ON MODIFIED DIAGONAL CYCLES AND THE BEAUVILLE DECOMPOSITION OF THE CERESA CYCLE

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Abstract. The choice of a degree 1 divisor e on a curve C of genus $g \geq 2$ gives an embedding of C into its Jacobian J; we denote by $[C]^e \in \mathrm{CH}(J;\mathbb{Q})$ the class defined by the image of C under this embedding. It is known that the vanishing of the Ceresa cycle $\mathrm{Cer}(C,e) := [C]^e - [-1]_*[C]^e$ is equivalent to that of the Gross-Kudla-Schoen modified diagonal cycle $\Gamma^3(C,e) \in \mathrm{CH}(C^3;\mathbb{Q})$. In this paper, we extend this result to show that the vanishing of the s-th Beauville component $[C]^e_{(s)}$, $s \geq 1$ is equivalent to the vanishing of the $(s+2)$-nd higher modified diagonal cycle $\Gamma^{s+2}(C,e) \in \mathrm{CH}(C^{s+2};\mathbb{Q})$. In the $s=1$ case, we show an integral refinement, relating the order of torsion of $\mathrm{Cer}(C,e) \in \mathrm{CH}(J;\mathbb{Z})$ to that of $\Gamma^3(C,e) \in \mathrm{CH}(C^3;\mathbb{Z})$. As a corollary to our results, we deduce that for a very general curve of genus $g \geq 4$, the cycle $[C]^e_{(2)}$ is non-zero for the standard choice of divisor $e = K_C/(2g-2)$. We also establish a "successive vanishing" for these cycles: for instance, if $\Gamma^n(C,e) = \Gamma^{n+1}(C,e) = 0$, then $\Gamma^k(C,e) = 0$ for all $k \geq n$.

Contents

1. Introduction

Let k be an algebraically closed field of arbitrary characteristic, and let C be a smooth connected projective curve of genus $g \geq 2$ over k. For any degree 1 divisor e with integer coefficients, there is an embedding ι_e of C into its Jacobian J, given by $x \mapsto \mathcal{O}_C(x-e)$. The image $[C]^e = \iota_{e,*}(C)$ is a natural class in the Chow group of the Jacobian, and is connected to interesting geometric invariants such as the Ceresa cycle, defined as $\mathrm{Cer}(C,e) := [C]^e - [-1]_*[C]^e$. This cycle is homologically trivial, but for a very general curve of genus $g \geq 3$, it has infinite order modulo algebraic equivalence (see [Cer83], [Fak96]). Since its discovery, determining the precise locus of curves C for which the Ceresa cycle is non-vanishing, finding vanishing criteria, and constructing

2020 *Mathematics Subject Classification.* 14C25, 14C15, 14H40.

families of curves on which the Ceresa cycle does vanish, have become active areas of research (see for instance [GZ24],[QZ24] [LS25], [LS24]).

The behaviour of these cycles is typically studied in the Chow ring with $\mathbb{Q}$-coefficients (to suppress torsion), which we will simply write as $\mathrm{CH}(X)$ for a variety X over k.

As we are working with rational coefficients, we have the Beauville decomposition

$$\mathrm{CH}(J) = \bigoplus_{i,s} \mathrm{CH}_{i,(s)}(J)$$

where $\mathrm{CH}_{i,(s)}(J) = \{c \in \mathrm{CH}_i(J) : [n]_* c = n^{2i+s} c, \forall n \in \mathbb{Z}\}$, for $[n] : J \to J$ the multiplication-by-n map. For any class $z \in \mathrm{CH}_i(J)$, we will denote its component in $\mathrm{CH}_{i,(s)}(J)$ by $z_{(s)}$. Applying the Beauville decomposition to the Ceresa cycle, we have $\mathrm{Cer}(C, e) = 2 \cdot \sum_{s \text{ odd}} [C]^e_{(s)}$. It therefore becomes interesting to study the Beauville components $[C]^e_{(s)}$ of the curve class in order to study the vanishing of the Ceresa cycle itself.

When working modulo algebraic equivalence, the classes $[C]^e_{(s)}$ are independent of the choice of e, and have been extensively studied. For example, E. Colombo and B. van Geemen [CvG93] showed that if C admits a morphism of degree d to $\mathbb{P}^1$, then $[C]^e_{(s)}$ is algebraically trivial for all $s \geq d-1$. Conversely, they conjectured that for a very general curve (over $\mathbb{C}$) of genus $g \geq 2s+1$, the cycle $[C]^e_{(s)}$ is algebraically non-trivial (this is spelled out in [Voi14, Conjecture 1.4]).

There is another family of classes attached to the pair (C, e), known as the modified diagonal or Gross-Kudla-Schoen cycles [GK92], [GS95]. For any non-empty subset $I \subset \{1, ..., n\}$ we first consider the following cycle in $\mathrm{CH}_1(C^n)$,

$$\Delta^n_I(C, e) = \mathrm{pr}_I^*(\Delta_C^{(|I|)}) \cdot \prod_{i \notin I} \mathrm{pr}_i^* e, \quad \text{where} \quad \mathrm{pr}_I : C^n \longrightarrow C^{|I|}, \tag{1.1}$$

and $\Delta_C^{(m)}$ denotes the small diagonal in C^m. Then, the n-th modified diagonal cycle attached to (C, e) is defined as

$$\Gamma^n(C, e) = \sum_{\emptyset \neq I \subset \{1,\dots,n\}} (-1)^{n-|I|} \Delta^n_I(C, e) \in \mathrm{CH}_1(C^n).$$

These cycles are notable due to their applications to number theory: when C is a modular curve, Gross and Kudla conjectured that $\Gamma^3(C, e)$ arises in special-value formulae for triple-product L-functions and their derivatives, which has been shown in various contexts (see [YZZ23], [DR14]).

The relation between the modified diagonal cycles and the Beauville components $[C]^e_{(s)}$ is first suggested by the following theorem of S.-W. Zhang, working again modulo rational equivalence:

Theorem 1.1 ([Zha10, Theorem 1.5.5])**.** *Let e be any degree 1 divisor on the curve C. Then there is an equivalence between vanishing of cycles as follows:*

$$\mathrm{Cer}(C, e) = 0 \iff [C]^e_{(1)} = 0 \iff \Gamma^3(C, e) = 0.$$

Moreover, this vanishing can occur only if $(2g-2)e = K_C \in \mathrm{CH}_0(C)$.

In light of this theorem, the choice of divisor $e = \xi := K_C/(2g-2)$ has become standard for problems involving the Ceresa cycle. We note that Zhang's original motivation for considering the cycle ξ is that this choice of cycle minimizes the Beilinson-Bloch height of the modified diagonal $\Gamma^3(C, e)$ [Zha10, Section 1.3].

One technical issue is that for a generic curve C of genus $g \geq 4$, the cycle ξ is not rationally equivalent to a point. One way to see this is by the generic non-vanishing of the Faber–Pandharipande cycle $(2g-2)^2 \cdot (\xi \times \xi - \Delta_{C,*}(\xi))$, which always vanishes when ξ is rationally equivalent to a point $x_0 \in C(k)$ (this cycle is introduced in [GG03]; for the generic non-vanishing see [Yin15]). However, much work on (higher) Beauville components and (higher) modified diagonal cycles has been restricted to the case where the embedding ι_e is taken with respect to a basepoint $e = x_0$. In particular, B. Moonen and Q. Yin [MY16] have studied analogues of Zhang's results to higher Beauville components under these restrictions: they prove that the vanishing of $[C]^{x_0}_{(s)}$ is equivalent to the vanishing of $\Gamma^{s+2}(C, x_0)$. Our first main theorem extends their work to an arbitrary choice of degree 1 divisor, e.

Theorem A. *For $s \geq 1$ and any choice of degree 1 divisor $e \in \mathrm{CH}(C)$, we have*

$$[C]^e_{(s)} = 0 \iff \Gamma^{s+2}(C, e) = 0.$$

This theorem is shown by studying the properties of the natural extension $\sigma_n \colon C^n \to J$ of the map ι_e. The backwards implication is given by generalizing the method of Zhang. For the forwards implication, we use a refinement $B^n(C, e) \in \mathrm{CH}_1(C^n)$ for the modified diagonal cycles $\Gamma^n(C, e)$, inspired by a motivic interpretation of these cycles in [MY16] (see Definition 3.1). We show that the vanishing of these cycles $B^n(C, e)$ is equivalent to the vanishing of $\Gamma^n(C, e)$, and we show the vanishing of $[C]^e_{(s)}$ implies the vanishing of the $B^n(C, e)$.

Zhang's theorem [Zha10, Theorem 1.5.5] also shows that the vanishing of $[C]^e_{(1)}$ must force the vanishing of $[C]^e_{(s)}$ for all $s > 0$. Restricting to the case where $e = x_0$, a more general statement can be deduced from the work of A. Polishchuk [Pol07], which shows that $[C]^{x_0}_{(s)} = 0$ implies $[C]^{x_0}_{(t)} = 0$ for all $t \geq s$. This has been shown separately for modified diagonals by K. O'Grady [O'G14], also restricting to the case where $e = x_0$. To extend Polishchuk and O'Grady's results to general e with our methods, we need either a stronger assumption, or get a weaker conclusion.

Theorem B. *For any $n \geq 3$, we have the following implication:*

$$\Gamma^n(C, e) = \Gamma^{n+1}(C, e) = 0 \implies \Gamma^k(C, e) = 0, \quad \textit{for all } k \geq n.$$

Equivalently, for any $s \geq 1$, if $[C]^e_{(s)} = [C]^e_{(s+1)} = 0$, then $[C]^e_{(t)} = 0$ for all $t \geq s$.

If $n \geq 3$ and $\Gamma^n(C, e) = 0$, then we have at least $\Gamma^k(C, e) = 0$ for all $k \geq 2n - 2$.

We make note of two interesting zero-cycles which naturally arise in the proofs of Theorem A and Theorem B. Firstly, we consider the cycle

$$\delta^e := \iota_{e,*}(e) - [0] \in \mathrm{CH}_0(J). \tag{1.2}$$

This cycle is trivial if $e = x_0$, a basepoint, but can be non-trivial for general e; it loosely measures the defect in how the behaviour of $[C]^e$ deviates from that of $[C]^{x_0}$. One can show that $\delta^e_{(0)} = \delta^e_{(1)} = 0$, but for $s \geq 2$, the appearance of s-th Beauville components $\delta^e_{(s)}$ in many formulas is an obstacle to generalizing Polishchuk and O'Grady's results about the implications of the vanishing of $[C]^e_{(s)}$ and $\Gamma^{s+2}(C, e)$. The second cycle of interest is denoted by $\gamma^s_e(e) \in \mathrm{CH}_0(C^s)$, $s \geq 2$ (see Definition 3.1); when $n = 2$, it is a multiple of the Faber–Pandharipande cycle. Relating these two cycles is key to our proof of Theorem A. A corollary to our results is that, for $s \geq 2$, $\delta^e_{(s)} = 0$ if and only if $\gamma^s_e(e) = 0$ (see Corollary 4.10). This is analogous to the relationship between $[C]^e_{(s)}$

and $\Gamma^{s+2}(C,e)$ in Theorem A. Analogously to Theorem B, we also obtain a successive vanishing result for $\gamma_e^s(e)$ and for $\delta^e_{(s)}$ (see Corollary 3.9).

We conclude with a closer investigation of our results in the cases of $s = 1, 2$, using the divisor $e = \xi = K_C/(2g-2)$. For $s = 2$, combining our results with those of Yin [Yin15], we show the following:

Theorem C. *For the generic curve C of genus $g \geq 4$, we have $[C]^{\xi}_{(2)} \neq 0$. On the other hand, for any curve C of genus $g \leq 3$, we have $[C]^{\xi}_{(2)} = 0$.*

Returning to Theorem A, in the case $s = 1$, we recover the statement of Zhang's theorem (see Section 5.1). Furthermore, we give an integral refinement of this result by establishing a relationship between the torsion order of the Ceresa cycle and that of the modified diagonal cycle, working with Chow groups with integral coefficients. To accomplish this, we combine our results with the decomposition of the integral Chow group $\mathrm{CH}(J;\mathbb{Z})$ constructed in [MP10a, Theorem 4], which is related to the Beauville decomposition.

Theorem D. *Let $\xi_{\mathrm{int}} \in \mathrm{CH}_0(C;\mathbb{Z})$ be an integral representative of $\xi = K_C/(2g-2)$, and let $d \in \mathbb{Z}$.*

(1) *If $\mathrm{Cer}(C,\xi_{\mathrm{int}}) \in \mathrm{CH}_1(J;\mathbb{Z})[d]$, then $\Gamma^3(C,\xi_{\mathrm{int}}) \in \mathrm{CH}_1(C^3;\mathbb{Z})[2\cdot d]$.*

(2) *If $\Gamma^3(C,\xi_{\mathrm{int}}) \in \mathrm{CH}_1(C^3;\mathbb{Z})[d]$, then $\mathrm{Cer}(C,\xi_{\mathrm{int}}) \in \mathrm{CH}_1(J;\mathbb{Z})[M_{g+1}\cdot d]$.*

Here, $M_{g+1} = \prod_{\text{prime } p \leq g+1} p^{\ell_p}$, with $\ell_p = \lfloor \frac{g}{p-1} \rfloor$ if $p \geq 3$, and $\ell_2 = g-1$.

Outline. The paper is structured as follows. In Section 2, we develop some basic intersection theory results. First, in order to allow a further generalisation to the case that e is an arbitrary degree 1 divisor with rational coefficients, rather than with integral coefficients; second, to easily calculate the terms arising in certain Fourier transforms. In Section 3, we recall the definition of the modified diagonal cycles $\Gamma^n(C,e)$ and $\gamma_e^n(e)$; inspired by [MY16], we also define a motivic variant, $B^n(C,e)$, which represents a component of $\Gamma^n(C,e)$ under a Künneth decomposition. In the same section, we also show the equivalence of the vanishing of $\Gamma^n(C,e)$ and $B^n(C,e)$, and establish the successive vanishing formulae. In Section 4, we show the equivalence of the vanishing of $[C]^e_{(s)}$ and $B^{s+2}(C,e)$, by studying the behaviour of these classes under Fourier transforms and natural addition maps $\sigma_n\colon C^n \to J$. In Section 5, we investigate an integral analogue of Zhang's result, and explore the behaviour of $[C]^e_{(s)}$ for general curves of genus at least 4.

Notation and Conventions. Unless explicitly stated otherwise, from now on all equivalences of algebraic cycles are in Chow rings $\mathrm{CH}(-)$ with $\mathbb{Q}$-coefficients, i.e., modulo rational equivalence up to torsion. The integral counterparts will be denoted by $\mathrm{CH}(-;\mathbb{Z})$.

We will work with an arbitrary degree 1 divisor $e \in \mathrm{CH}_0(C)$ in Sections 2 to 4, and specialize to the case $e = \xi = K_C/(2g-2)$ for some applications in Section 5.

Given two integers $i < j$, we will often write $[i,j]$ for the interval $\{i, i+1, \dots, j\}$. If $I \subseteq \{1,\dots,n\}$ is a subset, let $\mathrm{pr}_I\colon C^n \longrightarrow C^{|I|}$ be the projection onto the factors indexed by I. We will keep writing $\mathrm{pr}_i\colon C^n \longrightarrow C$ for the projection onto the i-th factor, and $\mathrm{pr}_{ij}\colon C^n \longrightarrow C^2$ for the projection onto the i-th and j-th factor. We will also write $\widehat{\mathrm{pr}}_I\colon C^n \longrightarrow C^{n-|I|}$ for the projection onto the factors indexed by elements not contained in I. Similarly, for any non-empty subset

$I \subseteq \{1,\dots,n\}$, we will denote by $m_I\colon J^n \longrightarrow J$ the sum of the components indexed by I. We will write m_{ij} if $I = \{i,j\}$, and $m_{[i,j]}$ if $I = \{i, i+1, \dots, j\}$.

For $n \geq 1$, we denote by $\Delta_C^{(n)}\colon C \to C^n$ the diagonal morphism $x \mapsto (x,\dots,x)$. For $n = 2$, we will write Δ_C instead of $\Delta_C^{(2)}$. We will also denote by $\Delta_C^{(n)}$ the small diagonal, namely the image of this diagonal morphism.

Acknowledgements. We are very grateful to Ben Moonen for suggesting this project at the Arizona Winter School 2024, for mentoring us during the workshop, and for providing us with valuable insights. We also wish to thank Jef Laga and Ziquan Yang for helpful conversations, and Ari Shnidman for raising the question that led to Theorem D. Finally, we extend our thanks to the organizers of the Arizona Winter School 2024, without whom we would not have had the opportunity to meet and work on this project.

JR was supported by the Warwick Mathematics Institute Centre for Doctoral Training during the preparation of this article, and gratefully acknowledges funding from the UK Engineering and Physical Sciences Research Council (Grant number: EP/W523793/1)

2. Cycles on Jacobians

2.1. **Embeddings of Curves.** Throughout, we denote by C a smooth projective connected curve of genus $g \geq 2$ over an algebraically closed field k. Denote by $J = \mathrm{Pic}^0_C$ the Jacobian of the curve C, and by J^t its dual abelian variety. We let e be a degree 1 divisor on C with rational coefficients. Given a positive integer n, we define the map

$$a_n : C^n \longrightarrow \mathrm{Pic}^n_C, \quad (x_1,\dots,x_n) \mapsto \mathcal{O}_C(x_1 + \cdots + x_n).$$

The map a_1 extends to an isomorphism $\mathrm{CH}_0(C;\mathbb{Z})^{\deg=0} \xrightarrow{\sim} J(k)$. We let $m\colon J \times J \to J$ be the summation map on J, and we define the *Pontryagin product* on $\mathrm{CH}(J)$ by $x \star y := m_*(x \times y)$.

