3.20) and (3.21), we obtain a solitary wave with a cusp at $\varphi = \frac{c}{\sigma}$ and decays exponentially.

Case 2: $\varphi(x) < 0$ near $-\infty$. In this case we are solving

$$\begin{cases} \varphi_x = -\sqrt{F(\varphi)}, \\ \varphi(x_0) = -\varepsilon \end{cases}$$

for some x_0 sufficiently negative and $\varepsilon > 0$ sufficiently small.

When $\sigma > 0$ we see that $F'(\varphi) < 0$, for $\varphi < 0$. Therefore in this case there is no solitary wave.

If $\sigma < 0$, then $\varphi = c/\sigma < 0$ is a pole of $F(\varphi)$. Argue as before, we obtain an anticusped solitary wave with $\min_{x\in\mathbb{R}} = c/\sigma$, which decays exponentially.

Finally, by the standard ODE theory and the fact that the equation (3.8) is invariant under the transformation $x \longrightarrow -x$, we conclude that the solitary waves obtained above are unique and up to translations. □

We conclude this section by providing an implicit formula for the peaked solitary waves. Let us consider only the case $c > M_1 > 0$. Denote

$$\widetilde{A}_1 = \frac{-A + \sqrt{A^2 - 8\Omega A + 4(1 - 2\Omega c)}}{2},$$

and

$$\widetilde{A}_2 = \frac{-A - \sqrt{A^2 - 8\Omega A + 4(1 - 2\Omega c)}}{2},$$

then

$$M_1 = c - \widetilde{A}_1, \qquad M_2 = c - \widetilde{A}_2.$$

By Theorem 4.2 we know that peaked solitary waves exist only when $M_1 = \frac{c}{\sigma}$. In this case we have

$$\varphi_x^2 = \frac{\varphi^2(M_2 - \varphi)}{c - \varphi}.$$

Since φ is positive, even with respect to some x_0 and decreasing on (x_0, ∞), so for $x > x_0$ we have

$$\varphi_x = -\varphi\sqrt{1 - \frac{\widetilde{A}_2}{c - \varphi}}.$$

Integrating the above equation, there appears

$$-(x - x_0) = \int_{c-\widetilde{A}_1}^{\varphi} \frac{ds}{s\sqrt{1 - \frac{\widetilde{A}_2}{c-s}}}.$$

Let $\omega = 1 - \frac{\widetilde{A}_2}{c-s}$ in the the above equation. Then

$$\begin{aligned} -(x-x_0) &= \int_{1-\frac{\widetilde{A}_2}{\widetilde{A}1}}^{1-\frac{\widetilde{A}_2}{c-\varphi}} \frac{-\widetilde{A}_2}{[c\omega-(c-\widetilde{A}_2)](\omega-1)\sqrt{\omega}} d\omega \\ &= \int_{1-\frac{\widetilde{A}_2}{\widetilde{A}1}}^{1-\frac{\widetilde{A}_2}{c-\varphi}} \frac{1}{\sqrt{\omega}} \left[\frac{c}{c\omega-(c-\widetilde{A}_2)} - \frac{1}{\omega-1} \right] d\omega \\ &= \left(\sqrt{\frac{c}{c-\widetilde{A}_2}} \ln \left| \frac{\sqrt{c\omega}-\sqrt{c-\widetilde{A}_2}}{\sqrt{c\omega}+\sqrt{c-\widetilde{A}_2}} \right| - \ln \left| \frac{\sqrt{\omega}-1}{\sqrt{\omega}+1} \right| \right)_{1-\frac{\widetilde{A}_2}{\widetilde{A}_1}}^{1-\frac{\widetilde{A}_2}{c-\varphi}}. \end{aligned}$$

An implicit formula for the peaked solitary waves is thus established in the following.

$$-|x-x_0| = \left(\sqrt{\frac{c}{c-\widetilde{A}_2}} \ln \left| \frac{\sqrt{c\omega}-\sqrt{c-\widetilde{A}_2}}{\sqrt{c\omega}+\sqrt{c-\widetilde{A}_2}} \right| - \ln \left| \frac{\sqrt{\omega}-1}{\sqrt{\omega}+1} \right| \right)_{1-\frac{\widetilde{A}_2}{\widetilde{A}_1}}^{1-\frac{\widetilde{A}_2}{c-\varphi}}.$$

Below is a graph of a peaked solitary wave. The blue is the velocity profile and the red is the surface profile.

