Lecture 17: more 10.1 Inference for Quantitative Variable: *z* Tests; begin 10.2: *t* Confidence Intervals

- z Test about Population Mean: 4 Steps
- Examples: 1-sided or 2-sided Alternative
- Relating Test and Confidence Interval
- Factors in Rejecting Null Hypothesis
- Inference Based on *t* vs. *z*
- Begin Confidence Intervals with *t*

Looking Back: *Review*

- **4 Stages of Statistics**
  - Data Production (discussed in Lectures 1-3)
  - Displaying and Summarizing (Lectures 3-8)
  - Probability (discussed in Lectures 9-14)
  - Statistical Inference
    - 1 categorical (discussed in Lectures 14-16)
    - 1 quantitative: conf. ints. L16, *z* hypothesis tests
    - categorical and quantitative
    - 2 categorical
    - 2 quantitative

Three Types of Inference Problem

Mean yearly earnings for sample of 446 students at a particular university was $3,776.

1. What is our best guess for the mean earnings of all students at that university?

   (Point Estimate)

2. What interval should contain mean earnings for all the students?

   (Confidence Interval)

3. Is this convincing evidence that mean earnings for all the students is less than $5,000?

   (Hypothesis Test)

Behavior of Sample Mean *(Review)*

For random sample of size n from population with mean μ and standard deviation σ, sample mean $\bar{X}$ has

- mean μ
- standard deviation $\frac{\sigma}{\sqrt{n}}$
- shape approximately normal for large enough n

→If σ is known, standardized $\bar{X}$ follows z (standard normal) distribution

Hypothesis Test About μ (with z)

Problem Statement $H_0 : \mu = \mu_0$ vs. $H_a : \left\{ \begin{array}{c} \mu > \mu_0 \\ \mu < \mu_0 \\ \mu \neq \mu_0 \end{array} \right\}$

1. Consider sampling and study design.
2. Summarize with $\bar{x}$, standardize to $z = \frac{\bar{x} - \mu_o}{\sigma / \sqrt{n}}$ assuming $H_o : \mu = \mu_o$ is true; is z "large"?
3. Find P-value (prob. of Z this far above/below/away from 0); is it "small"?
4. Based on size of P-value, choose H_0 or H_a.

Hypothesis Test About μ with z *(Details)*

1. Consider sampling and study design.
2. Summarize with $\bar{x}$, standardize to $z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$ assuming H_0 is true; is z "large"?
3. Find prob. of z this far above/below/away from 0 (P-value); consider if it is "small".
4. Based on size of P-value, choose H_0 or H_a.

- If sample is biased, mean of $\bar{X}$ is not μ_0.
- If pop<10n, s.d. of $\bar{X}$ is not $\sigma/\sqrt{n}$.
- If n is too small, distribution of $\bar{X}$ is not normal, won't standardize to z: graph data, see guidelines.