Definition 2.1. Given an integral divisor γ of degree d on C, the *translation by* γ is defined as the isomorphism

$$t_\gamma : \mathrm{Pic}^n_C \longrightarrow \mathrm{Pic}^{n+d}_C, \quad \mathcal{O}_C(z) \mapsto \mathcal{O}_C(z+\gamma).$$

At the level of Chow groups, the pushforward map $t_{\gamma,*}\colon \mathrm{CH}(\mathrm{Pic}^n_C;\mathbb{Z}) \to \mathrm{CH}(\mathrm{Pic}^{n+d}_C;\mathbb{Z})$ coincides with the Pontryagin product $[\gamma] \star (-)$.

Since $J(k)\otimes\mathbb{Q} = \mathrm{CH}_0(C)_{\deg=0}$ is divisible (because $k = \overline{k}$), for any rational divisor $\gamma \in \mathrm{CH}_0(C)$, there exists an integral divisor $\gamma_{\mathrm{int}} \in \mathrm{CH}_0(C;\mathbb{Z})$ representing it. Moreover, the choice of an integral representative is unique up to torsion. We now show that this choice does not affect the map $t_{\gamma,*}$ in the Chow group with $\mathbb{Q}$-coefficients.

Proposition 2.2. *Let γ and γ' be two integral divisors on C of degree d. If $\tau = \gamma - \gamma'$ is torsion as an element of $\mathrm{CH}_0(C;\mathbb{Z})$, then for every $z \in \mathrm{CH}(\mathrm{Pic}^n_C)$:*

$$t_{\gamma,*}(z) = t_{\gamma',*}(z) \ \text{ in } \ \mathrm{CH}(\mathrm{Pic}^{n+d}_C).$$

Proof. Since τ is torsion, $n\tau = 0$ for some $n \in \mathbb{N}$. In particular, $[n]_*([\tau] - [0])$ is zero in $\mathrm{CH}_0(J)$. Given that $[n]_*$ is injective on $\mathrm{CH}(J)$, we deduce that $[\tau] = [0]$. We then get the result from the equality $t_{\gamma,*}(z) = t_{\gamma',*}(z) \star [\tau]$. □

Definition 2.3. For a rational zero-cycle γ of degree d, we define $t_{\gamma,*} : \mathrm{CH}(\mathrm{Pic}^n_C) \to \mathrm{CH}(\mathrm{Pic}^{n+d}_C)$ to be $t_{\gamma_{\mathrm{int}},*}$, for any integral representative γ_{int}. This is well-defined by the previous proposition. In particular, the pushforward map

$$\iota_{e,*} := t_{-e,*} \circ (a_1)_* \colon\ \mathrm{CH}(C) \longrightarrow \mathrm{CH}(J)$$

is well-defined. Similarly, we define the pull-back map ι_e^* as

$$\iota_e^* : \mathrm{CH}(J) \longrightarrow \mathrm{CH}(C), \qquad z \mapsto \mathrm{pr}_{C,*}(\mathrm{pr}_J^* z \cdot \Gamma_{e_{\mathrm{int}}}),$$

where e_{int} is any integral representative of e, and $\Gamma_{e_{\mathrm{int}}}$ is the graph of $\iota_{e_{\mathrm{int}}}$. Indeed, by the same principle as before, different choices of e_{int} produce graphs that differ only by torsion.

2.2. **Recollection of the Beauville decomposition and Fourier transform.** For further details on the results in this subsection, one may consult [Moo24].

Let $\mathscr{P}$ be the Poincaré bundle on $J \times J^t$. Consider the canonical principal polarization $\phi\colon J \xrightarrow{\sim} J^t$ of J. Let Θ be the theta divisor in Pic^{g-1}_C. The Mumford bundle is the line bundle on $J \times J$ defined as

$$\Lambda_\Theta := (\mathrm{id} \times \phi)^* \mathscr{P} = (\phi \times \mathrm{id})^* \mathscr{P}^t,$$

where $\mathscr{P}^t$ denotes the pull-back of the Poincaré bundle along the swap-map $\sigma : J^t \times J \to J \times J^t$. It can explicitly described as

$$\Lambda_\Theta = \mathcal{O}_{J\times J}(m^*\Theta_\gamma - \mathrm{pr}_1^*\Theta_\gamma - \mathrm{pr}_2^*\Theta_\gamma), \tag{2.1}$$

where Θ_γ is the translation of Θ by a divisor γ of degree $g-1$. In particular, the right hand side of (2.1) is independent of the choice of γ. We will frequently use the fact that for all $n \in \mathbb{Z}$,

$$([n] \times \mathrm{id})^* \Lambda_\Theta = (\mathrm{id} \times [n])^* \Lambda_\Theta = \Lambda_\Theta^{\otimes n},$$

which follows from the analogous statement on $\mathscr{P}$ (see [Moo24, Cor. 4.10]).

Now, recall that the Chow ring $\mathrm{CH}(J)$ can be decomposed as a direct sum

$$\mathrm{CH}(J) = \bigoplus_{i,s} \mathrm{CH}_{i,(s)}(J) = \bigoplus_{i,s} \mathrm{CH}^i_{(s)}(J),$$

called the Beauville decomposition. Here, $\mathrm{CH}_{i,(s)}(J) = \{c \in \mathrm{CH}_i(J) : [n]_* c = n^{2i+s} c, \forall n \in \mathbb{Z}\}$, and the groups $\mathrm{CH}_{i,(s)}(J) = \mathrm{CH}^{g-i}_{(s)}(J)$ vanish outside the range $-i \le s \le g-i$ (see [Bea83]). Given $z \in \mathrm{CH}_i(J)_{\mathbb{Q}}$, denote by $z_{(s)}$ its s-th Beauville component in $\mathrm{CH}_{i,(s)}(J)$.

Beauville conjectured that $\mathrm{CH}_{i,(s)}(J) = 0$ whenever $s < 0$, and proved this vanishing for $i \in \{0,1\}$ [Bea83, Proposition 3(a)]. In particular, we can write

$$[C]^e = [C]^e_{(0)} + \dots + [C]^e_{(g-1)},$$

and for any $z \in \mathrm{CH}_0(J)$, we have $z = z_{(0)} + \cdots + z_{(g)}$.

We recall the existence of two maps from the group $\mathrm{CH}_0(J)$: the *degree map* $\deg\colon \mathrm{CH}_0(J) \to \mathbb{Q}$, and the *Albanese map*

$$\mathrm{Alb}\colon\ \mathrm{CH}_0(J) \to J(k) \otimes \mathbb{Q}, \quad \sum n_i x_i \mapsto \sum n_i [x_i],$$

where the second summation uses the group law on J. Let I denote the kernel of deg. Then $I = \oplus_{s\ge1} \mathrm{CH}_{0,(s)}(J)$, and it forms an ideal with respect to the Pontryagin product. The kernel of Alb is then given by $I^{\star 2} = \oplus_{s\ge2} \mathrm{CH}_{0,(s)}(J)$. Note that in general, we have that $I^{\star n} = \oplus_{s\ge n} \mathrm{CH}_{0,(s)}(J)$.

The Chow group also admits an invertible Fourier transform and its dual

$$\mathscr{F} := \mathrm{ch}(\mathscr{P})_* : \mathrm{CH}(J) \xrightarrow{\sim} \mathrm{CH}(J^t), \quad z \mapsto \mathrm{pr}_{J^t,*}\left(\mathrm{ch}(\mathscr{P}) \cdot \mathrm{pr}_J^*(z)\right),$$
$$\mathscr{F}^t := \mathrm{ch}(\mathscr{P}^t)_* : \mathrm{CH}(J^t) \xrightarrow{\sim} \mathrm{CH}(J), \quad z \mapsto \mathrm{pr}_{J,*}\left(\mathrm{ch}(\mathscr{P}^t) \cdot \mathrm{pr}_{J^t}^*(z)\right).$$

Here, ch denotes the Chern character. These transforms satisfy $\mathscr{F}^t \circ \mathscr{F} = (-1)^g \cdot [-1]_*$. Moreover, $\mathscr{F}$ and $\mathscr{F}^t$ restrict to isomorphisms between $\mathrm{CH}_{i,(s)}(J) \xrightarrow{\sim} \mathrm{CH}_{g-i-s,(s)}(J^t)$.

2.3. Preliminary intersection theory results.

Notation 2.4. We denote by κ a theta characteristic of C, that is, a cycle in $\mathrm{CH}_0(C;\mathbb{Z})$ satisfying $2\kappa = K_C$. Its associated Theta divisor Θ_κ is symmetric. We also fix an integral representative $\varepsilon = e_{\mathrm{int}}$ for e.

We state the following two lemmas with integral coefficients rather than rational ones, as they will be needed for our integral results in Section 5.3.

Lemma 2.5. *Given $\gamma \in \mathrm{CH}_0(C;\mathbb{Z})$ of degree $g-1$, we have the following equality in $\mathrm{CH}_0(C;\mathbb{Z})$:*

$$\iota_\varepsilon^*(\Theta_\gamma) = K_C - \gamma + \varepsilon.$$

In particular, $\iota_\varepsilon^*(\Theta_\kappa) = \kappa + \varepsilon$.

Proof. By [EvdGM, Cor. 14.21], for a degree $g-2$ line bundle $\mathscr{L}$ on C, we have

$$(t_{\mathscr{L}} \circ a_1)^* \mathcal{O}_{\mathrm{Pic}_C^{g-1}}(\Theta) \cong \mathcal{O}_C(K_C) \otimes \mathscr{L}^{-1}.$$

Setting $\mathscr{L} = \mathcal{O}_C(\gamma - \varepsilon)$, we get

$$(t_{\gamma-\varepsilon} \circ a_1)^* \Theta = (t_{-\varepsilon} \circ a_1)^*(t_\gamma^* \Theta) = \iota_\varepsilon^*(\Theta_\gamma) = K_C - \gamma + \varepsilon. \qquad \square$$

The following identity is well known in the case that e is a k-point. We verify it in the general case.

Lemma 2.6. *The following equality holds in $\mathrm{CH}(C^2;\mathbb{Z})$:*

$$-(\iota_\varepsilon \times \iota_\varepsilon)^* c_1(\Lambda_\Theta) = \Delta_C - (\varepsilon \times C) - (C \times \varepsilon),$$

where $c_1(\Lambda_\Theta)$ is the first Chern class of Λ_Θ.

Proof. Let $x_0 \in C(k)$, and let $(\mathscr{P}_C, \alpha_C)$ be the universal line bundle of degree 0 on $C \times J$ rigidified at $\{x_0\} \times J$. We first show that

$$-(\iota_\varepsilon \times \mathrm{id}_J)^* c_1(\Lambda_\Theta) = c_1(\mathscr{P}_C) + \mathrm{pr}_J^*(-\Theta_{(g-1)\varepsilon} + \Theta_{g\varepsilon - x_0}). \tag{2.2}$$

To do this, consider the map

$$f : C \times \mathrm{Pic}_C^g \longrightarrow \mathrm{Pic}_C^{g-1}, \qquad (x, \mathscr{L}) \mapsto \mathscr{L}(-x).$$

By [EvdGM, Prop. 14.20], the line bundle $f^*(\mathcal{O}_{\mathrm{Pic}_C^{g-1}}(\Theta)) \otimes \mathrm{pr}_{\mathrm{Pic}_C^g}^* \mathcal{O}_{\mathrm{Pic}_C^g}(-t_{-x_0}^* \Theta)$, with a fixed trivialization (the identity) at $\{x_0\} \times \mathrm{Pic}_C^g$, is the universal line bundle of degree g on $C \times \mathrm{Pic}_C^g$. So by the See-Saw Principle, the pull-back

$$(\mathrm{id}_C \times t_{g\cdot\varepsilon})^*[f^*(\mathcal{O}_{\mathrm{Pic}_C^{g-1}}(\Theta)) \otimes \mathrm{pr}_{\mathrm{Pic}_C^g}^* \mathcal{O}_{\mathrm{Pic}_C^g}(-t_{-x_0}^* \Theta)]$$

is isomorphic to $\mathscr{P}_C \otimes \mathrm{pr}_C^* \mathcal{O}_C(g \cdot \varepsilon)$. Since f fits in the commutative diagram

$$\begin{array}{ccc} C \times J & \xrightarrow{m\circ([-1]\circ\iota_\varepsilon\times \mathrm{id}_J)} & J \\ \Big\downarrow{\scriptstyle \mathrm{id}_C\times t_{g\cdot\varepsilon}} & & \Big\downarrow{\scriptstyle t_{(g-1)\varepsilon}} \\ C\times \mathrm{Pic}^g_C & \xrightarrow{f} & \mathrm{Pic}^{g-1}_C, \end{array}$$

we can write

$$\begin{aligned} c_1(\mathscr{P}_C) + \mathrm{pr}_C^*(g\cdot\varepsilon) &= \big(m\circ([-1]\circ\iota_\varepsilon\times\mathrm{id}_J)\big)^* t^*_{(g-1)\varepsilon}\Theta - (\mathrm{id}_C\times t_{g\varepsilon})^*\,\mathrm{pr}^*_{\mathrm{Pic}^g_C}(t^*_{-x_0}\Theta) \\ &= ([-1]\circ\iota_\varepsilon\times\mathrm{id}_J)^* m^*\Theta_{(g-1)\varepsilon} - \mathrm{pr}_J^*(\Theta_{g\varepsilon-x_0}). \end{aligned}$$

By definition of Λ_Θ, we have that

$$m^*\Theta_{(g-1)\varepsilon} = \mathrm{pr}_1^*\Theta_{(g-1)\varepsilon} + \mathrm{pr}_2^*\Theta_{(g-1)\varepsilon} + c_1(\Lambda_\Theta).$$

So the term $([-1]\circ\iota_\varepsilon\times\mathrm{id}_J)^* m^*\Theta_{(g-1)\varepsilon}$ can be rewritten as

$$\iota_\varepsilon^*[-1]^*\Theta_{(g-1)\varepsilon} + \mathrm{pr}_J^*(\Theta_{(g-1)\varepsilon}) + (\iota_\varepsilon\times\mathrm{id}_J)^*([-1]\times\mathrm{id}_J)^* c_1(\Lambda_\Theta).$$

Using the equalities $([-1]\times\mathrm{id}_J)^*\Lambda_\Theta \cong \Lambda_\Theta^{-1}$ and $[-1]^*\Theta_{(g-1)\varepsilon} = \Theta_{K_C-(g-1)\varepsilon}$, we find that

$$-(\iota_\varepsilon\times\mathrm{id}_J)^* c_1(\Lambda_\Theta) = c_1(\mathscr{P}_C) + \mathrm{pr}_J^*(-\Theta_{(g-1)\varepsilon} + \Theta_{g\varepsilon-x_0}) + \mathrm{pr}_C^*(g\cdot\varepsilon - \iota_\varepsilon^*\Theta_{K_C-(g-1)\varepsilon}).$$

By Lemma 2.5, we have that $g\cdot\varepsilon - \iota_\varepsilon^*\Theta_{K_C-(g-1)\varepsilon} = 0$. Hence equation (2.2) follows.

Now in order to compute $-(\iota_\varepsilon\times\iota_\varepsilon)^* c_1(\Lambda_\Theta)$, we use the See-Saw Principle (see e.g. [Moo24, Section 2.1]). By universality of $\mathscr{P}_C$, the line bundle $\mathscr{P}_C|_{C\times\iota_\varepsilon(x)}$ corresponds to the k-point $\mathcal{O}_C(x-\varepsilon)$ for any $x\in C(k)$. Then by equation (2.2) and Lemma 2.5, the restriction to $C\times\{x\}$ of $-(\iota_\varepsilon\times\iota_\varepsilon)^* c_1(\Lambda_\Theta)$ is

$$-(\iota_\varepsilon\times\iota_\varepsilon)^* c_1(\Lambda_\Theta)|_{C\times x} = c_1(\mathscr{P}_C|_{C\times\iota_\varepsilon(x)}) + \mathrm{pr}_2^*(x_0-\varepsilon)|_{C\times\{x\}} = x-\varepsilon.$$

On the other hand

$$(\Delta_C - \varepsilon\times C - C\times\varepsilon)|_{C\times x} \cong x - \varepsilon\times x - 0 \cong x-\varepsilon.$$

Moreover, the restriction of $-(\iota_\varepsilon\times\iota_\varepsilon)^* c_1(\Lambda_\Theta)$ to $\{x_0\}\times C$ is

$$c_1(\mathscr{P}_C|_{\{x_0\}\times C}) + \mathrm{pr}_2^*(x_0-\varepsilon)|_{\{x_0\}\times C} = x_0-\varepsilon,$$

and the restriction of $(\Delta_C - \varepsilon\times C - C\times\varepsilon)|_{\{x_0\}\times C}$ is $x_0-\varepsilon$ as well. We conclude by another application of the See-Saw principle. ☐

Definition 2.7. We define

$$\delta^e := \iota_{e,*}(e) - [0] \in \mathrm{CH}_0(J).$$

In particular, if $e = x_0$ is a k-point, then $\delta^{x_0} = 0$.