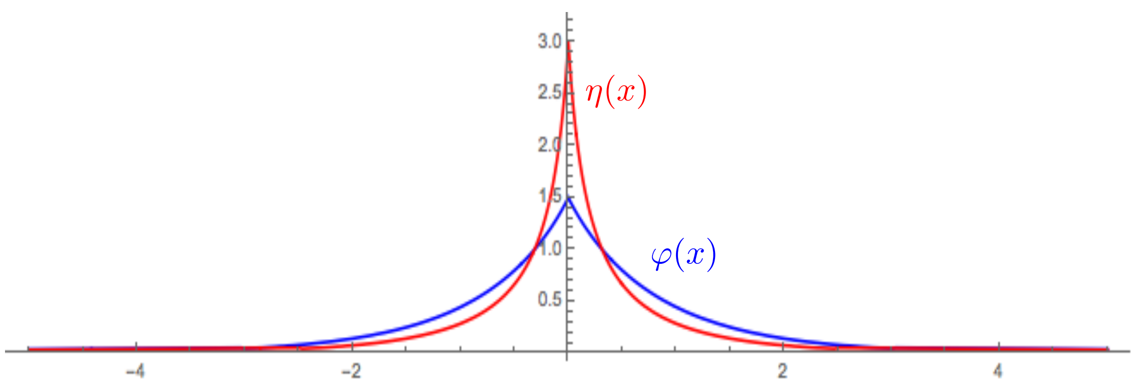

FIGURE 1. The plot of a peakon with $x_0 = 0, A = 0, \Omega = \dfrac{3}{16}, c = 2, \sigma = \dfrac{4}{3}$.

5. TRAVELING-WAVE SOLUTIONS

Attention in this section is restricted to a unique x-symmetric weak solution of system (1.1). We will prove that such a solution must be a traveling wave. First, we define what we mean by an x-symmetric solution.

Definition 5.1. *A function $\vec{u}(t,x) = (u(t,x), \eta(t,x))$ is x-symmetric if there exists a function $b(t) \in C^1(\mathbb{R}_+)$ such that*

$$\vec{u}(t,x) = (u(t, 2b(t)-x), \eta(t, 2b(t)-x)),$$

for almost every $x \in \mathbb{R}$. We say that $b(t)$ is the symmetric axis of $\vec{u}(t,x)$.

In the subsequent discussion, we will use $\langle\ ,\ \rangle$ for distributions and we can rewrite (3.3) in Definition 3.1 as follows

$$\begin{cases} \langle u, (1-\partial_x^2)\psi_t\rangle + \Omega\langle (1+\eta)^2, \psi_t\rangle \\ \quad -\langle Au - \frac{3}{2}u^2 - \frac{\sigma}{2}u_x^2 - \frac{1-2\Omega A}{2}(1+\eta)^2, \psi_x\rangle - \langle \frac{\sigma}{2}u^2, \psi_{xxx}\rangle = 0, \\ \langle \eta, \psi_t\rangle + \langle (1+\eta)u, \psi_x\rangle = 0. \end{cases} \tag{5.1}$$

Lemma 5.1. *Assume that $\vec{U}(x) = (U(x), V(x)) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ and satisfies*

$$\begin{cases} \int_{\mathbb{R}} [cU(1-\partial_x^2)\phi_x + \Omega c(1+V)^2\phi_x \\ \quad +(AU - \frac{3}{2}U^2 - \frac{\sigma}{2}U_x^2 - \frac{1-2\Omega A}{2}(1+V)^2)\phi_x + \frac{\sigma}{2}U^2\phi_{xxx}]dx = 0, \\ \int_{\mathbb{R}} [-cV\phi_x + (1+V)U\phi_x]dx = 0 \end{cases} \tag{5.2}$$

for all $\phi \in C_0^\infty(\mathbb{R})$. Then $\vec{u}$ given by

$$\vec{u}(t,x) = \vec{U}(x - c(t-t_0)) \tag{5.3}$$

is a weak solution of system (1.1)*, for any fixed $t_0 \in \mathbb{R}$.*

Proof. Without loss of generality, we can assume $t_0 = 0$. Following the arguments in [18], we get the $\vec{u}(t,x)$ belongs to $C(\mathbb{R}, H^1(\mathbb{R}) \times L^2(\mathbb{R}))$. For any $\zeta \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R})$, letting $\zeta_c = \zeta(t, x+ct)$, it follows that

$$\begin{cases} \partial_x(\zeta_c) = (\zeta_x)_c, \\ \partial_t(\zeta_c) = (\zeta_t)_c + c(\zeta_x)_c. \end{cases} \tag{5.4}$$