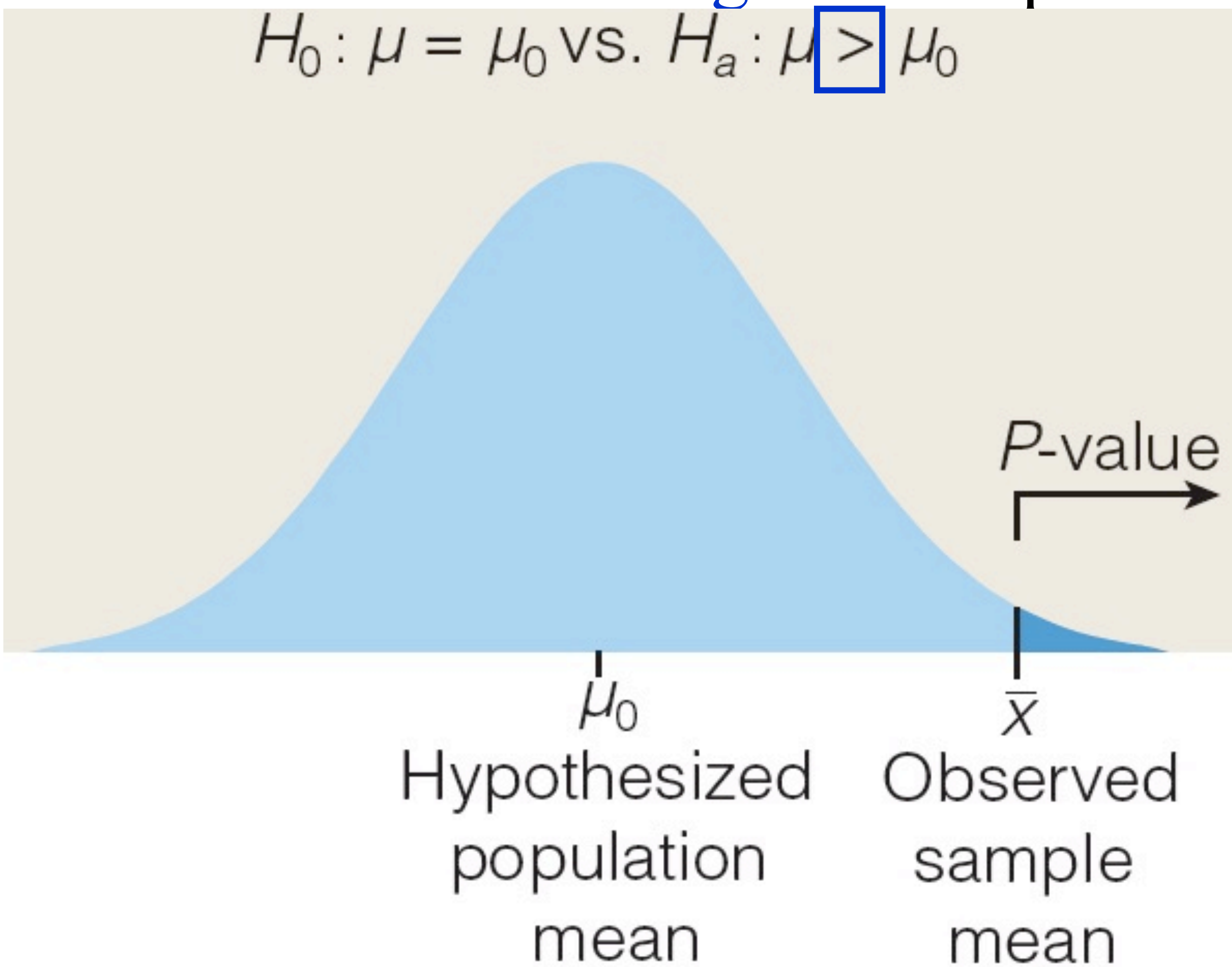
Hypothesis Test About μ with *z (Details)*

1. Consider sampling and study design
2. Summarize with $\bar{x}$, standardize to $z = \frac{\bar{x} - \mu_o}{\sigma / \sqrt{n}}$ assuming $H_o : \mu = \mu_o$ is true; is *z* "large"?
3. Find prob. of *z* this far above/below/away from 0 (*P*-value); consider if it is "small".
4. Based on size of *P*-value, choose H_0 or H_a.

- Assess *P*-value based on form of alternative hypothesis (greater, less, or not equal)