Remark 2.8. The cycle δ^e is actually in the ideal $I^{\star 2}$. Indeed, write $e = \sum_i m_i x_i$ with $m_i\in\mathbb{Q}$ and $x_i\in C(k)$, then $\iota_{e,*}(e) = \sum_i m_i[\mathcal{O}_C(x_i-e)]$. In particular, $\iota_{e,*}(e)$ has degree 1, so $\iota_{e,*}(e)-[0] = \delta^e\in I$. Moreover,

$$\mathrm{Alb}(\delta^e) = \mathrm{Alb}(\sum_i m_i[\mathcal{O}_C(x_i-e)] - [0]) = \mathcal{O}_C(\sum_i m_i(x_i-e)) = \mathcal{O}_C(e-e) = 0.$$

Hence $\delta^e\in\ker(\mathrm{Alb}) = I^{\star 2}$, and so in particular $\delta^e_{(0)} = \delta^e_{(1)} = 0$.

3. Modified Diagonal Cycles

In this section, we recall the definition of the modified diagonal cycles $\Gamma^n(C,e)$ in the motivic formulation of Moonen–Yin [MY16]. These can be viewed as a special case (for $z=[C]$) of the more general cycles $\gamma_e^n(z)$. This perspective naturally leads to the introduction of another family of cycles, denoted $\beta_e^n(z)$, arising as distinguished components of $\gamma_e^n(z)$ under a motivic Künneth decomposition. In fact $\beta_e^n(z)$ only differs from $\gamma_e^n(z)$ when $z=[C]$, in which case we denote it by $B^n(C,e)$. This cycle will play a central role, as we will later show in Section 4 that it is directly related to the curve class $[C]^e_{(n-2)}$.

We establish in Theorem 3.7 the equivalence between the vanishing of $\Gamma^n(C,e)$ and of $B^n(C,e)$. We also prove successive vanishing results for the γ_e^n's and the β_e^n's in Corollary 3.9 and Corollary 3.10, which will serve as important ingredients in recovering Zhang's theorem together with its integral refinement in Section 5.

Throughout this section, it will be useful to keep in mind the statement of the adjunction formula in this context:

$$\Delta_C^*(\Delta_C) = -K_C \in \mathrm{CH}_0(C).$$

In particular this implies that $\Delta_C \cdot \Delta_C = -\Delta_{C,*}(K_C) \in \mathrm{CH}_0(C^2)$.

3.1. **Definitions of Modified Diagonal Classes.** We begin by recalling some aspects of the theory of motives that will be needed. We then give the motivic description of the modified diagonal cycles as introduced by Moonen–Yin [MY16].

For two smooth projective varieties X and Y over k, where X is connected, we denote

$$\mathrm{Corr}_i(X,Y) := \mathrm{CH}_{\dim(X)+i}(X\times Y).$$

The push-forward along such a correspondence $\rho \in \mathrm{Corr}(X,Y)$ is defined as

$$\rho_* : \mathrm{CH}(X) \longrightarrow \mathrm{CH}(Y), \quad \tau \mapsto \rho_*(\tau) := \mathrm{pr}_{Y,*}(\mathrm{pr}_X^*(\tau)\cdot\rho),$$

and the tensor product of $\rho,\tau \in \mathrm{Corr}(X,Y)$ is defined as

$$\rho\otimes\tau = \mathrm{pr}_{1,3}^*(\rho)\cdot \mathrm{pr}_{2,4}^*(\tau) \in \mathrm{Corr}(X\times Y, X\times Y).$$

We recall the definition of the category Mot_k of covariant Chow motives over k. The objects in this category are triples (X,p,m), where X is a smooth projective variety over k, $p\in \mathrm{Corr}_0(X,X)$ is an idempotent correspondence (also called a "projector"), and $m\in\mathbb{Z}$. The morphisms from (X,p,m) to (Y,q,n) are correspondences of the form $q\circ \mathrm{Corr}_{m-n}(X,Y)\circ p$. In particular, the identity morphism of (X,p,m) is $p\circ\Delta_X\circ p$. There is a covariant functor $\mathfrak{h}:\mathrm{SmProj}_k\to\mathrm{Mot}_k$ sending X to $(X,\Delta_X,0)$ and $f:X\to Y$ to the class of its graph $[\Gamma_f]$. Given a Chow motive M, we define its Chow group $\mathrm{CH}_i(M) := \mathrm{Hom}_{\mathrm{Mot}_k}\big(\mathbf{1}(i),M\big)$ for $i\geq 0$, where $\mathbf{1}(i) = (\mathrm{Spec}(k),\mathrm{id},i)$.

We do not make much use of the motivic structure. The most important fact is that projectors give rise to a direct sum decomposition of Chow groups, in particular, we have the following decomposition. For the curve C, we use the 0-cycle e of degree 1 to define *orthogonal projectors*

$$\pi_0 := C\times e,\quad \pi_2 := e\times C,\quad \pi_1 := \Delta_C - \pi_0 - \pi_2$$

in $\mathrm{CH}_1(C^2) = \mathrm{Corr}_0(C,C)$. These give a decomposition of $\mathfrak{h}(C)$

$$\mathfrak{h}(C) \cong \mathfrak{h}_0(C)\oplus\mathfrak{h}_1(C)\oplus\mathfrak{h}_2(C),$$

where $\mathfrak{h}_i(C) := (C, \pi_i, 0)$. In particular, $\mathrm{CH}(C) = \mathrm{CH}(\mathfrak{h}(C)) = \oplus_i \pi_{i,*} \mathrm{CH}(\mathfrak{h}(C))$. We also set $\pi_+ := \pi_1 + \pi_2 = \Delta_C - \pi_0$, and $\mathfrak{h}_+(C) := (C, \pi_+, 0) = \mathfrak{h}_1(C) \oplus \mathfrak{h}_2(C)$. Similarly, there is a motivic Künneth decomposition of $\mathfrak{h}(C^n)$ which induces a direct sum decomposition of $\mathrm{CH}(C^n)$: of particular interest for our purposes are the direct summands $\mathrm{CH}(\mathfrak{h}_{+\dots+}(C^n)) = (\pi_+^{\otimes n})_* \mathrm{CH}(C^n)$ and $\mathrm{CH}(\mathfrak{h}_{1\dots1}(C^n)) = (\pi_1^{\otimes n})_* \mathrm{CH}(C^n)$. We note that $\mathrm{CH}(\mathfrak{h}_{1\dots1}(C^n))$ is a direct summand of $\mathrm{CH}(\mathfrak{h}_{+\dots+}(C^n))$.

We briefly recall a convenient property of tensor products of motives that will be of use later. If we let $\rho \in \mathrm{CH}(C^n)$, $\tau \in \mathrm{CH}(C^m)$ and $\pi \in \mathrm{CH}(C^2)$, then by definition of the tensor of correspondences we have the equality

$$(\pi^{\otimes(n+m)})_*(\rho \times \tau) = (\pi^{\otimes n})_*(\rho) \times (\pi^{\otimes m})_*(\tau). \tag{3.1}$$

We can now give the motivic definition of the cycles we will be interested in.

Definition 3.1. Given $n \geq 1$, we define the homomorphisms $\beta_e^n, \gamma_e^n \colon \mathrm{CH}(C) \longrightarrow \mathrm{CH}(C^n)$ by

$$\beta_e^n := (\pi_1^{\otimes n})_* \circ \Delta_{C,*}^{(n)}, \quad \gamma_e^n := (\pi_+^{\otimes n})_* \circ \Delta_{C,*}^{(n)}.$$

In particular, we define the modified diagonal cycles

$$B^n(C,e) := \beta_e^n([C]), \qquad \Gamma^n(C,e) := \gamma_e^n([C]).$$

Example 3.2. For $n = 1$, we have

(1) $B^1(C,e) = 0$ and $\Gamma^1(C,e) = [C]$.
(2) $\beta_e^1(z) = \gamma_e^1(z) = z - \deg(z)e$, for any 0-cycle $z \in \mathrm{CH}_0(C)$.

For $n = 2$, we have

(1) $B^2(C,e) = \Gamma^2(C,e) = \Delta_C - (C \times e) - (e \times C)$.
(2) $\beta_e^2(e) = \gamma_e^2(e) = \Delta_{C,*}(e) - (e \times e)$. This cycle coincides, up to a scalar, with the Faber–Pandharipande cycle (see [GG03]). Since C is connected, it is algebraically trivial.

Unwinding the definition of γ_e^n and β_e^n, we obtain an explicit description of these operators.

Lemma 3.3. *We have the following expressions.*

(1) *If $z \in \mathrm{CH}_0(C)$ and $n \geq 1$, then*

$$\gamma_e^n(z) = \beta_e^n(z) = (-1)^n \deg(z) \cdot \prod_{j=1}^n \mathrm{pr}_j^*(e) + \sum_{k=1}^n (-1)^{n-k} \cdot \Big(\sum_{|I|=k} \mathrm{pr}_I^*(\Delta_{C,*}^{(k)}(z)) \cdot \prod_{j \notin I} \mathrm{pr}_j^* e \Big).$$

(2) *If $n \geq 1$, then*

$$\Gamma^n(C,e) = \sum_{k=1}^n (-1)^{n-k} \cdot \Big(\sum_{|I|=k} \mathrm{pr}_I^*(\Delta_C^{(k)}) \cdot \prod_{j \notin I} \mathrm{pr}_j^* e \Big) = \sum_{\varnothing \neq I \subseteq \{1,\dots n\}} (-1)^{n-|I|} \cdot \Delta_I^n(C,e),$$

where $\Delta_I^n(C,e)$ is as in (1.1).

(3) *If $n \geq 3$, then*

$$B^n(C,e) = \Gamma^n(C,e) - \sum_{i=1}^n \widehat{\mathrm{pr}}_i^*(\gamma_e^{n-1}(e)).$$

Proof. This is a direct computation. By definition, $\gamma_e^n(z)$ is equal to

$$\begin{aligned}&\mathrm{pr}_{[n+1,2n],*}\Big(\mathrm{pr}_{[1,n]}^*\,\Delta_{C,*}^{(n)}(z)\cdot\prod_{i=1}^n \mathrm{pr}_{i\,n+i}^*(\Delta_C - C\times e)\Big)\\ &= \sum_{I\subseteq\{1,\dots,n\}} (-1)^{n-|I|}\cdot \mathrm{pr}_{[n+1,2n],*}\Big(\mathrm{pr}_{[1,n]}^*\,\Delta_{C,*}^{(n)}(z)\cdot\prod_{i\in I}\mathrm{pr}_{i\,n+i}^*(\Delta_C)\cdot\prod_{i\notin I}\mathrm{pr}_{n+i}^*(e)\Big).\end{aligned}$$

Moreover, if z is a 0-cycle, then $\mathrm{pr}_{[1,n]}^*\,\Delta_{C,*}^{(n)}(z)\cdot \mathrm{pr}_{i\,n+i}(\pi_2) = 0$, so the projection along $\pi_+^{\otimes n}$ coincides with the one along $\pi_1^{\otimes n}$. This yields the first formulas.

If $z=[C]$, then the term corresponding to the case $I=\varnothing$ vanishes for dimension reasons, hence the second ones. Furthermore, by definition we have

$$B^n(C,e) = \Gamma^n(C,e) - \sum_{i=1}^n \mathrm{pr}_{[n+1,2n],*}\Big(\mathrm{pr}_{[1,n]}^*\,\Delta_C^{(n)}\cdot \mathrm{pr}_{i\,n+i}^*(\pi_2)\cdot\prod_{j\neq i}\mathrm{pr}_{n+j}^*(\pi_+)\Big). \tag{3.2}$$

Given the identity $\Delta_C^{(n)}\cdot \mathrm{pr}_i^*(e) = \Delta_{C,*}^{(n)}(e)$, the second term on the RHS in (3.2) is equal to $\sum_{i=1}^n \widehat{\mathrm{pr}}_i^*(\gamma_e^{n-1}(e))$. Hence the third formula. □

Remark 3.4. The first two formulas of Lemma 3.3, can be combined into one, for any cycle z:

$$\gamma_e^n(z) = (-1)^n\deg(z)\cdot\prod_{j=1}^n \mathrm{pr}_j^*(e) + \sum_{k=1}^n(-1)^{n-k}\cdot\Big(\sum_{|I|=k}\mathrm{pr}_I^*(\Delta_{C,*}^{(k)}(z))\cdot\prod_{j\notin I}\mathrm{pr}_j^*\,e\Big),$$

where for $\dim z=0$, the degree $\deg(z)$ is defined as in [Ful84, Definition 1.4].

3.2. **Vanishing Results.** We now study the consequences of the vanishing of $B^n(C,e)$.

Since $B^n(C,e)$ is the projection of $\Delta_C^{(n)}$ onto the component $\mathfrak{h}_{1\dots1}(C^n)$, which is contained in $\mathfrak{h}_{+\dots+}(C^n)$, we have $\pi_{1,*}^{\otimes n}(\Gamma^n(C,e)) = B^n(C,e)$. In particular, the vanishing of $\Gamma^n(C,e)$ implies the vanishing of $B^n(C,e)$. In this subsection, we prove that the converse is true as well. In order to do so, we first establish three consequences of the hypothesis $B^n(C,e)=0$ (Proposition 3.5 and Proposition 3.6), and then use these results to show that if $B^n(C,e)=0$ then $\Gamma^n(C,e)=0$ as well (see Theorem 3.7).

The essence of the following proofs is to use $B^n(C,e)$ as a correspondence between C and C^{n-1}, and as a correspondence between C^2 and C^{n-2}. These calculations could be done using the explicit formulae in Lemma 3.3, but this becomes cumbersome. Instead, we use the projectors to simplify the book-keeping at the cost of the argument being slightly less transparent.

Proposition 3.5. *Let $n\geq 3$, and assume that $B^n(C,e)=0$, then*

$$\gamma_e^{n-2}(K_C+2e)=0, \qquad \textit{and} \qquad \gamma_e^{n-1}(K_C+4e)=0.$$

Proof. In this proof we will consider projections from C^{2n}, C^n and C^{2n-2}. To avoid confusion, we reserve $\mathrm{pr}_I : C^{2n}\longrightarrow C^{|I|}$ for the I-projection from C^{2n}, and denote by $\overline{\mathrm{pr}}_I : C^n\longrightarrow C^{|I|}$ the one from C^n and by $\widetilde{\mathrm{pr}}_I : C^{2n-2}\longrightarrow C^{|I|}$ the one from C^{2n-2}.

We establish the vanishing of both cycles simultaneously. Let us remark that one could also prove the first vanishing directly, viewing $B^n(C,e)$ as a correspondence between C^2 and C^{n-2}.

Indeed, unwinding the definitions, we have the following equality

$$B^n(C,e)_*(\Delta_C) = \overline{\mathrm{pr}}_{[3,n],*}\big(B^n(C,e)\cdot \overline{\mathrm{pr}}^*_{12}\Delta_C\big)$$

and non-trivial computations (similar to what will do in this proof) show that $B^n(C,e)_*(\Delta_C) = -\gamma_e^{n-2}(K_C+2e)$. Instead we first refine this calculation by computing

$$\overline{\mathrm{pr}}_{[2,n],*}\big(B^n(C,e)\cdot \overline{\mathrm{pr}}^*_{12}\Delta_C\big) = \overline{\mathrm{pr}}_{[2,n],*}\left(\mathrm{pr}_{[n+1,2n],*}\big(\mathrm{pr}^*_{[1,n]}\,\Delta_C^{(n)}\cdot \pi_1^{\otimes n}\big)\cdot \overline{\mathrm{pr}}^*_{1,2}\Delta_C\right). \tag{3.3}$$

We then deduce the formula for $B^n(C,e)_*(\Delta_C)$ by projecting (3.3) onto the last $(n-2)$ factors.