Assume $\vec{u}(t,x) = \vec{U}(x-ct)$. One can easily check that

$$\begin{cases} \langle u, \zeta\rangle = \langle U, \zeta_c\rangle, \quad \langle u^2, \zeta\rangle = \langle U^2, \zeta_c\rangle, \quad \langle u_x^2, \zeta\rangle = \langle U_x^2, \zeta_c\rangle, \quad \langle \eta, \zeta\rangle = \langle V, \zeta_c\rangle, \\ \langle (1+\eta)^2, \zeta\rangle = \langle (1+V)^2, \zeta_c\rangle, \quad \langle (1+\eta)u, \zeta\rangle = \langle (1+V)U, \zeta_c\rangle, \end{cases} \tag{5.5}$$

where $\vec{U} = (U, V) = (U(x), V(x))$. In view of (5.4) and (5.5), we obtain

$$\begin{aligned} &\langle u, (1-\partial_x^2)\zeta_t\rangle = \langle U, \left((1-\partial_x^2)\partial_t\zeta\right)_c\rangle = \langle U, (1-\partial_x^2)(\partial_t\zeta_c - c\partial_x\zeta_c)\rangle, \\ &\langle (1+\eta)^2, \zeta_t\rangle = \langle (1+V)^2, (\zeta_t)_c\rangle = \langle (1+V)^2, \partial_t\zeta_c - c\partial_x\zeta_c\rangle, \\ &\langle Au - \frac{3}{2}u^2 - \frac{\sigma}{2}u_x^2 - \frac{1-2\Omega A}{2}(1+\eta)^2,\ \zeta_x\rangle \\ &\qquad\qquad = \langle AU - \frac{3}{2}U^2 - \frac{\sigma}{2}U_x^2 - \frac{1-2\Omega A}{2}(1+V)^2, \partial_x\zeta_c\rangle, \\ &\left\langle \frac{\sigma}{2}u^2,\ \zeta_{xxx}\right\rangle = \left\langle \frac{\sigma}{2}U^2,\ \partial_x^3\zeta_c\right\rangle, \\ &\langle \eta,\ \zeta_t\rangle = \langle V, (\zeta_t)_c\rangle = \langle V,\ \partial_t\zeta_c - c\partial_x\zeta_c\rangle, \\ &\langle (1+\eta)u, \zeta_x\rangle = \langle (1+V)U,\ \partial_x\zeta_c\rangle. \end{aligned} \tag{5.6}$$

Noting that $\vec{U}$ is independent of time, for T large enough such that it does not belong to the support of ζ_c, we deduce that

$$\begin{aligned}
\langle U,(1-\partial_x^2)\partial_t\zeta_c\rangle &= \int_{\mathbb{R}} U(x)\int_{\mathbb{R}_+}\partial_t(1-\partial_x^2)\zeta_c dtdx \\
&= \int_{\mathbb{R}} U(x)[(1-\partial_x^2)\zeta_c(T,x)-(1-\partial_x^2)\zeta_c(0,x)]dx = 0, \\
\langle U,\partial_t\zeta_c\rangle &= \int_{\mathbb{R}} U(x)\int_{\mathbb{R}_+}\partial_t\zeta_c dtdx \\
&= \int_{\mathbb{R}} U(x)[\zeta_c(T,x)-\zeta_c(0,x)]dx = 0, \\
\langle V,\partial_t\zeta_c\rangle &= 0.
\end{aligned} \tag{5.7}$$