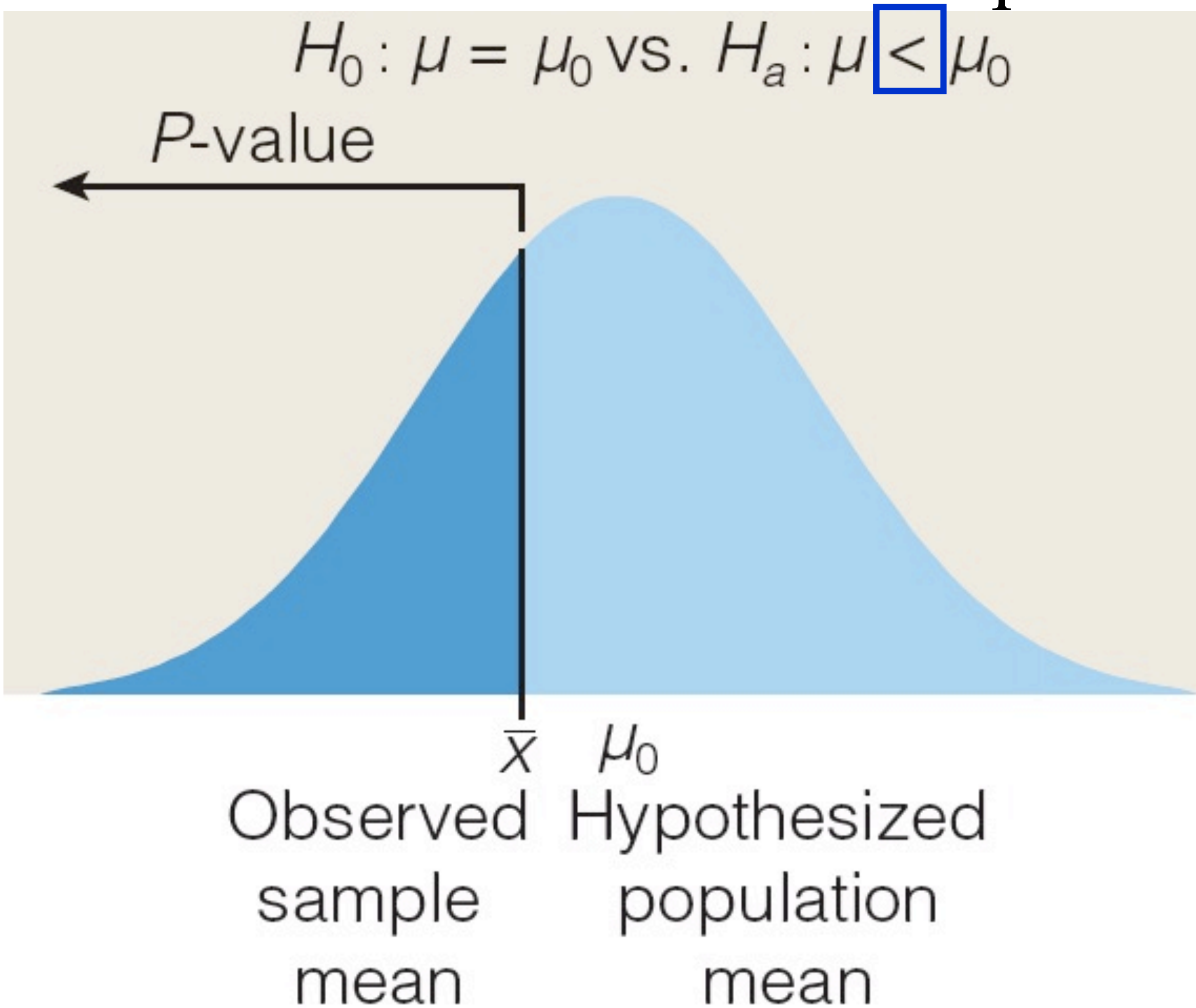
Hypothesis Test About μ with z *(Details)*

Alternative ">": P-value is right-tailed probability



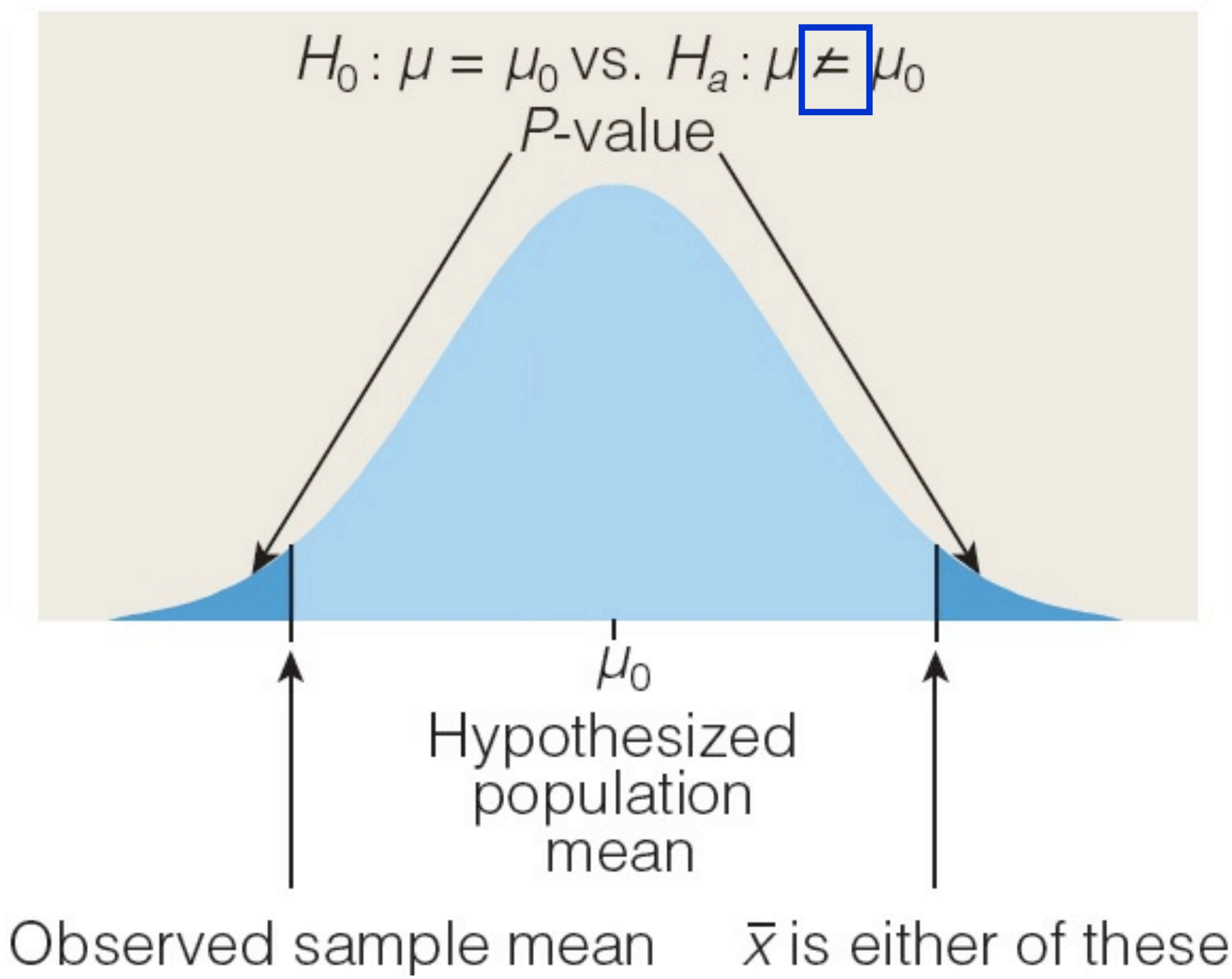
Hypothesis Test About μ with z *(Details)*