Let us write $\mathrm{pr}^*_{[1,n]}\,\Delta_C^{(n)}$ as $\mathrm{pr}^*_{1,2}\,\Delta_C\cdot \mathrm{pr}^*_{[2,n]}\,\Delta_C^{(n-1)}$. Applying projection formula with respect to $\mathrm{pr}_{[n+1,2n]}$, we find that (3.3) is equal to

$$\mathrm{pr}_{[n+2,2n],*}\left(\mathrm{pr}^*_{1,2}\,\Delta_C\cdot \mathrm{pr}^*_{[2,n]}\,\Delta_C^{(n-1)}\cdot \pi_1^{\otimes n}\cdot \mathrm{pr}^*_{n+1,n+2}(\Delta_C)\right).$$

We replace $\pi_1^{\otimes n}$ by its expression as

$$\mathrm{pr}^*_{\{1,2,n+1,n+2\}}(\pi_1^{\otimes 2})\cdot \mathrm{pr}^*_{[3,n]\sqcup[n+3,2n]}(\pi_1^{\otimes(n-2)}).$$

Using the explicit formula for π_1 together with the adjunction formula, a straightforward computation gives

$$\pi_1^{\otimes 2}\cdot(\Delta_C\times\Delta_C) = (\Delta_C\times\Delta_C)_*\big(\Delta_{C,*}(-K_C-4e)+2e\times e\big).$$

Moreover, using proper-flat base change, we have the equality

$$\mathrm{pr}^*_{\{1,2,n+1,n+2\}}(\Delta_C\times\Delta_C)_* = (\Delta_C\times \mathrm{id}_{C^{n-2}}\times\Delta_C\times\mathrm{id}_{C^{n-2}})_*\widetilde{\mathrm{pr}}^*_{1n}.$$

Putting everything together, and then applying the projection formula with respect to the map $(\Delta_C\times \mathrm{id}_{C^{n-2}}\times\Delta_C\times\mathrm{id}_{C^{n-2}})\colon C^{2n-2}\to C^{2n}$, the cycle (3.3) becomes

$$\begin{aligned}&\widetilde{\mathrm{pr}}_{[n,2n-2],*}\left(\widetilde{\mathrm{pr}}^*_{1n}\big(\Delta_{C,*}(-K_C-4e)+2e\times e\big)\cdot\widetilde{\mathrm{pr}}^*_{[1,n+1]}\Delta_C^{(n+1)}\cdot\widetilde{\mathrm{pr}}^*_{[2,n-1]\sqcup[n+1,2n-2]}\pi_1^{\otimes(n-2)}\right)\\ &\quad=\widetilde{\mathrm{pr}}_{[n,2n-2],*}\left(\widetilde{\mathrm{pr}}^*_{[1,n]}\big(\Delta^{(n)}_{C,*}(-K_C-4e)+\Delta^{(n-1)}_{C,*}(2e)\times e\big)\cdot\widetilde{\mathrm{pr}}^*_{[2,n-1]\sqcup[n+1,2n-2]}\pi_1^{\otimes(n-2)}\right).\end{aligned}$$

On the other hand, by the definition of β_e^{n-1}, the cycle $\beta_e^{n-1}(-K_C-4e)$ is equal to

$$\begin{aligned}&\widetilde{\mathrm{pr}}_{[n,2n-2],*}\left(\widetilde{\mathrm{pr}}^*_{[1,n-1]}\Delta^{(n-1)}_{C,*}(-K_C-4e)\cdot\widetilde{\mathrm{pr}}^*_{1n}(\pi_1)\cdot\widetilde{\mathrm{pr}}^*_{[2,n-1]\sqcup[n+1,2n-2]}(\pi_1^{\otimes(n-2)})\right)\\ &=\widetilde{\mathrm{pr}}_{[n,2n-2],*}\left(\widetilde{\mathrm{pr}}^*_{[1,n]}\big(\Delta^{(n)}_{C,*}(-K_C-4e)+\Delta^{(n-1)}_{C,*}(K_C+4e)\times e\big)\cdot\widetilde{\mathrm{pr}}^*_{[2,n-1]\sqcup[n+1,2n-2]}(\pi_1^{\otimes(n-2)})\right).\end{aligned}$$

Therefore, the difference $\beta_e^{n-1}(-K_C-4e)-\overline{\mathrm{pr}}_{[2,n],*}(B^n(C,e)\cdot\overline{\mathrm{pr}}^*_{12}\Delta_C)$ is equal to

$$\widetilde{\mathrm{pr}}_{[n,2n-2],*}\left(\widetilde{\mathrm{pr}}^*_{[1,n-1]}\Delta^{(n-1)}_{C,*}(K_C+2e)\cdot\widetilde{\mathrm{pr}}^*_{n}e\cdot\widetilde{\mathrm{pr}}^*_{[2,n-1]\sqcup[n+1,2n-2]}\pi_1^{\otimes(n-2)}\right)=e\times\beta_e^{n-2}(K_C+2e).$$

In other words, we found that (3.3) is equal to $\beta_e^{n-1}(-K_C-4e)-(e\times\beta_e^{n-2}(K_C+2e))$.

By the explicit expression of β_e^{n-1} given in Lemma 3.3, it is straightforward to see that the projection onto the last $(n-2)$ factors of $\beta_e^{n-1}(-K_C-4e)$ is trivial. Therefore, we find that

$$B^n(C,e)_*(\Delta_C) = -\beta_e^{n-2}(K_C+2e).$$

Combining this with the expression of (3.3), we deduce that the vanishing of $B^n(C,e)$ implies the desired vanishings. We also recall that by (1) of Lemma 3.3, β_e and γ_e agree on zero-cycles. □

The second proposition is of a slightly different flavour.

Proposition 3.6. *Let $n \geq 3$, and assume that $B^n(C,e) = 0$, then*

$$\gamma_e^{n-1}(z) = \deg(z) \cdot \gamma_e^{n-1}(e)$$

for any $z \in \mathrm{CH}_0(C)$.

Proof. As in the previous proposition we denote by pr_I the projections from C^{2n}, by $\overline{\mathrm{pr}}_I$ the projections from C^n, and by $\widetilde{\mathrm{pr}}_I$ the projections from C^{2n-2}.

We view $B^n(C,e)$ as a correspondence from C to C^{n-1}, and compute $B^n(C,e)_*(z)$. As before, this can be written as $\mathrm{pr}_{[n+2,2n],*}\left(\pi_1^{\otimes n} \cdot \mathrm{pr}_{n+1}^* z \cdot \mathrm{pr}_{[1,n]}^* \Delta_C^{(n)}\right)$. Using that $\pi_1 = \Delta_C - (e \times C) - (C \times e)$, we obtain

$$B^n(C,e)_*(z) = \mathrm{pr}_{[n+2,2n],*}\left(\pi_1^{\otimes(n-1)} \cdot \mathrm{pr}_{1\,n+1}^*\left(\Delta_C(z) - e \times z\right) \cdot \mathrm{pr}_{[1,n]}^* \Delta_C^{(n)}\right).$$

We then can write $\mathrm{pr}_{1\,n+1}^*\left(\Delta_C(z) - e \times z\right) \cdot \mathrm{pr}_{[1,n]}^* \Delta_C^{(n)}$ as

$$\mathrm{pr}_{1\,n+1}^*(\Delta_C) \cdot \mathrm{pr}_{[1,n]}^* \Delta_{C,*}^{(n)}(z) - \mathrm{pr}_{n+1}^*(z) \cdot \mathrm{pr}_{[1,n]}^* \Delta_{C,*}^{(n)}(e)$$

and obtain that $B^n(C,e)_*(z)$ is equal to

$$\mathrm{pr}_{[n+2,2n],*}\left(\pi_1^{\otimes(n-1)} \cdot \mathrm{pr}_{1\,n+1}^*(\Delta_C) \cdot \mathrm{pr}_{[1,n]}^* \Delta_{C,*}^{(n)}(z)\right) - \mathrm{pr}_{[n+2,2n],*}\left(\pi_1^{\otimes(n-1)} \cdot \mathrm{pr}_{n+1}^*(z) \cdot \mathrm{pr}_{[1,n]}^* \Delta_{C,*}^{(n)}(e)\right).$$

After factoring the diagonal $\Delta_{C,*}^{(n)}(z)$ as $\overline{\mathrm{pr}}_{1\,2}^*\Delta_C \cdot \overline{\mathrm{pr}}_{[2,n]}^*\Delta_{C,*}^{(n-1)}(z)$, and expressing $\mathrm{pr}_{[n+2,2n]}$ as $\widetilde{\mathrm{pr}}_{[n,2n-2]} \circ \mathrm{pr}_{[2,n]\sqcup[n+2,2n]}$, we apply projection formula with respect to $\mathrm{pr}_{[2,n]\sqcup[n+2,2n]}$ and get that $B^n(C,e)_*(z)$ is equal to

$$\widetilde{\mathrm{pr}}_{[n,2n-2],*}\left(\widetilde{\mathrm{pr}}_{[1,n-1]}^*\Delta_{C,*}^{(n-1)}(z) \cdot \pi_1^{\otimes(n-1)}\right) - \widetilde{\mathrm{pr}}_{[n+2,2n],*}\left(\deg(z) \cdot \widetilde{\mathrm{pr}}_{[2,n]}^*\Delta_{C,*}^{(n-1)}(e) \cdot \pi_1^{\otimes(n-1)}\right)$$
$$= \beta_e^{n-1}(z) - \deg(z) \cdot \beta_e^{n-1}(e) = \gamma_e^{n-1}(z) - \deg(z) \cdot \gamma_e^{n-1}(e).$$

Therefore if $B^n(C,e)$ vanishes, we get the desired equality. □

Now we are in a position to prove the main result of this section.

Theorem 3.7. *If $n \geq 2$, $\Gamma^n(C,e) = 0$ if and only if $B^n(C,e) = 0$.*

Proof. The forward direction is clear by the remarks at the beginning of this subsection. If $B^n(C,e) = 0$ and $n \geq 3$ then by Proposition 3.5 and Proposition 3.6 we have

$$\gamma_e^{n-1}(K_C + 4e) = \deg(K_C + 4e) \cdot \gamma_e^{n-1}(e) = (2g+2) \cdot \gamma_e^{n-1}(e) = 0 \implies \gamma_e^{n-1}(e) = 0.$$

However, the difference between $B^n(C,e)$ and $\Gamma^n(C,e)$ is precisely $\sum_{i=1}^n \widehat{\mathrm{pr}}_i^{\,*}\gamma_e^{n-1}(e)$. If $n = 2$, then $B^2(C,e)$ is equal to $\Gamma^2(C,e)$ as noted in Example 3.2. □

3.3. **Successive Vanishing.** By [MY16, Prop. 3.4], if $e = x_0$ is a k-point and $z \in \mathrm{CH}(C)$ any cycle, then the vanishing of $\gamma_{x_0}^n(z)$ forces the vanishing of $\gamma_{x_0}^{n+1}(z)$. However the method in *loc. cit.* does not extend to the case of a general divisor e.

In this section we prove a similar result under a slightly stronger assumption: for any divisor e of degree 1, the vanishing of $\gamma_e^n(z)$ and $\gamma_e^{n+1}(z)$ implies the vanishing of all the following $\gamma_e^k(z)$ for $k \geq n+2$. To do this, we establish a relationship between $\gamma_e^{n+m}(z)$ and $\gamma_e^{n-1}(z)$, under the assumption $\gamma_e^n(z) = 0$.

The main idea is to replace $\Delta_C^{(n+m)}(z)$ with $\mathrm{pr}_{[1,n]}^* \Delta_C^{(n)}(z) \cdot \mathrm{pr}_{[n,n+m]}^* \Delta_C^{(m+1)}$, and then use the vanishing of $\gamma_e^n(z)$ to rewrite the first term. This could be done using the formulae in Lemma 3.3, but as in the previous section, this is notationally complicated, and the proof is simplified using projectors.

Theorem 3.8. *Let $z \in \mathrm{CH}(C)$ be such that $\gamma_e^n(z) = 0$, and let $n \geq 2$. Then for all $m \in \mathbb{N}$,*

$$\gamma_e^{n+m}(z) = \gamma_e^{n-1}(z) \times \gamma_e^{m+1}(e).$$

Similarly, if $\beta_e^n(z) = 0$, then

$$\beta_e^{n+m}(z) = \beta_e^{n-1}(z) \times \beta_e^{m+1}(e) + \beta_e^{n-1}(z \cdot e) \times \beta_e^{m+1}(C).$$

Proof. By definition, $\gamma_e^{n+m}(z) = \pi_{+,*}^{\otimes(n+m)} \Delta_{C,*}^{(n+m)}(z)$ which can be rewritten as

$$\pi_{+,*}^{\otimes(n+m)}\Big(\mathrm{pr}_{[1,n]}^* \Delta_{C,*}^{(n)}(z) \cdot \mathrm{pr}_{[n,n+m]}^* \Delta_C^{(m+1)}\Big). \tag{3.4}$$

We will first rewrite $\Delta_{C,*}^{(n)}(z)$ as a sum of explicit terms. Then we will plug in this "new expression" of $\Delta_{C,*}^{(n)}(z)$ in equation (3.4) and see that only one term contributes in a non-trivial way. We then finish by interpreting that term as $\gamma_e^{n-1}(z) \times \gamma_e^{m+1}(e)$.

Since $\gamma_e^n(z) = 0$, by definition of $\gamma_e^n(z)$ we have

$$\Delta_{C,*}^{(n)}(z) = \Delta_{C,*}^{(n)}(z) - \gamma_e^n(z) = (\Delta_C^{\otimes n} - \pi_+^{\otimes n})_*\big(\Delta_{C,*}^{(n)}(z)\big).$$

Writing π_+ as $\Delta_C - (C \times e)$, we can expand the difference $\Delta_C^{\otimes n} - \pi_+^{\otimes n}$ as

$$\sum_{I \subsetneq \{1,\ldots,n\}} (-1)^{n-1-|I|} \cdot \prod_{i \in I} \mathrm{pr}_{i\,n+i}^*(\Delta_C) \cdot \prod_{i \notin I} \mathrm{pr}_{i\,n+i}^*(C \times e).$$

Therefore, the pushforward $(\Delta_C^{\otimes n} - \pi_+^{\otimes n})_*(\Delta_{C,*}^{(n)}(z))$ coincides with the sum over $I \subsetneq \{1,\ldots,n\}$ of the terms

$$(-1)^{n-1-|I|} \cdot \Big(\prod_{i \in I} \mathrm{pr}_{i\,n+i}^*(\Delta_C) \cdot \prod_{i \notin I} \mathrm{pr}_{i\,n+i}^*(C \times e)\Big)_* \big(\Delta_{C,*}^{(n)}(z)\big). \tag{3.5}$$

Carrying on the computations, one can see that equation (3.5) is equal to

$$\begin{cases} (-1)^{n-1} \deg(z) \cdot \prod_{i=1}^n \overline{\mathrm{pr}}_i^*(e) & \text{if } I = \varnothing, \\ (-1)^{n-1-|I|} \cdot \overline{\mathrm{pr}}_I^*(\Delta_{C,*}^{(|I|)})(z) \cdot \prod_{i \notin I} \overline{\mathrm{pr}}_i^*(e) & \text{if } I \neq \varnothing, \end{cases} \tag{3.6}$$

where $\overline{\mathrm{pr}}\colon C^n \to C$ is the projection onto the i-th component. Since $I \subsetneq \{1,\ldots,n\}$ is a proper subset, this expression always has the shape $\widehat{\overline{\mathrm{pr}}}_j^*(z_j') \cdot \overline{\mathrm{pr}}_j^*(e)$ for some $j \notin I$ and cycle $z_j' \in \mathrm{CH}(C^{n-1})$. We now note that for any $j \neq n$, the pushforward $\pi_{+,*}^{\otimes(n+m)}(\widehat{\mathrm{pr}}_j^*(z_j') \cdot \mathrm{pr}_j^*(e) \cdot \mathrm{pr}_{[n,n+m]}^* \Delta_C^{(m+1)})$ is zero, as $\pi_{+,*}(e) = 0$. Therefore the only non-trivial contribution comes from $I = \{1,\ldots,n-1\}$. Hence, the original equation (3.4) is equal to $\pi_{+,*}^{\otimes(n+m)}(\Delta_{C,*}^{(n-1)}(z) \times \Delta_{C,*}^{(m+1)}(e))$, which is exactly $\gamma_e^{n-1}(z) \times \gamma_e^{m+1}(e)$.

The argument for β_e is analogous. We use $\pi_1 = \pi_+ - \pi_2$ and the assumption $\beta_e^n(z) = 0$ to rewrite

$$\Delta_{C,*}^{(n)}(z) = \big(\Delta_C^{\otimes n} - (\pi_+ - \pi_2)^{\otimes n}\big)_* \Delta_{C,*}^{(n)}(z).$$

For dimension reasons, $\pi_{2,*}^{\otimes j}(\Delta_{C,*}^{(j)}(z')) = 0$ for any $z' \in \mathrm{CH}(C)$ and $j \geq 2$. Hence the only non-zero terms in this expression come from pushforward by the cycle

$$\Delta_C^{\otimes n} - \Big(\pi_+^{\otimes n} - \sum_{i=1}^{n} \widehat{\mathrm{pr}}_{i\,n+i}^*(\pi_+^{\otimes(n-1)}) \cdot \mathrm{pr}_{i\,n+i}^*(\pi_2)\Big).$$

Therefore the cycle $\Delta_{C,*}^{(n)}(z)$ is

$$(\Delta_C^{\otimes n} - \pi_+^{\otimes n})_* \Delta_{C,*}^{(n)}(z) + \sum_{i=1}^{n} \mathrm{pr}_{[n+1,2n],*}\left(\mathrm{pr}_{[1,n]}^*\left(\Delta_{C,*}^{(n)}(z\cdot e)\right) \cdot \widehat{\mathrm{pr}}_{i\,n+i}^*(\pi_+^{\otimes(n-1)}) \cdot \mathrm{pr}_{n+i}(C)\right).$$

As done before, we then plug in this new expression of $\Delta_{C,*}^{(n)}(z)$ in the equation computing $\beta_e^{n+m}(z)$ (that is, (3.4) with π_+ replaced by π_1). The first summand contributes in the same way as $\Delta_{C,*}^{(n-1)}(z)\times e$; the only non-trivial contribution from the second sum is from the term $i = n$, because $\pi_{1,*}(C) = 0$. Therefore this second piece contributes in the same way as $\Delta_{C,*}^{(n-1)}(z\cdot e) \times C$. $\square$

As an immediate application, we get the first statement of Theorem B.

Corollary 3.9. *Let $n \geq 1$. If $z \in \mathrm{CH}(C)$ is such that both $\gamma_e^n(z)$ and $\gamma_e^{n+1}(z)$ vanish, then $\gamma_e^k(z) = 0$ for all $k \geq n$. In particular for $z = [C]$, we get*

$$\Gamma_e^n(C,e) = \Gamma_e^{n+1}(C,e) = 0 \implies \Gamma^k(C,e) = 0 \quad \forall k \geq n.$$

Corollary 3.10. *Let $n \geq 2$. If $\gamma_e^n(e) = 0$, then $\gamma_e^{k(n-1)+1}(e) = 0$ for all $k \geq 0$.*

Proof. We proceed by induction. The case $k = 0$ is always true, and the case $k = 1$ is the hypothesis. Let us then assume that $\gamma_e^{k(n-1)+1}(e) = 0$ for $k \geq 1$. By Theorem 3.8, we have

$$\gamma_e^{k(n-1)+1+m}(e) = \gamma_e^{k(n-1)}(e) \times \gamma_e^{m+1}, \quad \forall\, m \in \mathbb{N}.$$

Setting $m = n-1$ we get $\gamma_e^{(k+1)(n-1)+1}(e) = 0$. $\square$

Theorem 3.11. *Assume $\Gamma^n(C,e) = 0$ for some $n \geq 3$. Then for any $z \in \mathrm{CH}_0(C)$, we have $\gamma_e^k(z) = 0$ for all $k \geq n-1$.*

Proof. The case when $k = n-1$ is given by Proposition 3.5 and Proposition 3.6. By Corollary 3.9, it is sufficient to prove this for $k = n$.