Combining (5.6) with (5.7), it follows that

$$\begin{aligned}
&\langle u,(1-\partial_x^2)\psi_t\rangle + \Omega\langle(1+\eta)^2,\psi_t\rangle - \langle Au - \frac{3}{2}u^2 - \frac{\sigma}{2}u_x^2 - \frac{1-2\Omega A}{2}(1+\eta)^2,\psi_x\rangle - \langle\frac{\sigma}{2}u^2,\psi_{xxx}\rangle \\
&= \langle U,-c(1-\partial_x^2)\partial_x\zeta_c\rangle + \Omega\langle(1+V)^2,-c\partial_x\zeta_c\rangle \\
&\quad - \langle AU - \frac{3}{2}U^2 - \frac{\sigma}{2}U_x^2 - \frac{1-2\Omega A}{2}(1+2V)^2,\partial_x\zeta_c\rangle - \langle\frac{\sigma}{2}U^2,\partial_x^3\zeta_c\rangle \\
&= \int_{\mathbb{R}_+}\int_{\mathbb{R}}[-cU(1-\partial_x^2)\partial_x\zeta_c - \Omega c(1+V)^2\partial_x\zeta_c \\
&\quad - \left(AU - \frac{3}{2}U^2 - \frac{\sigma}{2}U_x^2 - \frac{1-2\Omega A}{2}(1+2V)^2\right)\partial_x\zeta_c - \frac{\sigma}{2}U^2\partial_x^3\zeta_c]dxdt = 0,
\end{aligned}$$

and

$$\begin{aligned}
&\langle\eta,\psi_t\rangle + \langle(1+\eta)u,\psi_x\rangle \\
&\quad = \langle V,-c\partial_x\zeta_c\rangle + \langle(1+V)U,\partial_x\zeta_c\rangle = \int_{\mathbb{R}_+}\int_{\mathbb{R}} -cV\partial_x\zeta_c + (1+V)U\partial_x\zeta_c dxdt = 0,
\end{aligned}$$

where we used (5.2) with $\phi(x) = \zeta_c(t,x)$, which belongs to $C_0^\infty(\mathbb{R})$, for every given $t \geq 0$. This completes the proof of Lemma 5.1. □

Finally, we give the main result of this section.

Theorem 5.1. *If $\vec{u}(t,x)$ is a unique weak solution of system* (1.1) *and is x-symmetric, then $\vec{u}(t,x)$ is a traveling wave.*

Proof. Recalling Definition 3.1 and noting that $C_0^\infty(\mathbb{R}_+\times\mathbb{R})$ is dense in $C_0^1(\mathbb{R}_+,C_0^3(\mathbb{R}))$, we can only consider the test functions ψ belong to $C_0^1(\mathbb{R}_+,C_0^3(\mathbb{R}))$. Let

$$\psi_b(t,x) = \psi(t,2b(t)-x), \qquad b(t)\in C^1(\mathbb{R}).$$

Then we obtain that $(\psi_b)_b = \psi$ and

$$\begin{cases} \partial_x u_b = -(\partial_x u)_b, \quad \partial_x\psi_b = -(\partial_x\psi)_b, \\ \partial_t\psi_b = (\partial_t\psi)_b + 2\dot{b}(\partial_x\psi)_b, \end{cases} \tag{5.8}$$

where $\dot{b}$ denote the time derivative of b. Moreover

$$\begin{cases} \langle u_b, \psi\rangle = \langle u, \psi_b\rangle, \quad \langle u_b^2, \psi\rangle = \langle u^2, \psi_b\rangle, \\ \langle (\partial_x u_b)^2, \psi\rangle = \langle (\partial_x u)^2, \psi_b\rangle, \quad \langle (1+\eta_b)^2, \psi\rangle = \langle (1+\eta)^2, \psi_b\rangle, \\ \langle \eta_b, \psi\rangle = \langle \eta, \psi_b\rangle, \quad \langle (1+\eta_b)u_b, \psi\rangle = \langle (1+\eta)u, \psi_b\rangle. \end{cases} \tag{5.9}$$

Since $\vec{u}$ is x-symmetric, by virtue of (5.8) and (5.7), we get

$$\begin{aligned} &\langle u, (1-\partial_x^2)\psi_t\rangle = \langle u, \big((1-\partial_x^2)\partial_t\psi\big)_b\rangle = \langle u, (1-\partial_x^2)(\partial_t\psi_b + 2\dot{b}\partial_x\psi_b)\rangle, \\ &\Omega\langle (1+\eta)^2, \psi_t\rangle = \Omega\langle (1+\eta)^2, \partial_t\psi_b + 2\dot{b}\partial_x\psi_b\rangle, \\ &\langle Au - \frac{3}{2}u^2 - \frac{\sigma}{2}u_x^2 - \frac{1-2\Omega A}{2}(1+\eta)^2, \psi_x\rangle \\ &\qquad\qquad = -\langle Au - \frac{3}{2}u^2 - \frac{\sigma}{2}u_x^2 - \frac{1-2\Omega A}{2}(1+\eta)^2, \partial_x\psi_b\rangle, \\ &\langle \frac{\sigma}{2}u^2, \psi_{xxx}\rangle = \langle \frac{\sigma}{2}u^2, -\partial_x^3\psi_b\rangle, \\ &\langle \eta, \psi_t\rangle = \langle \eta, \partial_t\psi_b + 2\dot{b}\partial_x\psi_b\rangle, \\ &\langle (1+\eta)u, \psi_x\rangle = \langle (1+\eta)u, -\partial_x\psi_b\rangle. \end{aligned} \tag{5.10}$$