Alternative "<": *P*-value is left-tailed probability



Hypothesis Test About μ with z *(Details)*

Alternative "≠": *P*-value is two-tailed probability



Example: *Assumptions for z Test*

- **Background**: Earnings of 446 surveyed university students had mean \$3,776. The mean of earnings for the population of students is unknown. Assume we know population standard deviation is \$6,500.
- **Question:** What aspect of the situation is unrealistic?
- **Response:**

Looking Ahead: *In real-life problems, we rarely know the value of the population standard deviation. Eventually, we'll learn how to proceed when all we know is the sample standard deviation s.*

Example: *Test with One-Sided Alternative*

- **Background**: Earnings of 446 surveyed university students had mean $3,776. Assume population s.d. $6,500.
- **Question:** Are we convinced that μ is less than $5,000?
- **Response:** State H_o : vs. H_a :

```
One-Sample Z: Earned
Test of mu = 5 vs mu < 5
The assumed sigma = 6.5
Variable          N     Mean    StDev   SE Mean
Earned          446    3.776    6.503     0.308
Variable      95.0% Upper Bound       Z       P
Earned                    4.282   -3.98   0.000
```

1. Data production issues were discussed for confidence interval.
2. Output shows sample mean ________ and z = _____. Large?______
3. P-value = __________________________. Small? ______
4. Conclude? ____

Example: *Notation*

- **Background**: Want to test if mean of all male shoe sizes could be 11.0, based on a sample mean 11.222 from 9 male students. Assume pop. s.d. 1.5.
- **Question:** How do we denote the numbers given?
- **Response:**
  - 11.0 is proposed value of population mean ____
  - 11.222 is sample mean ____
  - 9 is sample size ____
  - 1.5 is population standard deviation ____

Practice: 10.4a p.477

Example: *Intuition Before Formal Test*

- **Background**: Want to test if mean of all male shoe sizes could be 11.0, based on a sample mean 11.222 from 9 male students. Assume pop. s.d. 1.5.
- **Question:** What conclusion do we anticipate, by "eye-balling" the data?
- **Response:**