Let us denote by pr_I the I-projection from C^n. As before, we write $\gamma_e^n(z) = \pi_{+,*}^{\otimes n}(\Delta_{C,*}^{(n)}(z))$. We now rewrite $\Delta_{C,*}^{(n)}(z)$ as $\mathrm{pr}_1^* z \cdot \Delta_C^{(n)}$. Since $\Gamma^n(C,e) = 0$, we have $\pi_{+,*}^{\otimes n}\Delta_C^{(n)} = 0$, and so $\Delta_C^{(n)} = (\Delta_C^{\otimes n} - \pi_+^{\otimes n})_* \Delta_C^{(n)}$. As seen in the proof of Theorem 3.8, equation (3.6) tells us that every term in this pushforward can be expressed as a product of the form $\widehat{\mathrm{pr}}_j(z_j') \cdot \mathrm{pr}_j^* e$ for some cycle $z_j' \in \mathrm{CH}(C^{n-1})$ and some index $1 \leq j \leq n$. Intersecting this with $\mathrm{pr}_1^* z$ does not change this fact. Indeed either a term contains $\mathrm{pr}_j^*(e)$ for $j \neq 1$; or it contains $\mathrm{pr}_1^*(e)$, and the intersection then vanishes for dimension reasons. By (3.1), the remaining terms will vanish under $\pi_{+,*}^{\otimes n}$ as $\pi_{+,*}(e) = 0$. $\square$

We can leverage this to get a version of Corollary 3.10, for $\Gamma^n(C,e)$. This gives the second statement of Theorem B.

Corollary 3.12. *Let $n \geq 3$, and suppose $\Gamma^n(C,e) = 0$. Then for all $k \geq 2n-2$, $\Gamma^k(C,e) = 0$.*

Proof. Let $k = n+m$, where $m \geq n-2$. By Theorem 3.8, $\Gamma^k(C,e) = \Gamma^{n-1}(C,e) \times \gamma_e^{m+1}(e)$. By the previous result, $\gamma_e^{m+1}(e) = 0$. □

Remark 3.13. The relations written above are not the only possible recurrence relations. For example, we can write $\Delta_C^{(n+2)}$ as $\mathrm{pr}_{12}^* \Delta_C \cdot \mathrm{pr}_{[2,n+1]}^* \Delta_C^{(n)} \cdot \mathrm{pr}_{n+1\,n+2}^* \Delta_C$. If for a 0-cycle z we have $\gamma_e^n(z) = 0$, then with similar steps as for Theorem 3.8 we would get

$$\gamma_e^{n+2}(z) = -\gamma_e^2(e) \times \gamma_e^{n-2}(z) \times \gamma_e^2(e) \quad \forall n \geq 3.$$

Using similar arguments, we could get additional consequences of the vanishing of $\Gamma^n(C,e)$ (or equivalently, $B^n(C,e)$).

4. From Jacobians to Products of Curves and Back Again

The aim of this section is to prove Theorem A. We recall that this theorem is proven in [MY16] in the case of a base point, but their method does not generalize to arbitrary e due to the fact that we do not necessarily have the vanishing of $\gamma_e^2(e)$ (or equivalently $\delta_{(2)}^e$, see Corollary 4.10).

In the course of the proof, we will see that the cycle $B^{s+2}(C,e)$ naturally appears in terms of the Beauville component $[C]_{(s)}$. This could be anticipated by the fact that the motive $\mathfrak{h}_1(C)$ corresponds to the Jacobian (see [Sch94, §3.2]), and so the Künneth decomposition for motives suggests that cycles on C^n related to the Jacobian should occur in $\mathfrak{h}_{1\ldots1}(C^n)$. This is one reason for introducing this cycle.

As a consequence, we show that the vanishing of $[C]_{(s)}$ implies the vanishing of $B^{s+2}(C,e)$ (see Corollary 4.7), which in turn is the same of $\Gamma^{s+2}(C,e)$ thanks to Theorem 3.7. We then show that the vanishing of $\Gamma^{s+2}(C,e)$ implies the vanishing of $[C]_{(s)}$ in Theorem 4.9, which then yields Theorem A.

Notation 4.1. In this section, we label the coordinates in $C \times C^n$ by $\{0,1,\ldots,n\}$ and the projections from it by $\overline{\mathrm{pr}}$. We abuse the notation and use the same one for projections from C^n (so forgetting first the 0-th coordinate). As in the previous section we reserve the symbol pr for projections from C^{2n}.

4.1. **From the Beauville Decomposition to Diagonal Cycles.**

In this subsection we show that for any $z \in \mathrm{CH}_i(C)$ and $s \geq 1$, the vanishing of $(\iota_{e,*}z)_{(s)}$ in $\mathrm{CH}_{i,(s)}(J)$ implies $\beta_e^{s+2i}(z) = 0$ in $\mathrm{CH}(C^{s+2i})$. To achieve this, we follow and extend the proof method of [Zha10, Theorem 5.3.1]. Specifically, our goal will be to compute $(\phi \circ \sigma_{s+2i})^* \mathscr{F}(\iota_{e,*}z)_{(s)}$, where σ_n denotes the composite

$$\sigma_n \colon C^n \xrightarrow{\iota_e^n} J^n \xrightarrow{m_{[1,n]}} J.$$

Let us recall that σ_n can also be described as

$$C^n \xrightarrow{a_n} \mathrm{Pic}_C^n \xrightarrow{t_{-ne}} \mathrm{Pic}_C^0 = J,$$

where a_n and t_{-ne} are as in Definition 2.1, and the discussion immediately preceding it.

Proposition 4.2. *Let $z \in \mathrm{CH}(C)$. We have the equality*

$$(\phi \circ \sigma_n)^* \mathscr{F}(\iota_{e,*}(z)) = \overline{\mathrm{pr}}_{[1,n],*}\left(\overline{\mathrm{pr}}_0^* z \cdot \mathrm{ch}\left((\iota_e \times \sigma_n)^* c_1(\Lambda_\Theta)\right)\right) \in \mathrm{CH}(C^n).$$

Proof. We have the following Cartesian diagram:

$$\begin{array}{ccccccc}
C & \xleftarrow{\mathrm{pr}_C} & C\times J^t & \xleftarrow{\mathrm{id}_C\times\phi} & C\times J & \xleftarrow{\mathrm{id}_C\times\sigma_n} & C\times C^n \\
\downarrow{\scriptstyle \iota_e} & & \downarrow{\scriptstyle \iota_e\times\mathrm{id}_{J^t}} & & \downarrow{\scriptstyle \iota_e\times\mathrm{id}_J} & & \downarrow{\scriptstyle \iota_e\times\mathrm{id}_{C^n}} \\
J & \xleftarrow{\mathrm{pr}_J} & J\times J^t & \xleftarrow{\mathrm{id}_J\times\phi} & J\times J & \xleftarrow{\mathrm{id}_J\times\sigma_n} & J\times C^n \\
 & & \downarrow{\scriptstyle \mathrm{pr}_{J^t}} & & \downarrow & & \downarrow{\scriptstyle \mathrm{pr}_{C^n}} \\
 & & J^t & \xleftarrow{\phi} & J & \xleftarrow{\sigma_n} & C^n.
\end{array}$$

We start with the maps at the bottom left of the diagram. Since all the squares are Cartesian, proper-flat base change allows us to work our way to the top right:

$$\begin{aligned}
\sigma_n^*\phi^*\left(\mathrm{pr}_{J^t,*}\left(\mathrm{pr}_J^*(\iota_{e,*}(z))\cdot\mathrm{ch}(\mathscr{P})\right)\right) &= \mathrm{pr}_{C^n,*}\left((\mathrm{id}_J\times\phi\circ\sigma_n)^*\left(\mathrm{pr}_J^*(\iota_{e,*}(z))\cdot\mathrm{ch}(\mathscr{P})\right)\right)\\
&= \mathrm{pr}_{C^n,*}\left((\iota_e\times\mathrm{id}_{C^n})_*(z\times C^n)\cdot(\mathrm{id}_J\times\sigma_n)^*\,\mathrm{ch}(\Lambda_\Theta)\right)\\
&= \overline{\mathrm{pr}}_{[1,n],*}\left(\overline{\mathrm{pr}}_0^* z\cdot\mathrm{ch}\left((\iota_e\times\sigma_n)^*\Lambda_\Theta\right)\right).
\end{aligned}$$

In the last two steps, we first use that $\Lambda_\Theta = (\mathrm{id}_J\times\phi)^*\mathscr{P}$, and then apply the projection formula with respect to $\iota_e\times\mathrm{id}_{C^n}$. This yields the desired equality. □

Now we would like to rewrite $(\iota_e\times\sigma_n)^*\Lambda_\Theta$ in other terms.

Lemma 4.3. *We have the following equality in* $\mathrm{CH}(J\times C^n)$,

$$(\mathrm{id}_J\times\sigma_n)^*c_1(\Lambda_\Theta) = \sum_{j=1}^n(\mathrm{id}_J\times\overline{\mathrm{pr}}_j)^*(\mathrm{id}_J\times\iota_e)^*c_1(\Lambda_\Theta),$$

and therefore the following one in $\mathrm{CH}(C\times C^n)$,

$$(\iota_e\times\sigma_n)^*c_1(\Lambda_\Theta) = -\sum_{j=1}^n\overline{\mathrm{pr}}_{0j}^*\pi_1.$$

Proof. Let $D = m_{[0,n]}^*\Theta_\kappa - m_0^*\Theta_\kappa - m_{[1,n]}^*\Theta_\kappa$, then we have the following equality in $\mathrm{CH}(J\times C^n)$

$$\begin{aligned}
(\mathrm{id}_J\times\iota_e^n)^*D &= (\mathrm{id}_J\times\iota_e^n)^*m_{[0,n]}^*\Theta_\kappa - \mathrm{pr}_J^*\,\Theta_\kappa - \mathrm{pr}_{C^n}^*\,\sigma_n^*\Theta_\kappa\\
&= (\mathrm{id}_J\times\sigma_n)^*m^*\Theta_\kappa - \mathrm{pr}_J^*\,\Theta_\kappa - \mathrm{pr}_{C^n}^*\,\sigma_n^*\Theta_\kappa\\
&= (\mathrm{id}_J\times\sigma_n)^*(m^* - \mathrm{pr}_{J,1}^* - \mathrm{pr}_{J,2}^*)\Theta_\kappa = (\mathrm{id}_J\times\sigma_n)^*c_1(\Lambda_\Theta).
\end{aligned}$$

Note that $m_{[0,k]}$ can be written as the sum of m_0, $m_{[1,k-1]}$ and m_k. Then the Theorem of the Cube for $(m_0, m_{[1,k-1]}, m_k)$ states that

$$(m_{[0,k]}^* - m_{[1,k]}^*)\Theta_\kappa = (m_{[0,k-1]}^* - m_{[1,k-1]}^*)\Theta_\kappa + (m_{0k}^* - m_0^* - m_k^*)\Theta_\kappa.$$

By induction on k, we can write

$$(m_{[0,k]}^* - m_{[1,k]}^*)\Theta_\kappa = (m_{01}^* - m_1^*)\Theta_\kappa + \Big(\sum_{j=2}^k m_{0j}^* - (k-1)m_0^* - \sum_{j=2}^k m_j^*\Big)\Theta_\kappa.$$

Since $D + m_0^*\Theta_\kappa = (m^*_{[0,n]} - m^*_{[1,n]})\Theta_\kappa$ we get

$$D = \Big(\sum_{j=1}^n m^*_{0\,i} - n\,\overline{\mathrm{pr}}^*_{J,0} - \sum_{j=1}^n \overline{\mathrm{pr}}^*_{J,j}\Big)\Theta_\kappa = \sum_{j=1}^n \overline{\mathrm{pr}}^*_{J,0j}(c_1(\Lambda_\Theta)),$$

where $\overline{\mathrm{pr}}_{J,j}$ is the projection on the j-th component $J \times J^n \longrightarrow J$ for $0 \le j \le n$. Therefore, the pullback $(\mathrm{id}_J \times \iota_e^n)^*D$ can be written as $\sum_{j=1}^n (\mathrm{id}_J \times \overline{\mathrm{pr}}_j)^*(\mathrm{id}_J \times \iota_e)^* c_1(\Lambda_\Theta)$. Hence the first equality. The second one follows by taking the pullback of the first one by $(\iota_e \times \mathrm{id}_{C^n})$ and using the identity $(\iota_e \times \iota_e)^* c_1(\Lambda_\Theta) = -\pi_1$ of Lemma 2.6. □

We can now compute the value of $(\phi \circ \sigma_{s+2i})^* \mathscr{F}(\iota_{e,*}(z)_{(s)})$. We record the following proposition to simplify the intersection calculations.

Proposition 4.4. *Let $z \in \mathrm{CH}(C)$. We have the following equality in $\mathrm{CH}(C^n)$:*

$$\overline{\mathrm{pr}}_{[1,n],*}\Big(\overline{\mathrm{pr}}_0^*\, z \cdot \prod_{j=1}^n \overline{\mathrm{pr}}^*_{0j}\, \pi_1\Big) = \beta_e^n(z).$$

Proof. We first note that $\overline{\mathrm{pr}}_0^* z$ is equal to $\mathrm{pr}_{[n,2n],*}\,\mathrm{pr}^*_{[1,n]}\,\Delta^{(n)}_{C,*}(z)$ in $\mathrm{CH}(C \times C^n)$. Applying the projection formula with respect to $\mathrm{pr}_{[n,2n]}$, the cycle

$$\overline{\mathrm{pr}}_{[1,n],*}\Big(\mathrm{pr}_{[n,2n],*}\,\mathrm{pr}^*_{[1,n]}\,\Delta^{(n)}_{C,*}(z) \cdot \prod_{j=1}^n \overline{\mathrm{pr}}^*_{0j}\pi_1\Big)$$

can be rewritten as

$$\mathrm{pr}_{[n+1,2n],*}\Big(\mathrm{pr}^*_{[1,n]}\,\Delta^{(n)}_{C,*}(z) \cdot \prod_{j=1}^n \mathrm{pr}^*_{n\,n+j}\,\pi_1\Big).$$

Since $\mathrm{pr}^*_{[1,n]}\,\Delta^{(n)}_C \cdot \mathrm{pr}^*_{n\,n+j}\,\pi_1 = \mathrm{pr}^*_{[1,n]}\,\Delta^{(n)}_C \cdot \mathrm{pr}^*_{j\,n+j}\,\pi_1$, the previous expression is equal to

$$\mathrm{pr}_{[n+1,2n],*}\Big(\mathrm{pr}^*_{[1,n]}\,\Delta^{(n)}_{C,*}(z) \cdot \prod_{j=1}^n \mathrm{pr}^*_{j\,n+j}\,\pi_1\Big)$$

which is precisely $\beta_e^n(z)$. □

Theorem 4.5. *Let $z \in \mathrm{CH}_0(C)$ and $n \ge 1$. Then*

$$(\phi \circ \sigma_n)^* \mathscr{F}(\iota_{e,*}(z)_{(n)}) = (-1)^n \gamma_e^n(z).$$

Proof. By Proposition 4.2 and Lemma 4.3, we have the equality

$$(\phi \circ \sigma_n)^* \mathscr{F}(\iota_{e,*}(z)) = \overline{\mathrm{pr}}_{[1,n],*}\Big(\overline{\mathrm{pr}}_0^* z \cdot \mathrm{ch}\,\big(-\sum_{j=1}^n \overline{\mathrm{pr}}^*_{0j}\pi_1\big)\Big),$$

where as before ch denotes the Chern character. The n-th Beauville component of $\iota_{e,*}(z)$ becomes the codimension n component under the Fourier transform. Thus we just have to take the codimension n component of the right hand side, which comes from the codimension n term of $\mathrm{ch}(-\sum_{j=1}^n \overline{\mathrm{pr}}^*_{0\,j}\pi_1)$. Write $\mathrm{ch}(-\sum_{j=1}^n \overline{\mathrm{pr}}^*_{0j}\pi_1)$ as $(-1)^n \prod_{j=1}^n \mathrm{ch}(\overline{\mathrm{pr}}^*_{0j}\pi_1)$. As π_1 has codimension one in C^2, we have $\pi_1^3 = 0$. Similarly, we have $\overline{\mathrm{pr}}_0^* z \cdot \overline{\mathrm{pr}}^*_{0i}\pi_1^2 = 0$. Therefore, we can restrict to the terms of the Chern character with no quadratic or higher terms in any $\overline{\mathrm{pr}}^*_{0i}\pi_1$, which is precisely the term $(-1)^n \prod_{j=1}^n \overline{\mathrm{pr}}^*_{0j}\pi_1$.

It remains to compute $\overline{\mathrm{pr}}_{[1,n],*}\big(\overline{\mathrm{pr}}_0^* z \cdot \prod_{j=1}^n \overline{\mathrm{pr}}_{0\,j}^*\pi_1\big)$, which is the content of Proposition 4.4. By Lemma 3.3, we know that $\beta_e^n = \gamma_e^n$ for zero-cycles, hence the result. □

We now fix $n = s+2$ in Notation 4.1.