In view of (5.1), we get

$$\begin{aligned} &\langle u, (1-\partial_x^2)\psi_t\rangle + \Omega\langle (1+\eta)^2, \psi_t\rangle - \langle Au - \frac{3}{2}u^2 - \frac{\sigma}{2}u_x^2 - \frac{1-2\Omega A}{2}(1+\eta)^2), \psi_x\rangle - \langle \frac{\sigma}{2}u^2, \psi_{xxx}\rangle \\ &\quad = \langle u, (1-\partial_x^2)(\partial_t\psi_b + 2\dot{b}\partial_x\psi_b)\rangle + \Omega\langle (1+\eta)^2, (\partial_t\psi_b + 2\dot{b}\partial_x\psi_b)\rangle \\ &\qquad + \langle Au - \frac{3}{2}u^2 - \frac{\sigma}{2}u_x^2 - \frac{1-2\Omega A}{2}(1+\eta)^2, \partial_x\psi_b\rangle + \langle \frac{\sigma}{2}u^2, \partial_x^3\psi_b\rangle = 0, \\ &\langle \eta, \psi_t\rangle + \langle (1+\eta)u, \psi_x\rangle \\ &\quad = \langle \eta, (\partial_t\psi_b + 2\dot{b}\partial_x\psi_b)\rangle + \langle (1+\eta)u, -\partial_x\psi_b\rangle = 0. \end{aligned} \tag{5.11}$$

Noting that $(\psi_b)_b = \psi$ and substituting ψ_b in (5.11) for ψ, we obtain

$$\begin{cases} \langle u, (1-\partial_x^2)(\partial_t\psi + 2\dot{b}\partial_x\psi)\rangle + \Omega\langle (1+\eta)^2, (\partial_t\psi + 2\dot{b}\partial_x\psi)\rangle \\ \quad +\langle Au - \frac{3}{2}u^2 - \frac{\sigma}{2}u_x^2 - \frac{1-2\Omega A}{2}(1+\eta)^2, \partial_x\psi\rangle + \langle \frac{\sigma}{2}u^2, \partial_x^3\psi\rangle = 0, \\ \langle \eta, (\partial_t\psi + 2\dot{b}\partial_x\psi)\rangle + \langle (1+\eta)u, -\partial_x\psi\rangle = 0. \end{cases} \tag{5.12}$$

Combining (5.12) with (5.1), we have

$$\begin{cases} \langle u, -2\dot{b}(1-\partial_x^2)\partial_x\psi\rangle + \Omega\langle (1+\eta)^2, -2\dot{b}\partial_x\psi\rangle \\ \quad -2\langle Au - \frac{3}{2}u^2 - \frac{\sigma}{2}u_x^2 - \frac{1-2\Omega A}{2}(1+\eta)^2, \partial_x\psi\rangle - \langle \sigma u^2, \partial_x^3\psi\rangle = 0, \\ \langle \eta, -2\dot{b}\partial_x\psi\rangle + 2\langle (1+\eta)u, \partial_x\psi\rangle = 0. \end{cases} \tag{5.13}$$

We consider a fixed but arbitrary time $t_0 > 0$. For any $\phi \in C_0^\infty(\mathbb{R})$, let $\psi_\varepsilon(t,x) = \phi(x)\rho_\varepsilon(t)$, where $\rho_\varepsilon \in C_0^\infty(\mathbb{R}_+)$ is a mollifier with the property that $\rho_\varepsilon \to \delta(t-t_0)$, the Dirac mass at t_0, as