Sample mean (11.222) seems close to proposed μ_O=11.0? ____

Sample size (9) small→ ______________________________

S.d. (1.5) not very small→ ____________________________

Anticipate standardized sample mean z large? _____

→*P*-value small? _____

→conclude population mean may be 11.0? _____

Example: *Test with Two-Sided Alternative*

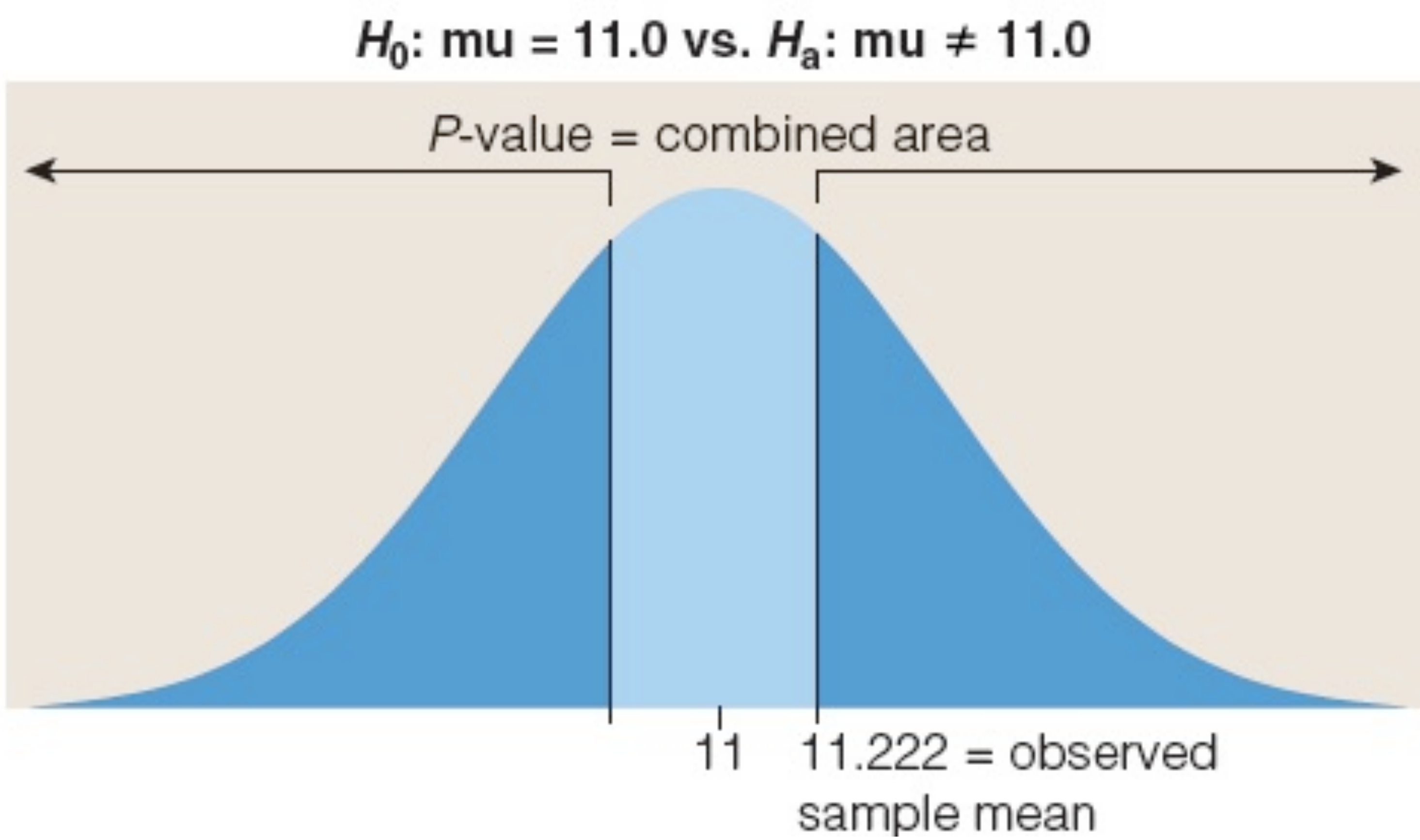
- **Background:** Want to test if mean of all male shoe sizes could be 11.0, based on a sample mean 11.222 from 9 male students. Assume pop. s.d. 1.5.
- **Question:** What do we conclude from the output?
- **Response:** $z = 0.44$. Large? ____
  P-value (two-tailed) = 0.657. Small? ____
  Conclude population mean may be 11.0? ____

```
One-Sample Z: Shoe
Test of mu = 11 vs mu not = 11
The assumed sigma = 1.5
Variable          N       Mean     StDev   SE Mean
Shoe              9     11.222     1.698     0.500
Variable              95.0% CI               Z       P
Shoe            (  10.242,  12.202)      0.44   0.657
```