Theorem 4.6. *Set $z = [C]$ and $s \geq 0$. The cycle $(\phi \circ \sigma_{s+2})^* \mathscr{F}([C]^e_{(s)})$ is equal to*

$$(-1)^s\Big(B^{s+2}(C,e) - \frac{(s+1)}{2}\sum_{j=1}^{s+2}\widehat{\overline{\mathrm{pr}}}_j^*\gamma_e^{s+1}(K_C+4e) - \frac{1}{2}\sum_{j=1}^{s+2}\sum_{k\neq j}\overline{\mathrm{pr}}_k^*(e)\cdot\widehat{\overline{\mathrm{pr}}}_{jk}^*\gamma_e^s(K_C+2e)\Big). \tag{4.1}$$

Proof. By Proposition 4.2, we have the equality

$$(\phi\circ\sigma_{s+2})^*\mathscr{F}([C]^e) = \overline{\mathrm{pr}}_{[1,s+2],*}\Big(\mathrm{ch}\,\big(-\sum_{i=1}^{s+2}\overline{\mathrm{pr}}_{0i}^*\pi_1\big)\Big).$$

The s-th Beauville component of $[C]^e$ becomes the codimension $(s+1)$ component under the Fourier transform. Hence, we only need to compute the codimension $(s+1)$ component of the right hand side, which comes from the codimension $(s+2)$ component of $\mathrm{ch}(-\sum_{i=1}^{s+2}\overline{\mathrm{pr}}_{0i}^*\pi_1)$. As before, since π_1 has codimension one in C^2, we have $\pi_1^3 = 0$. Unlike in the previous case, we do get contributions from $\overline{\mathrm{pr}}_{0i}(\pi_1^2)$; however we can use the fact that $\overline{\mathrm{pr}}_{0i}(\pi_1^2)\cdot\overline{\mathrm{pr}}_{0j}(\pi_1^2) = 0$ for dimension reasons. Therefore, we can restrict to the terms of the Chern character with at most one quadratic term in any $\overline{\mathrm{pr}}_{0j}^*\pi_1$. These are given by the expression

$$(-1)^{s+2}\prod_{j=1}^{s+2}\overline{\mathrm{pr}}_{0j}^*\pi_1 + \frac{(-1)^s}{2}\sum_{j=1}^{s+2}\sum_{k\neq j}\big(\overline{\mathrm{pr}}_{0k}^*\pi_1^2\prod_{l\neq j,k}\overline{\mathrm{pr}}_{0l}^*\pi_1\big). \tag{4.2}$$

The first term contributes $(-1)^{s+2}\beta_e^{s+2}(C) = (-1)^{s+2}B^{s+2}(C,e)$ by Proposition 4.4. Hence it only remains to compute the double sum.

By the adjunction formula $\Delta_C^2 = -\Delta_{C,*}(K_C)$ and the explicit expression of π_1, we find that

$$\pi_1^2 = -\Delta_{C,*}(K_C+4e) + 2\cdot(e\times e).$$

Writing $\Delta_{C,*}(K_C+4e)$ as $\Delta_C\cdot((K_C+4e)\times C)$, and in turn Δ_C as $\pi_1 + (C\times e) + (e\times C)$, we have that

$$\pi_1^2 = -\pi_1\cdot\big((K_C+4e)\times C\big) - (K_C+2e)\times e.$$

Using this expression, we find that the double sum in (4.2) is equal to

$$\frac{(-1)^{s+1}}{2}\sum_{j=1}^{s+2}\sum_{k\neq j}\Bigg[\overline{\mathrm{pr}}_{[1,s+2],*}\Big(\overline{\mathrm{pr}}_0^*(K_C+4e)\cdot\prod_{l\neq j}\overline{\mathrm{pr}}_{0l}^*\pi_1\Big) + \overline{\mathrm{pr}}_{[1,s+2],*}\Big(\overline{\mathrm{pr}}_0^*(K_C+2e)\cdot\overline{\mathrm{pr}}_k^*e\cdot\prod_{l\neq j,k}\overline{\mathrm{pr}}_{0l}^*\pi_1\Big)\Bigg].$$

Consider the Cartesian square

$$\begin{array}{ccc} C\times C^{s+2} & \xrightarrow{\widehat{\overline{\mathrm{pr}}}_j} & C\times C^{s+1} \\ {\scriptstyle \overline{\mathrm{pr}}_{[1,s+2]}}\downarrow & & \downarrow{\scriptstyle \overline{\mathrm{pr}}_{[1,s+1]}} \\ C^{s+2} & \xrightarrow{\widehat{\overline{\mathrm{pr}}}_j} & C^{s+1}. \end{array}$$

By proper-flat base change, we have that

$$\sum_{k\neq j}\overline{\mathrm{pr}}_{[1,s+2],*}\Big(\overline{\mathrm{pr}}_0^*(K_C+4e)\cdot\prod_{l\neq j}\overline{\mathrm{pr}}_{0l}^*\pi_1\Big) = (s+1)\widehat{\overline{\mathrm{pr}}}_j^*\,\overline{\mathrm{pr}}_{[1,s+1],*}\Big(\overline{\mathrm{pr}}_0^*(K_C+4e)\cdot\prod_{l\neq j}\overline{\mathrm{pr}}_{0l}^*\pi_1\Big)$$
$$= (s+1)\widehat{\overline{\mathrm{pr}}}_j^*\beta_e^{s+1}(K_C+4e).$$

where the last equality uses Proposition 4.4. Similarly, we have

$$\overline{\mathrm{pr}}_{[1,s+2],*}\Big(\overline{\mathrm{pr}}_0^*(K_C+2e)\cdot\overline{\mathrm{pr}}_k^*e\prod_{l\neq j,k}\overline{\mathrm{pr}}_{0l}^*\pi_1\Big) = \overline{\mathrm{pr}}_k^*e\cdot\widehat{\overline{\mathrm{pr}}}_{jk}^*\beta_e^s(K_C+2e).$$

Combining these three terms, we have the statement of the theorem. □

Corollary 4.7. *For $s\geq 0$, the projection of $(\phi\circ\sigma_{s+2})^*\mathscr{F}([C]^e_{(s)})$ under $\pi_1^{\otimes(s+2)}$ is $B^{s+2}(C,e)$. In particular if $[C]^e_{(s)}=0$, then $B^{s+2}(C,e)=\Gamma^{s+2}(C,e)=0$.*

Proof. Let us consider equation (4.1). As $B^{s+2}(C,e)$ is already in the subspace $\mathfrak{h}_{1\ldots1}(C^{s+2})$, it is fixed by $\pi_1^{\otimes s+2}$. Note that all the other terms in (4.1) are of the from $C\times z'$ for some cycle $z'\in\mathrm{CH}_0(C^{s+1})$. Since $\pi_{1,*}(C)=0$, by (3.1) all of the other terms vanish. Therefore,

$$\pi_1^{\otimes(s+2)}\big((\phi\circ\sigma_{s+2})^*\mathscr{F}([C]^e_{(s)})\big) = B^{s+2}(C,e).$$

The rest of the statement follows from Theorem 3.7. □

Corollary 4.8. *We have that $\Gamma^n(C,e)=0$ for all $n>g+1$.*

4.2. **From Diagonal Classes to Beauville Components.** We now show the converse of the results of the previous section, namely that the vanishing of diagonal classes imply the vanishing of some corresponding Beauville components. More precisely, we have the following theorem.

Theorem 4.9. *Let $z\in\mathrm{CH}_i(C)$ and $s\geq 1$. We have*

$$\iota_{e,*}(z)_{(s-2i)} = \text{ codimension } (s-i) \text{ part of } \mathscr{F}^t\left((\phi\circ\sigma_s)_*\gamma_e^s(z)\right).$$

In particular, we have the implication

$$\gamma_e^s(z)=0 \quad\Longrightarrow\quad \iota_{e,*}(z)_{(s-2i)}=0.$$

Proof. We fix the value $n=s$ in Notation 4.1. Using the Cartesian diagram

$$\begin{array}{ccccccc} C^s\times J & \xrightarrow{\sigma_s\times\mathrm{id}} & J\times J & \xrightarrow{\phi\times\mathrm{id}} & J^t\times J & \xrightarrow{\mathrm{pr}_J} & J \\ \downarrow{\scriptstyle \mathrm{pr}_{C^s}} & & \downarrow{\scriptstyle \mathrm{pr}_{J,1}} & & \downarrow{\scriptstyle \mathrm{pr}_{J^t}} & & \\ C^s & \xrightarrow{\sigma_s} & J & \xrightarrow{\phi} & J^t, & & \end{array}$$

by proper-flat base change, we have that

$$\mathscr{F}^t\big((\phi\circ\sigma_s)_*\gamma_e^s(z)\big) = \mathrm{pr}_{J,*}\left(\mathrm{ch}(\mathscr{P}^t)\cdot\mathrm{pr}_{J^t}^*(\phi\circ\sigma_s)_*(\gamma_e^s(z))\right)$$
$$= \mathrm{pr}_{J,*}\left(\mathrm{ch}(\mathscr{P}^t)\cdot(\phi\times\mathrm{id})_*(\sigma_s\times\mathrm{id})_*\,\mathrm{pr}_{C^s}^*(\gamma_e^s(z))\right).$$

By the projection formula with respect to $(\phi \times \mathrm{id}) \circ (\sigma_s \times \mathrm{id})$ and the fact that $\mathrm{pr}_J \circ (\phi \times \mathrm{id}) \circ (\sigma_s \times \mathrm{id})) = \mathrm{pr}_J$, we get that

$$\begin{aligned}\mathscr{F}^t\big((\phi \circ \sigma_s)_*\gamma_e^s(z)\big) &= \mathrm{pr}_{J,*}\left(\mathrm{ch}((\sigma_s \times \mathrm{id})^*(\phi \times \mathrm{id})^*\mathscr{P}^t) \cdot \mathrm{pr}_{C^s}^*(\gamma_e^s(z))\right)\\ &= \mathrm{pr}_{J,*}\left(\mathrm{ch}((\sigma_s \times \mathrm{id})^*\Lambda_\Theta) \cdot \mathrm{pr}_{C^s}^*(\gamma_e^s(z))\right),\end{aligned}$$

where the last equality uses the definition of Λ_Θ. By Lemma 4.3, we have

$$(\sigma_s \times \mathrm{id})^*\Lambda_\Theta = \sum_{j=1}^{s} \mathscr{L}_j, \text{ where } \mathscr{L}_j := (\overline{\mathrm{pr}}_j \times \mathrm{id})^*(\mathscr{L}) \text{ and } \mathscr{L} := (\iota_e \times \mathrm{id})^*\Lambda_\Theta.$$

Unwinding the definition of $\gamma_e^s(z)$, we get that

$$\mathscr{F}^t\left((\phi \circ \sigma_s)_*\gamma_e^s(z)\right) = \mathrm{pr}_{J,*}\Big(\prod_{j=1}^{s}\mathrm{ch}(\mathscr{L}_j) \cdot \mathrm{pr}_{C^s}^*\,\mathrm{pr}_{[s+1,2s],*}\left(\mathrm{pr}_{[1,s]}^*\,\Delta_C^{(s)}(z) \cdot \prod_{j=1}^{s}\mathrm{pr}_{j\,j+s}^*(\pi_+)\right)\Big).$$

By the Cartesian square

$$\begin{array}{ccc} C^{2s} \times J & \xrightarrow{\mathrm{pr}_{[s+1,2s]} \times \mathrm{id}_J} & C^s \times J \\ \downarrow{\scriptstyle \mathrm{pr}_{C^{2s}}} & & \downarrow{\scriptstyle \mathrm{pr}_{C^s}} \\ C^{2s} & \xrightarrow{\mathrm{pr}_{[s+1,2s]}} & C^s, \end{array}$$

and proper-flat base change, we have

$$\mathscr{F}^t\big((\phi\circ\sigma_s)_*\gamma_e^s(z)\big) = \mathrm{pr}_{J,*}\Big(\prod_{j=1}^{s}\mathrm{ch}(\mathscr{L}_j)\cdot(\mathrm{pr}_{[s+1,2s]} \times \mathrm{id}_J)_*\,\mathrm{pr}_{C^{2s}}^*\left(\mathrm{pr}_{[1,s]}^*\,\Delta_C^{(s)}(z)\cdot\prod_{j=1}^{s}\mathrm{pr}_{j\,j+s}^*(\pi_+)\right)\Big).$$

By the projection formula with respect to $\mathrm{pr}_{[s+1,2s]} \times \mathrm{id}_J$, we get

$$\mathscr{F}^t\big((\phi \circ \sigma_s)_*\gamma_e^s(z)\big) = \mathrm{pr}_{J,*}\Big(\prod_{j=1}^{s}\mathrm{ch}\left((\mathrm{pr}_{s+j} \times \mathrm{id})^*\mathscr{L}\right)\cdot\mathrm{pr}_{[1,s]}^*\,\Delta_{C,*}^{(s)}(z)\cdot\prod_{j=1}^{s}\mathrm{pr}_{j\,j+s}^*(\pi_+)\Big). \tag{4.3}$$

We simplify the expression (4.3) further. If in the product $\prod_{j=1}^{s}\mathrm{ch}((\mathrm{pr}_{s+j} \times \mathrm{id}_J)^*\mathscr{L})$ there is a term with no contribution from $c_1((\mathrm{pr}_{s+k} \times \mathrm{id}_J)^*\mathscr{L})$ for some $1 \le k \le s$, then it can be written as

$$\mathrm{pr}_{J,*}\left((\widehat{\mathrm{pr}}_{k+s} \times \mathrm{id}_J)^*(z') \cdot \mathrm{pr}_{k\,k+s}^*(\pi_+)\right),$$

where $z' \in \mathrm{CH}(C^{2s-1} \times J)$. Therefore writing pr_J as $\mathrm{pr}_J \circ (\widehat{\mathrm{pr}}_{k+s} \times \mathrm{id}_J)$, we can use projection formula with respect to $\widehat{\mathrm{pr}}_{k+s} \times \mathrm{id}_J$ and get

$$\mathrm{pr}_{J,*}\left(z' \cdot \left(\widehat{\mathrm{pr}}_{k+s,*}\,\mathrm{pr}_{k\,k+s}^*(\pi_+)\right) \times J\right) = \mathrm{pr}_{J,*}\left(z' \cdot \left(\widetilde{\mathrm{pr}}_k^*\,\mathrm{pr}_{1,*}(\pi_+)\right) \times J\right),$$

where $\widetilde{\mathrm{pr}}_k$ is the k-th projection $C^{2s-1} \to C$. Since $\mathrm{pr}_{1,*}(\pi_+) = C - \deg(e)C = 0$, this term gives no contribution. Therefore, $\mathscr{F}^t\left((\phi \circ \sigma_s)_*\gamma_e^s(z)\right)$ is equal to

$$\mathrm{pr}_{J,*}\Big(\prod_{j=1}^{s}(\mathrm{pr}_{s+j} \times \mathrm{id}_J)^*c_1(\mathscr{L}) \cdot \mathrm{pr}_{[1,s]}^*\,\Delta_C^{(s)}(z)\prod_{j=1}^{s}\mathrm{pr}_{j\,j+s}^*(\pi_+)\Big). \tag{4.4}$$

All the terms in (4.4) can be written as a pullback $(\widehat{\mathrm{pr}}_{1+s} \times \mathrm{id}_J)(z')$ with $z' \in \mathrm{CH}(C^{2s-1} \times J)$ except for $(\mathrm{pr}_{s+1} \times \mathrm{id}_J)^*c_1(\mathscr{L}) \cdot \mathrm{pr}_{1\,1+s}^*(\pi_+)$: writing pr_J as $\mathrm{pr}_J \circ (\widehat{\mathrm{pr}}_{1+s} \times \mathrm{id}_J)$, we can use projection formula with respect to $\widehat{\mathrm{pr}}_{1+s} \times \mathrm{id}_J$ in (4.4) and get

$$\begin{aligned}\mathrm{pr}_{J,*}&\Big(z' \cdot (\widehat{\mathrm{pr}}_{s+1} \times \mathrm{id}_J)_*\big((\mathrm{pr}_{s+1} \times \mathrm{id}_J)^* c_1(\mathscr{L}) \cdot \mathrm{pr}^*_{1\,s+1}(\pi_+)\big)\Big)\\ &= \mathrm{pr}_{J,*}\Big(z' \cdot (\widehat{\mathrm{pr}}_{s+1} \times \mathrm{id}_J)_*\big((\mathrm{pr}_{1\,s+1} \times \mathrm{id}_J)^*(C \times c_1(\mathscr{L}) \cdot \pi_+ \times J)\big)\Big)\\ &= \mathrm{pr}_{J,*}\Big(z' \cdot (\widetilde{\mathrm{pr}}_1 \times \mathrm{id}_J)^*(\mathrm{pr}_1 \times \mathrm{id}_J)_*(C \times c_1(\mathscr{L}) \cdot \pi_+ \times J)\Big),\end{aligned}$$

where the last equality comes from proper-flat base change. Iterating, we get that (4.4) is equal to

$$\mathrm{pr}_{J,*}\Big(\Delta^{(s)}_{C,*}(z) \times J \cdot \prod_{j=1}^{s} (\overline{\mathrm{pr}}_j \times \mathrm{id}_J)^*(\mathrm{pr}_{C,1} \times \mathrm{id}_J)_*(C \times c_1(\mathscr{L}) \cdot \pi_+ \times J)\Big),$$

where here $\mathrm{pr}_{C,1}$ is the projection on the first component $C \times C \to C$. Writing $\pi_+ = \Delta_C - (C \times e)$, we see that

$$\begin{aligned}&(\mathrm{pr}_{C,1} \times \mathrm{id})_*(C \times c_1(\mathscr{L}) \cdot \pi_+ \times J)\\ &= (\mathrm{pr}_{C,1} \times \mathrm{id})_*\big(C \times c_1(\mathscr{L}) \cdot (\Delta_C \times \mathrm{id}_J)_*(C \times J)\big) - C \times \mathrm{pr}_{J,*}(c_1(\mathscr{L}) \cdot e \times J)\\ &= (\Delta_C \times \mathrm{id}_J)^*(C \times c_1(\mathscr{L})) - C \times \mathrm{pr}_{J,*}(c_1(\mathscr{L}) \cdot e \times J),\end{aligned}$$

where the last equality is given by projection formula with respect to $\Delta_C \times \mathrm{id}_J$. The first term is equal to $c_1(\mathscr{L})$ and the term $\mathrm{pr}_{J,*}(c_1(\mathscr{L}) \cdot e \times J)$ is the codimension one part of $\mathscr{F}^t((\phi \circ \iota_e)_* e)$. By Fourier duality, $\mathscr{F}^t$ induces an isomorphism $\mathrm{CH}_{0,(s)}(J^t) \xrightarrow{\sim} \mathrm{CH}^s_{(s)}(J)$. Therefore, we find that $\mathrm{pr}_{J,*}(c_1(\mathcal{L}) \cdot e \times J) = \mathscr{F}^t((\phi \circ \iota_e)_* e)_{(1)})$. However, we have that $((\phi \circ \iota_e)_* e)_{(1)} = \phi((\iota_{e,*} e)_{(1)}) = 0$ since $(\iota_{e,*} e)_{(1)} = \delta^e_{(1)} = 0$ by Remark 2.8. Therefore the second term vanishes, and we get that (4.4) is equal to

$$\mathrm{pr}_{J,*}\Big(\Delta^{(s)}_{C,*}(z) \times J \cdot \prod_{j=1}^{s} c_1(\mathscr{L}_j)\Big).$$