$\varepsilon \to 0$. From (5.13), by using the test function $\phi_\varepsilon(t,x)$, we have

$$\begin{cases} \int_{\mathbb{R}} \left(-2(1-\partial_x^2)\partial_x\phi \int_{\mathbb{R}_+} \dot{b}u\rho_\varepsilon(t)dt\right)dx + \Omega\int_{\mathbb{R}}\left(-2\partial_x\phi\int_{\mathbb{R}_+}\dot{b}(1+\eta)^2\rho_\varepsilon(t)dt\right)dx \\ -\int_{\mathbb{R}}\left(2\partial_x\phi\int_{\mathbb{R}_+}(Au-\frac{3}{2}u^2-\frac{\sigma}{2}u_x^2-\frac{1-2\Omega A}{2}(1+\eta)^2)\rho_\varepsilon(t)dt\right)dx \\ -\int_{\mathbb{R}}\left(\sigma\partial_x^3\phi\int_{\mathbb{R}_+}u^2\rho_\varepsilon(t)dt\right)dx = 0, \\ \int_{\mathbb{R}}\left(-2\partial_x\phi\int_{\mathbb{R}_+}\dot{b}\eta\rho_\varepsilon(t)dt\right)dx + \int_{\mathbb{R}}\left(2\partial_x\phi\int_{\mathbb{R}_+}(1+\eta)u\rho_\varepsilon(t)dt\right)dx = 0. \end{cases} \tag{5.14}$$

Note that

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}_+}\dot{b}u\rho_\varepsilon(t)dt = \dot{b}(t_0)u(t_0,x), \quad \lim_{\varepsilon\to 0}\int_{\mathbb{R}_+}\dot{b}\eta\rho_\varepsilon(t)dt = \dot{b}(t_0)\eta(t_0,x), \quad \text{in } L^2(\mathbb{R}),$$

and

$$\begin{aligned} &\lim_{\varepsilon\to 0}\int_{\mathbb{R}_+}\dot{b}(1+\eta)^2\rho_\varepsilon(t)dt = \dot{b}(t_0)(1+\eta(t_0,x))^2, \\ &\lim_{\varepsilon\to 0}\int_{\mathbb{R}_+}(Au-\frac{3}{2}u^2-\frac{\sigma}{2}u_x^2-\frac{1-2\Omega A}{2}(1+\eta)^2)\rho_\varepsilon(t)dt \\ &\quad = Au(t_0,x)-\frac{3}{2}u^2(t_0,x)-\frac{\sigma}{2}(\partial_x u(t_0,x))^2-\frac{1-2\Omega A}{2}(1+\eta(t_0,x))^2, \\ &\lim_{\varepsilon\to 0}\int_{\mathbb{R}_+}u^2\rho_\varepsilon(t)dt = u^2(t_0,x), \\ &\lim_{\varepsilon\to 0}\int_{\mathbb{R}_+}(1+\eta)u\rho_\varepsilon(t)dt = (1+\eta(t_0,x))u(t_0,x) \end{aligned}$$

in $L^1(\mathbb{R})$. Therefore, letting $\varepsilon\to 0$, (5.14) implies that

$$\begin{cases} \int_{\mathbb{R}}\dot{b}(t_0)u(t_0,x)(1-\partial_x^2)\partial_x\phi dx + \Omega\int_{\mathbb{R}}\dot{b}(t_0)(1+\eta(t_0,x))^2\partial_x\phi dx \\ +\int_{\mathbb{R}}\left(Au(t_0,x)-\frac{3}{2}u^2(t_0,x)-\frac{\sigma}{2}(\partial_x u(t_0,x))^2-\frac{1-2\Omega A}{2}(1+\eta(t_0,x))^2\right)\partial_x\phi dx \\ +\int_{\mathbb{R}}\frac{\sigma}{2}u^2(t_0,x)\partial_x^3\phi dx = 0, \\ -\int_{\mathbb{R}}\dot{b}(t_0)\eta(t_0,x)\partial_x\phi dx + \int_{\mathbb{R}}(1+\eta(t_0,x))u(t_0,x)\partial_x\phi dx = 0. \end{cases} \tag{5.15}$$

Thus, we deduce that $u(t_0,x)$ satisfies (5.2) for $c=\dot{b}(t_0)$. Applying Lemma 5.1, we get $\tilde{u}(t,x) = u(t_0, x-\dot{b}(t_0)(t-t_0))$ is a traveling wave solution of system (1.1). Since $\tilde{u}(t_0,x)=u(t_0,x)$, by the uniqueness assumption of the solution of system (1.1), we obtain $\tilde{u}(t,x)=u(t,x)$ for all time t. This completes the proof of Theorem 5.1. □

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