Practice: 10.13a p.480

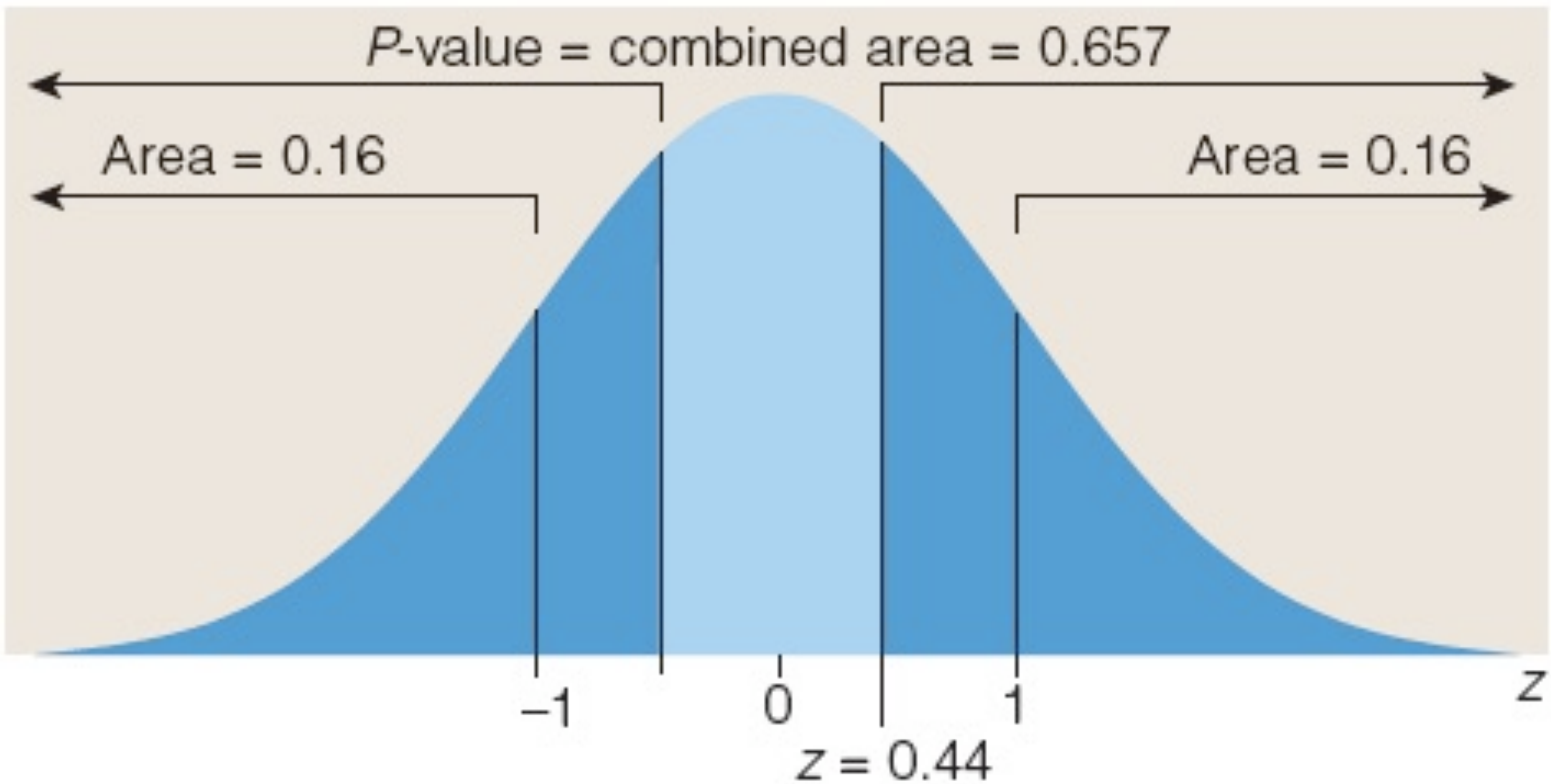
P-value as Nonstandard Normal Probability

P-value is probability of sample mean as far from 11.0 (in either direction) as 11.222.