Rewriting the diagonal term $\Delta^{(s)}_{C,*}(z)$ as $\overline{\mathrm{pr}}^*_1(z) \cdot \prod_{j=2}^{s} \overline{\mathrm{pr}}^*_{1j}(\Delta_C)$, and noting that $c_1(\mathscr{L}_j) \cdot \overline{\mathrm{pr}}^*_{1j}(\Delta_C) \times J = c_1(\mathscr{L}_1) \cdot \mathrm{pr}^*_{1j}(\Delta_C) \times J$, we find that (4.4) is equal to

$$\begin{aligned}\mathrm{pr}_{J,*}\big(\overline{\mathrm{pr}}^*_1(z) \times J \cdot c_1(\mathscr{L}_1)^s \cdot \mathrm{pr}^*_{C^s}(\Delta^{(s)}_C)\big) &= \mathrm{pr}_{J,*}\big((\overline{\mathrm{pr}}_1 \times \mathrm{id})^*(z \times J \cdot c_1(\mathscr{L})^s) \cdot \mathrm{pr}^*_{C^s}(\Delta^{(s)}_C)\big)\\ &= \mathrm{pr}_{J,*}\big(z \times J \cdot c_1(\mathscr{L})^s \cdot (\overline{\mathrm{pr}}_1 \times \mathrm{id})_* \mathrm{pr}^*_{C^s}(\Delta^{(s)}_C)\big)\\ &= \mathrm{pr}_{J,*}(z \times J \cdot c_1(\mathscr{L})^s)\end{aligned}$$

which is $s!$ times the codimension $(s-i)$ part of $\mathscr{F}^t((\phi \circ \iota_e)_* z)$. Again, by Fourier duality, we have an isomorphism $\mathscr{F}^t : \mathrm{CH}_{i,(s)}(J^t) \xrightarrow{\sim} \mathrm{CH}^{s+i}_{(s)}(J)$. It follows that the codimension $(s-i)$ part of $\mathscr{F}^t((\phi \circ \iota_e)_* z)$ coincides with $\mathscr{F}^t((\phi \circ \iota_e)_* z)_{(s-2i)}) = \mathscr{F}^t \circ \phi_*(\iota_{e,*}(z)_{(s-2i)})$.

Thus, the codimension $(s-i)$ part of $\mathscr{F}^t((\phi \circ \sigma_s)_* \gamma^s_e(z)))$ is equal to $\mathscr{F}^t \circ \phi_*(\iota_{e,*}(z)_{(s-2i)})$. In particular, if $\gamma^s_e(z) = 0$, then we get that $\iota_{e,*}(z)_{(s-2i)} = 0$. □

Besides establishing Theorem A, we also get the following consequence of Theorem 4.9 and Theorem 4.5.

Corollary 4.10. *If $s \geq 1$, then $\delta^e_{(s)} = 0 \iff \gamma^s_e(e) = 0$.*

Corollary 4.11. *If $\delta^e_{(2)} = 0$, then $\delta^e = 0$.*

Proof. Given the equivalence between the vanishing of $\delta^e_{(s)}$ and that of $\gamma^s_e(e)$, the statement follows from Corollary 3.10. Alternatively, we can provide a different proof by observing that

$$\begin{aligned}\sigma_{2,*}(\gamma^2_e(e)) &= m_*\iota^2_{e,*}(\Delta_{C,*}(e) - e\times e) = [2]_*\iota_{e,*}(e) - \iota_{e,*}(e)\star\iota_{e,*}(e)\\ &= [2]_*(\delta^e + [0]) - (\delta^e + [0])\star(\delta^e+[0]) = [2]_*\delta^e + [0] - \delta^e\star\delta^e - 2\delta^e - [0]\\ &= [2]_*\delta^e - 2\delta^e + \delta^e\star\delta^e.\end{aligned}$$

By hypothesis, this cycle is zero. Now, in order to prove $\delta^e = 0$, it is enough to show that $\delta^e \in I^n$ for any $n \geq 1$ because $\cap_{n=1}^{\infty} I^{\star n} = \{0\}$. Let now $\delta^e \in I^{\star n}$ for some $n \geq 2$. Since $I^n = \oplus_{s\geq n}\mathrm{CH}_{0,(s)}(J)$, this means that $\delta^e_{(s)} = 0$ for all $s < n$. Therefore

$$[2]_*\delta^e - 2\delta^e + \delta^e\star\delta^e \equiv 2^n\delta^e - 2\delta^e \equiv 0 \pmod{I^{\star(n+1)}}$$

which then implies $\delta^e \in I^{\star(n+1)}$. □

5. Applications

5.1. New proof of Zhang's result.

We will briefly explain how our results recover Theorem 1.1, following an argument suggested to us by Ben Moonen.

By Corollary 4.7 and Theorem 4.9, the vanishing of $[C]^e_{(1)}$ is equivalent to the vanishing of $\Gamma^3(C,e)$, which gives the second equivalence. Clearly, if $\mathrm{Cer}(C,e) = 0$ then $[C]^e_{(1)}$, and therefore $\Gamma^3(C,e)$ also vanish. Thus it remains to prove the converse.

Assume that $\Gamma^3(C,e)$ vanishes, then so does $B^3(C,e)$. By Proposition 3.5, we have

$$0 = \gamma^1_e(K_C + 2e) = K_C + 2e - (2g)e,$$

which gives $e = \xi = K_C/(2g-2)$. Furthermore, the same proposition implies that $\gamma^2_e(e) = 0$. Consequently, by Theorem 3.8 we deduce that $\Gamma^n(C,e) = 0$ for all $n \geq 3$. Interpreting this in terms of the Beauville decomposition of the curve class, this means that $[C]^e_{(s)} = 0$ for all $s \geq 1$, and in particular $\mathrm{Cer}(C,e) = 0$. Alternatively, we can recover the vanishing of the Ceresa cycle by computing $\sigma_{3,*}(\Gamma^3(C,e))$, and using the fact that $\delta^e = 0$ by Corollary 4.10 and Corollary 4.11.

5.2. The $[C]_{(2)}$-component.

For this subsection, we work with the canonical choice of divisor $e = \xi = K_C/(2g-2)$.

The work of Yin [Yin15] gives us an immediate application of our results to the generic non-vanishing of $[C]^{\xi}_{(2)}$ for $g \geq 4$. We note that by the main theorem of [CvG93], $[C]^{\xi}_{(2)}$ vanishes algebraically for the generic curve of genus 4.

Proposition 5.1. *For the generic curve C of genus $g \geq 4$, we have $[C]^{\xi}_{(2)} \neq 0$. If k is uncountable, then the same result holds for a very general curve C with $g \geq 4$.*

Proof. By Theorem A, the vanishing of $[C]^{\xi}_{(2)}$ is equivalent to that of $B^4(C,\xi)$, which in turn implies the vanishing of $\gamma^2_{\xi}(K_C + 2\xi) = 2g\cdot\gamma^2_{\xi}(\xi)$. As we have already noted, $\gamma^2_{\xi}(\xi)$ is, up to a scalar, the Faber–Pandharipande cycle. Consequently, the result follows from the main theorem of [Yin15], which states that the Faber–Pandharipande cycle is generically non-vanishing. □

On the other hand, we have the opposite behaviour in genus 3. To prove this, we borrow some machinery from the work of Polishchuk [Pol07]. For $e = x_0$ a k-point, Polishchuk studies the action of the Lie algebra $\mathfrak{sl}_2$ on $\mathrm{CH}(J)$ (see e.g. [Moo24, Section 9]) and on its so-called *tautological subring* $\mathcal{T}_{x_0}\operatorname{CH}(J)$. We recall that this subring contains the Beauville components $[C]^{x_0}_{(s)}$, and by [Pol07, Theorem 0.2] is generated by the classes $p_n := \mathscr{F}([C]^{x_0}_{(n-1)})$ $(n \geq 1)$ and $q_n := \mathscr{F}(\eta_{(n)})$ $(n \geq 0)$, where $\eta := \iota_{x_0,*}(K_C/2) + [0]$. Although this work is restricted to the case when $e = x_0$ is a k-point, we can still recover sufficient information when $e = \xi$ is not necessarily a k-point.

Theorem 5.2. *Let C be a genus 3 curve, then $[C]^{\xi}_{(2)} = 0$.*

Proof. We use the relation $[C]^{\xi} = [x_0 - \xi] \star [C]^{x_0}$. Taking the Fourier transform gives $\mathscr{F}([C]^{\xi}) = \mathscr{F}([x_0-\xi]) \cdot \mathscr{F}([C]^{x_0})$. As remarked in [Pol07, §1], we have $\mathscr{F}([x_0-\xi]) = \exp(\Theta_{\xi-x_0} - \Theta)$ (where $\Theta_{\xi-x_0}$ and Θ are as in Section 2.3). In particular, taking the codimension 4 component of these classes gives the following relation:

$$\mathscr{F}([C]^{\xi}_{(2)}) = \mathscr{F}([C]^{x_0}_{(2)}) + (\Theta_{\xi-x_0} - \Theta)\mathscr{F}([C]^{x_0}_{(1)}) + \frac{1}{2}(\Theta_{\xi-x_0} - \Theta)^2\mathscr{F}([C]^{x_0}_{(0)}).$$

Since $\eta = \iota_{x_0,*}(K_C/2) + [0]$ maps to $2(\xi - x_0)$ under the Albanese map, the divisor $\Theta_{\xi-x_0} - \Theta$ is equal to $-\frac{1}{2}\mathscr{F}(\eta_{(1)})$ (see [Pol07, §1]). We thus obtain the following expression for $\mathscr{F}([C]^{\xi}_{(2)})$ in terms of p_n and q_n:

$$8\mathscr{F}([C]^{\xi}_{(2)}) = 8p_3 - 4q_1p_2 + q_1^2p_1.$$

This expression can be simplified using the relations in the tautological ring to give the vanishing of the right hand side. Explicitly, as in [Yin15, Proposition 2.1, Corollary 2.2 (ii)], we have $4q_2 - q_1^2 = 0$. Moreover, by [Pol07, Theorem 0.2], we have $\mathbf{f}(p_2^2 - 2p_1q_2p_2 + 2p_1p_3) = 8p_3 - 4p_2q_1 - 4q_2p_1 + 2p_1q_1^2$, where $\mathbf{f}$ denotes the usual generator for the Lie algebra $\mathfrak{sl}_2$. But the cycle $p_2^2 - 2p_1q_2p_2 + 2p_1p_3$ vanishes for dimension reasons, so $[C]^{\xi}_{(2)} = 0$. □

Remark 5.3. Under the assumption $[C]_{(2)} = 0$, and working with the cycle $e = \xi$, we can strengthen some of the successive vanishing results of Section 3. Indeed the vanishing of $B^4(C,\xi)$ implies the vanishing of $\gamma^2_{\xi}(\xi)$. But as we saw in Corollary 3.9 and Corollary 4.10, the vanishing of $\gamma^2_{\xi}(\xi)$ implies the vanishing of $\gamma^n_{\xi}(\xi)$ for all $n \geq 2$, and that of δ^{ξ}. By Theorem 3.8, this also implies $B^5(C,\xi) = 0$, and so $[C]^{\xi} = [C]^{\xi}_{(0)} + [C]^{\xi}_{(1)}$.

For general e, the situation is slightly murkier. By Theorem 3.11, we have $\gamma^n_e(e) = 0$ for all $n \geq 3$, and so δ^e has at most one non-trivial Beauville component $(s = 2)$. We also get $\Gamma^n(C,e) = 0$ for all $n \geq 6$ by Corollary 3.12, which forces $[C]^e = [C]^e_{(0)} + [C]^e_{(1)} + [C]^e_{(3)}$.

Remark 5.4. In forthcoming work, we will generalize Polishchuk's results on the tautological subring $\mathcal{T}_{x_0}\operatorname{CH}(J)$ to the case where the basepoint x_0 is replaced by an arbitrary divisor e. In particular, we compute an explicit set of generators and relations for $\mathcal{T}_e\operatorname{CH}(J)$. However, the embedding with respect to the basepoint x_0 was sufficient for the proof of Theorem 5.2, since the computations simplify considerably for curves of small genus.

5.3. **Integral aspects.** In this subsection, we prove the integral refinement Theorem D for Zhang's result. We combine our methods with results of Moonen and Polishchuk [MP10a], who introduce a grading for the ring $\operatorname{CH}(J;\mathbb{Z})$ satisfying properties similar to those of the usual Beauville decomposition.

We let $\xi_{\text{int}} \in \mathrm{CH}_0(C;\mathbb{Z})$ be an integral representative for ξ, so that there exists an integer $N \neq 0$ satisfying $N(2g-2)\cdot\xi_{\text{int}} = N\cdot K_C$. We fix an integer $d \in \mathbb{Z}$.

Proposition 5.5. *If* $\mathrm{Cer}(C,\xi_{\text{int}}) \in \mathrm{CH}_1(J;\mathbb{Z})[d]$*, then* $\Gamma^3(C,\xi_{\text{int}}) \in \mathrm{CH}_1(C^3;\mathbb{Z})[2\cdot d]$.

Proof. Motivated by the proof of the corresponding statement with rational coefficients given in Theorem 4.6 and Proposition 4.2, we are lead to compute the cycle

$$\mathrm{pr}_{C^3,*}\Big(\prod_{j=1}^{3} c_1(\mathscr{L}_j)\cdot \mathrm{pr}_J^*\,\mathrm{Cer}(C,\xi_{\text{int}})\Big), \tag{5.1}$$

where $\mathscr{L}_j = (\mathrm{id}_J \times(\iota_{\xi_{\text{int}}}\circ \mathrm{pr}_j))^*\Lambda_\Theta$ is a line bundle on $J\times C^3$. Given that $([-1]\times\mathrm{id}_J)^*\Lambda_\Theta = -\Lambda_\Theta$, it follows that (5.1) is equal to

$$2\,\mathrm{pr}_{C^3,*}\Big(\prod_{j=1}^{3} c_1(\mathscr{L}_j)\cdot \mathrm{pr}_J^*[C]^{\xi_{\text{int}}}\Big).$$

By the Cartesian diagram from the proof of Proposition 4.2, we find that

$$\begin{aligned} 2\,\mathrm{pr}_{C^3,*}\Big(\prod_{j=1}^{3} c_1(\mathscr{L}_j)\cdot \mathrm{pr}_J^*[C]^{\xi_{\text{int}}}\Big) &= 2\,\mathrm{pr}_{C^3,*}\Big(\prod_{j=1}^{3} c_1(\mathscr{L}_j)\cdot (\iota_{\xi_{\text{int}}}\times \mathrm{id}_{C^3})_*[C\times C^3]\Big)\\ &= 2\,\mathrm{pr}_{C^3,*}\Big(\prod_{j=1}^{3}(\iota_{\xi_{\text{int}}}\times \mathrm{id}_{C^3})^* c_1(\mathscr{L}_j)\Big)\\ &= 2\,\mathrm{pr}_{C^3,*}\Big(\prod_{j=1}^{3}\overline{\mathrm{pr}}_{0j}^*\pi_1\Big). \end{aligned}$$

Here, we use the projection formula for the second equality, and Lemma 2.6 for the third equality. By Proposition 4.4, this is equal to $2B^3(C,\xi_{\text{int}})$. The equivalence between the vanishing of $B^3(C,\xi_{\text{int}})$ and $\Gamma^3(C,\xi_{\text{int}})$ remains, since $B^3(C,\xi_{\text{int}})$ is torsion, and by the proof of Proposition 3.5, $\gamma^2_{\xi_{\text{int}}}(\xi_{\text{int}})$ is also torsion. However, as $\gamma^2_{\xi_{\text{int}}}(\xi_{\text{int}})$ is in the kernel of the Albanese map, by Roĭtman's Theorem [Roj80] (completed by Milne in positive characteristic [Mil82]), this cycle vanishes integrally. ☐

Remark 5.6. This is the optimal result one can expect at this level of generality. Indeed, Moonen and Polishchuk provided an example of a hyperelliptic curve (so that $\mathrm{Cer}(C,\xi_{\text{int}}) = 0$) for which $\Gamma^3(C,\xi_{\text{int}})$ is non-trivial and 2-torsion (see [MP10b, Remark 2.6]).