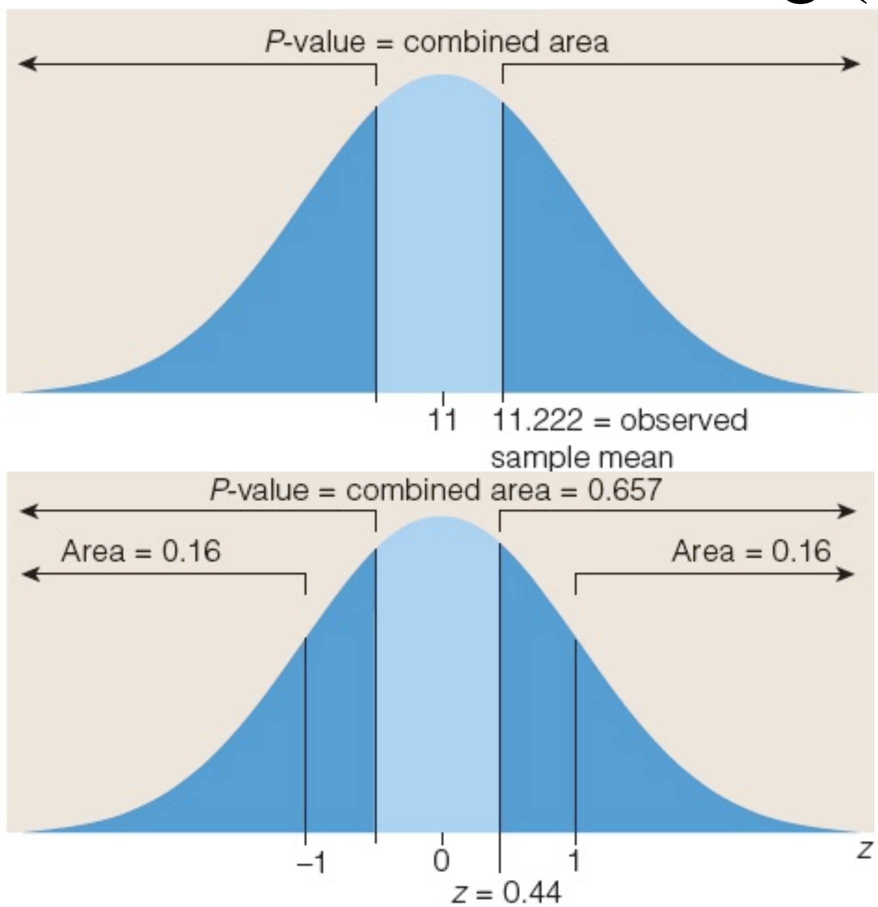
P-value as Standard Normal Probability

P-value as probability of standardized sample mean z as far from 0 (in either direction) as 0.44.



Comparing *P*-value Based on $\bar{x}$ vs. z

Same area under curve, just different scales on horizontal axis due to standardizing (below).



Example: *Test Results and Confidence Interval*

- **Background:** Tested if mean of all male shoe sizes could be 11.0, based on a sample mean 11.222 from 9 male students. Assumed pop. s.d. 1.5. *P*-value was 0.657; didn't reject null.
- **Question:** Would we expect 11.0 to be contained in a confidence interval for μ ?
- **Response:** Test showed 11.0 to be plausible for $\mu \rightarrow$ ____ (In fact, 11.0 is ______ contained in the confidence interval.)

```
One-Sample Z: Shoe
Test of mu = 11 vs mu not = 11
The assumed sigma = 1.5
Variable            N      Mean     StDev   SE Mean
Shoe                9    11.222     1.698     0.500
Variable               95.0% CI              Z       P
Shoe             (  10.242,  12.202)      0.44   0.657
```

Example: *Test Results and Confidence Interval*

- **Background:** Tested if mean earnings of all students at a university could be \$5,000, based on a sample mean \$3,776 for n=446. Assumed pop. s.d. \$6,500. P-value was 0.000; rejected null hypothesis.
- **Question:** Would 5,000 be contained in the confidence interval for μ?
- **Response:** ____

Factors That Lead to Rejecting Ho

Statistically significant data produce *P*-value small enough to reject H_0. z plays a role:

$$z = \frac{\bar{x} - \mu_o}{\sigma / \sqrt{n}} = \frac{(\bar{x} - \mu_o)\sqrt{n}}{\sigma}$$

Reject H_0 if *P*-value small; if $|z|$ large; if…

- Sample mean far from μ_o
- Sample size n large
- Standard deviation σ small

Role of Sample Size *n*

- **Large *n***: may reject H_0 even if sample mean is not far from proposed population mean, from a practical standpoint.

Very small *P*-value→strong evidence against Ho but $\overline{x}$ not necessarily very far from μ_o.

- **Small *n***: may fail to reject H_0 even though it is false.

Failing to reject false Ho is 2nd type of error.