Remark 5.7. Let us highlight that the last argument in the proof of Proposition 5.5 shows that the order of $B^3(C,e)$ coincides with the one of $\Gamma^3(C,e)$ in $\mathrm{CH}^2(C^2;\mathbb{Z})$.

Theorem 5.8. *Let* $M_{g+1} = \prod_{\text{prime } p\le g+1} p^{\ell_p}$*, with* $\ell_p = \lfloor\frac{g}{p-1}\rfloor$ *if* $p \geq 3$*, and* $\ell_2 = g-1$*. If* $\Gamma^3(C,\xi_{\text{int}}) \in \mathrm{CH}_1(C^3;\mathbb{Z})[d]$*, then* $\mathrm{Cer}(C,\xi_{\text{int}}) \in \mathrm{CH}_1(J;\mathbb{Z})[M_{g+1}\cdot d]$.

Proof. By [MP10a, Theorem 4], the group $\mathrm{CH}_i(J;\mathbb{Z})$ admits a direct sum decomposition

$$\mathrm{CH}_i(J;\mathbb{Z}) = \bigoplus_{m=0}^{g+i}\mathrm{CH}_i^{[m]}(J;\mathbb{Z}) \tag{5.2}$$

with the following property: for all $z \in \mathrm{CH}_i^{[m]}(J;\mathbb{Z})$ and all $n \in \mathbb{Z}$,

$$[n]_*(z) = n^m \cdot z + y, \quad y \in \bigoplus_{m'>m} \mathrm{CH}_i^{[m']}(J;\mathbb{Z}).$$

It follows that the pushforward $[n]_*$ respects the filtration $\mathrm{Fil}^m \mathrm{CH}_i(J;\mathbb{Z}) = \mathrm{CH}_i^{[\geq m]}(J;\mathbb{Z})$, and furthermore, $[n]_*$ acts on $\mathrm{gr}^m_{\mathrm{Fil}}\mathrm{CH}_i(J;\mathbb{Z})$ by multiplication by n^m. This is analogous to the action of $[n]_*$ on the Beauville decomposition (which is only defined over $\mathbb{Q}$-coefficients), though the decompositions are in fact distinct even with $\mathbb{Q}$-coefficients. We note also that the indexing is shifted by $2i$.

By [MP10a, §7] and by [MP10a, Corollary 3.8(i)], for dimension $i = 1$ we can start the decomposition (5.2) at $m = 2$ and write

$$[C]^{\xi_{\mathrm{int}}} = [C]^{[2]} + \cdots + [C]^{[g+1]}.$$

Assume now that $\Gamma^3(C, \xi_{\mathrm{int}})$ is d-torsion. Since $\gamma^2_{\xi_{\mathrm{int}}}(\xi_{\mathrm{int}}) = 0$, we deduce from Theorem 3.8 (which holds integrally) that $\Gamma^n(C, \xi_{\mathrm{int}})$ is d-torsion for $n \geq 3$. Since $\delta^{\xi_{\mathrm{int}}} = \iota_{\xi_{\mathrm{int}},*}(\xi_{\mathrm{int}}) - [0]$ is torsion and in the kernel of the Albanese, it vanishes integrally. Therefore, we find that

$$0 = d \cdot \sigma_{n,*}(\Gamma^n(C, \xi_{\mathrm{int}})) = d \cdot \sum_{k=0}^{n-1} (-1)^k \binom{n}{k} [n-k]_*([C]^{\xi_{\mathrm{int}}}). \tag{5.3}$$

Let us write

$$\mathrm{Cer}(C, \xi_{\mathrm{int}}) = \mathrm{Cer}(C, \xi_{\mathrm{int}})^{[3]} + \cdots + \mathrm{Cer}(C, \xi_{\mathrm{int}})^{[g+1]}.$$

The $\mathrm{CH}_1^{[3]}(J)$ component of $(\mathrm{id} - [-1]_*)\big(d \cdot \sigma_{3,*}(\Gamma^3(C, \xi_{\mathrm{int}}))\big)$ is then equal to

$$\left(d \cdot \sum_{k=0}^{2} (-1)^k \binom{3}{k} (3-k)^3\right) \cdot \mathrm{Cer}(C, \xi_{\mathrm{int}})^{[3]} = 6d \cdot \mathrm{Cer}(C, \xi_{\mathrm{int}})^{[3]} = 0. \tag{5.4}$$

For each integer $s \geq 3$, denote by $\mathcal{P}_s$ the following set of primes

$$\mathcal{P}_s = \{p \leq s : (p-1) \mid (s-1)\}.$$

For $m \geq 3$, let M_m be the positive integer $\prod_{s=3}^m \prod_{p \in \mathcal{P}_s} p$. We note that $v_p(M_m) = \lfloor \frac{m-1}{p-1} \rfloor$ for primes $p \geq 3$ and $v_2(M_m) = m-2$. Of course also $M_{m'} \mid M_m$ for $m' \leq m$.

We claim that for all $m \geq 3$, we have $dM_m \cdot \mathrm{Cer}(C, \xi_{\mathrm{int}})^{[m]} = 0$. We proceed by induction on m; the equation (5.4) implies the claim is true for $m = 3$. Assume the claim holds for all $m' \leq m$. If $m+1$ is even, we have $\mathcal{P}_{m+1} = \{2\}$. Note that

$$\big([-1]_* \mathrm{Cer}(C, \xi_{\mathrm{int}})\big)^{[m+1]} = \mathrm{Cer}(C, \xi_{\mathrm{int}})^{[m+1]} + \left([-1]_* \mathrm{Cer}(C, \xi_{\mathrm{int}})^{[\leq m]}\right)^{[m+1]},$$

where the superscript $[m+1]$ denotes taking the $\mathrm{CH}_1^{[m+1]}(J;\mathbb{Z})$-component. Since $[-1]_* \mathrm{Cer}(C, \xi_{\mathrm{int}}) = -\mathrm{Cer}(C, \xi_{\mathrm{int}})$, we find that $dM_m \cdot \mathrm{Cer}(C, \xi_{\mathrm{int}})^{[m+1]}$ is 2-torsion, and so $dM_{m+1} \cdot \mathrm{Cer}(C, \xi_{\mathrm{int}})^{[m+1]} = 0$ (as $2dM_m = dM_{m+1}$). On the other hand if m is odd, then by (5.3) we have that for each $n \geq 3$, the $\mathrm{CH}_1^{[m+1]}(J)$-component of $(\mathrm{id} - [-1]_*)(dM_m \cdot \sigma_{n,*}(\Gamma^n(C, \xi_{\mathrm{int}})))$ is equal to

$$\left(dM_m \cdot \sum_{k=0}^{n-1} (-1)^k \binom{n}{k} (n-k)^{m+1}\right) \cdot \mathrm{Cer}(C, \xi_{\mathrm{int}})^{[m+1]} = 0.$$

Note that the coefficient is non-zero if and only if $n \leq m+1$ (see [Voi15, Lemma 4.10]). Using Lemma 5.10, we deduce that $dM_{m+1} \cdot \mathrm{Cer}(C, \xi_{\mathrm{int}})^{[m+1]} = 0$. □

Remark 5.9. The constants M_m are closely related to the Hirzebruch numbers T_m, which appear as the denominators of the degree m component of the Todd power series (see [Hir95], Lemma 1.7.3). More precisely, we have $2M_m = T_{m-1}$. These constants play a role in the integral formulation of the Grothendieck-Riemann-Roch Theorem (see [Pap07]), and also feature in the recent construction of an integral Fourier transform [HHL$^+$25]. This is suggestive that this might be the limit of our methods.

Lemma 5.10. *Given $s \geq 3$ odd, the greatest common divisor of all $f(n) = \big(\sum_{k=0}^{n-1}(-1)^k\binom{n}{k}(n-k)^s\big)$ for $3 \leq n \leq s$ is given by the product $\prod_{p\in\mathcal{P}_s} p$, where $\mathcal{P}_s = \{p \leq s \text{ prime } : \ (p-1) \mid (s-1)\}$.*

Proof. First, assume p is a prime with $(p-1) \mid (s-1)$. Then, $(n-k)^s \equiv (n-k) \pmod p$. It follows that $f(n) \equiv \sum_{k=0}^{n-1}(-1)^k\binom{n}{k}(n-k) = 0 \pmod p$; the second equality is by identifying this sum with $\frac{d}{dx}(x-1)^n$ at $x = 1$. Hence every $p \in \mathcal{P}_s$ divides $f(n)$ for each $3 \leq n \leq s$.

Now assume p is an odd prime dividing $f(n)$ for all $3 \leq n \leq s$. In particular, p divides

$$f(4) + 4f(3) = 4^s - 6\cdot 2^s - 8 = (2^s-2)(2^s-4).$$

Therefore, the integer $\operatorname{ord}_p(2)$ (the multiplicative order of 2 modulo p) divides either $s-1$ or $s-2$. In the first case, we show inductively that $\operatorname{ord}_p(i) \mid (s-1)$ for all $3 \leq i \leq p-1$. Indeed as p divides $f(i)$, we have $i^s + \sum_{k=1}^{i-1}(-1)^k\binom{i}{k}(i-k)^s \equiv 0 \pmod p$; applying the inductive hypothesis,

$$i = -\sum_{k=1}^{i-1}(-1)^k\binom{i}{k}(i-k) \equiv i^s \pmod p.$$

An analogous argument holds when $\operatorname{ord}_p(2) \mid (s-2)$. As $i^{s-1} \equiv 1$ (resp. $i^{s-2} \equiv 1$) holds for all $1 \leq i \leq p-1$, it follows that $(p-1) \mid (s-1)$ (resp. $(p-1) \mid (s-2)$). As s and p are odd, we must be in the first case, and so $p \in \mathcal{P}_s$.

Finally, we show that the greatest common divisor is square-free. At the prime 2 we simply note that $f(3) \equiv 3^s + 3 \equiv 2 \pmod 4$. For odd primes $p \in \mathcal{P}_s$, the above argument showing $i^{s-1} \equiv 1$, also holds modulo p^2. Then, we see

$$f(p) = p^s + \sum_{k=1}^{p-1}(-1)^k\binom{p}{k}(n-k)^s \equiv \sum_{k=1}^{p-1}(-1)^k\binom{p}{k}(n-k) \equiv -p \pmod{p^2}.$$

Hence the greatest common divisor is not divisible by p^2 for any $p \in \mathcal{P}_s$, as desired. ☐

Let us recall the vanishing criterion of [QZ24]: if a curve C is endowed with the action of a group G by automorphisms such that $(H^1(C)^{\otimes 3})^G = 0$, then $\Gamma^3(C,\xi)$ vanishes in $\mathrm{CH}^2(C^3)$. This criterion admits an *integral refinement*, thereby providing an upper bound on the order of the modified diagonal whenever it applies.

Theorem 5.11. *Let G be a subgroup of Aut(C) such that $(H^1(C)^{\otimes 3})^G = 0$, where $H^1(C)$ is any Weil cohomology theory on which* End(J) *acts faithfully (for instance, Betti cohomology in characteristic 0). Then we have the following torsion bound,*

$$N\cdot(2g-2)|G|\cdot\Gamma^3(C,\xi_{\mathrm{int}}) = 0 \text{ in } \mathrm{CH}^2(C^3;\mathbb{Z}),$$

where as before N is such that $N(2g-2)\cdot\xi_{\mathrm{int}} = N\cdot K_C$.

Proof. Let us check that all the steps in the proof of [QZ24, Theorem 1.2.1] remain valid integrally:

1. Viewing elements of $\mathrm{CH}_1(C^2;\mathbb{Z})$ as self correspondences on C, we have an exact sequence
$$0 \to \mathrm{CH}_0(C;\mathbb{Z}) \oplus \mathrm{CH}_0(C;\mathbb{Z}) \xrightarrow{\mathrm{pr}_1^* \oplus \mathrm{pr}_2^*} \mathrm{CH}_1(C^2;\mathbb{Z}) \to \mathrm{End}(J) \to 0.$$
The section $f \mapsto (\iota^2_{\xi_{\mathrm{int}}})^* c_1((f \times \mathrm{id}_J)^* \Lambda_\Theta)$ induces an isomorphism $\mathrm{End}(J) \xrightarrow{\sim} \pi_{1,*} \mathrm{CH}_1(C^2;\mathbb{Z})$, where we recall that $\pi_1 = \Delta_C - (C \times \xi_{\mathrm{int}}) - (\xi_{\mathrm{int}} \times C)$.
2. For $z \in [\pi_{1,*} \mathrm{CH}_1(C^2;\mathbb{Z})]^{\otimes n}$, let $\mathrm{cl}(z) \in \mathrm{End}(H^1(C)^{\otimes n})$ be the corresponding endomorphism by functoriality. Since $\mathrm{End}(J)$ acts faithfully on $H^1(C)$, we have that $\mathrm{cl}(z) = 0$ if and only if $z = 0$.
3. For $g \in G$, let $\Gamma_g \in \mathrm{CH}_1(C^2;\mathbb{Z})$ be the graph of the automorphism of C defined by g. Given that $N \cdot K_C = N(2g-2) \cdot \xi_{\mathrm{int}}$ is G-invariant, we have that
$$N(2g-2) \cdot \Gamma_{g,*} \circ \pi_{1,*} = N(2g-2) \cdot \pi_{1,*} \circ \Gamma_{g,*}.$$
Consider the correspondence $z = \sum_{g \in G} (\Gamma_{g,*} \circ \pi_{1,*})^{\otimes 3}$. Since $\Gamma_{g,*}^{\otimes 3}(\Delta_C^{(3)}) = \Delta_C^{(3)}$, we deduce that $N \cdot (2g-2) \cdot z(\Delta_C^{(3)}) = N \cdot (2g-2)|G| \cdot B^3(C, \xi_{\mathrm{int}})$. On the other hand, by definition of z and the hypothesis $(H^1(C)^{\otimes 3})^G = 0$, we find that $\mathrm{cl}(z)$ acts trivially on $H^1(C)^{\otimes 3}$, hence $z = 0$. Therefore, we get that $N \cdot (2g-2)|G| \cdot B^3(C, \xi_{\mathrm{int}}) = 0$.

One concludes by Remark 5.7. □

Example 5.12. Consider the bielliptic Picard curve $C : y^3 = x^4 + 1$ which has automorphism group of order 48, and for which the criterion applies ([LS24, Example 3.1]). Then by Theorem 5.11, the order of $\Gamma^3(C, \infty)$ divides $(2 \cdot 3 - 2) \cdot 48 = 192$, where ∞ is the unique point at infinity. So by Theorem 5.8, $\mathrm{Cer}(C, \infty)$ has order dividing 2304. We remark that these bounds hold for any divisor e with $4e = K_C$. One may expect $B^3(C, \infty)$ to have a smaller order than $B^3(C, e)$ for a general e of this type, but we cannot prove this at present.

Example 5.13. Let n be a positive integer. We consider the hypergeometric curves $C_{n,\lambda}$: $(x^n - 1)(y^n - 1) = \lambda$, for a parameter $\lambda \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$, as defined by M. Asakura and N. Otsubo [AO24, Section 2.1]. In [EN25], P Eskandari and Y Nemoto show that for $p \geq 5$ a prime, and e a "cusp" (one of the points at infinity), $\Gamma^3(C_{p,\lambda}, e)$ is not l-torsion for any positive integer $l \leq \frac{p-1}{2}$. Combining this with Proposition 5.5 shows that $\mathrm{Cer}(C_{p,\lambda}, e)$ is not l-torsion for any $l \leq \frac{p-1}{4}$. In particular as $l = 1$ is in this range, the Ceresa cycle is non-zero for this choice of basepoint.

On the other hand when $p = 3$, Eskandari and Nemoto show that $\Gamma^3(C_{3,\lambda}, e)$ is torsion for every λ and every choice of basepoint e. We note that the curves $(x^3-1)(y^3-1) = \lambda$ are a 1-dimensional family of genus 4 curves with automorphism group containing S_3^2; indeed the automorphism group contains $(x, y) \mapsto (\omega x, \omega y)$, $(x, y) \mapsto (\frac{\mu}{x}, \frac{\mu}{y})$, $(x, y) \mapsto (\omega x, \omega^{-1} y)$ and $(x, y) \mapsto (y, x)$, where ω is a cube root of unity, and $\mu^3 = 1 - \lambda$. This explicit family therefore realises the genus 4 family from [QZ24]. Then by Theorem 5.11, if e is any divisor satisfying $(2 \cdot 4 - 2)e = K_{C_{3,\lambda}}$, the cycle $\Gamma^3(C_{3,\lambda}, e)$ has order dividing $(2 \cdot 4 - 2) \cdot 36 = 216$, and so by Theorem 5.8 the cycle $\mathrm{Cer}(C_{3,\lambda}, e)$ has order dividing $216 \cdot 360 = 77760$.

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