Definition *(Review)*

- **Type I Error:** reject null hypothesis even though it is true (false positive)
- **Type II Error:** fail to reject null hypothesis even though it's false (false negative)

Test conclusions determine possible error:

- Reject H_0: correct or Type I
- Do not reject H_0: correct or Type II

Example: *Errors in a Medical Context*

- **Background:** A medical test is carried out for a disease (HIV).
- **Questions:**
  - What does the null hypothesis claim?
  - What are the implications of a Type I Error?
  - What are the implications of a Type II Error?
  - Which type of error is more worrisome?

Responses:

- Null hypothesis: ________________________________
- False ________: conclude _________________________
- False ________: conclude _________________________
- Type ___ is more worrisome.

Practice: 10.64 p.513

Example: *Errors in a Legal Context*

- **Background:** A defendant is on trial.
- **Questions:**
  - What does H_0 claim?
  - What does a Type I Error imply?
  - What does a Type II Error imply?
  - Which type is more worrisome?
- **Responses:**
  - H_0: ______________________________
  - Type I: Conclude _____________________
  - Type II: Conclude ____________________
  - Type ___ is more worrisome.

Behavior of Sample Mean *(Review)*

For random sample of size n from population with mean μ, standard deviation σ, sample mean $\bar{X}$ has

- mean μ
- standard deviation $\frac{\sigma}{\sqrt{n}}$
- shape approximately normal for large enough n

Sample Mean Standardizing to *z*

→If σ is known, standardized $\bar{X}$ follows *z* (standard normal) distribution:

$$\frac{\bar{x}-\mu}{\sigma/\sqrt{n}} = z$$

If σ is unknown and *n* is large enough (20 or 30), then $s \approx \sigma$ and $\frac{\bar{x}-\mu}{s/\sqrt{n}} \approx z$

Can use *z* if σ is known or *n* is large.

What if σ is unknown **and** *n* is small?

Sample mean standardizing to *t*

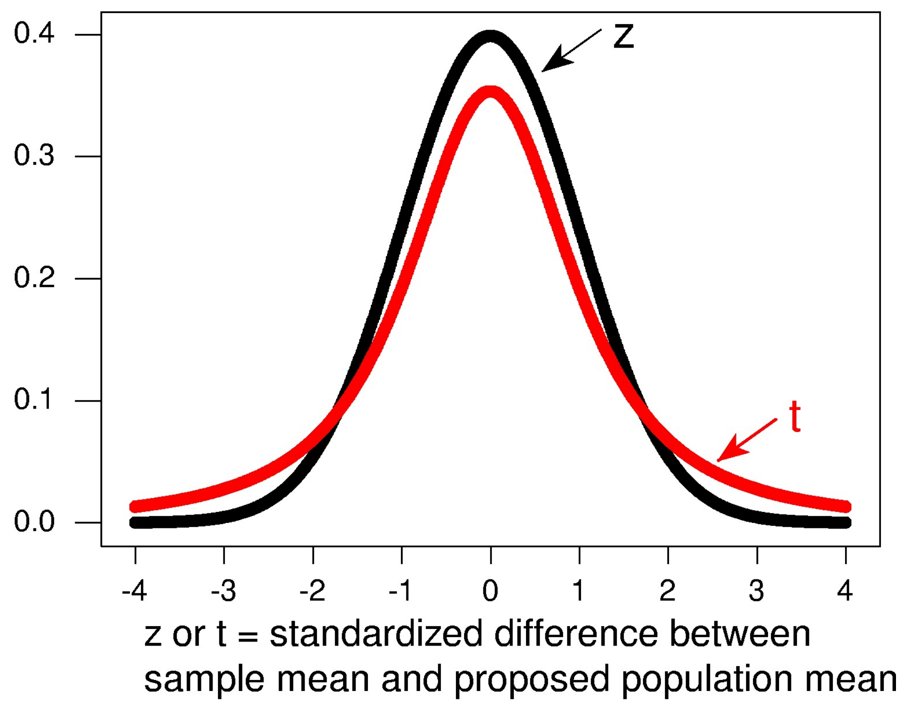
For σ unknown and *n* small, $\dfrac{\bar{x}-\mu}{s/\sqrt{n}} = t$

- *t* (like *z*) centered at 0 since $\bar{X}$ centered at μ
- *t* (like *z*) symmetric and bell-shaped if $\bar{X}$ normal
- *t* more spread than *z* (s.d.>1) [*s* gives less info]

t has "*n*-1 degrees of freedom"(spread depends on *n*)

Inference About Mean Based on *z* or *t*

- σ known→ standardized $\bar{x}$ is z (may use z if σ unknown but n large)
- σ unknown→ standardized $\bar{x}$ is t



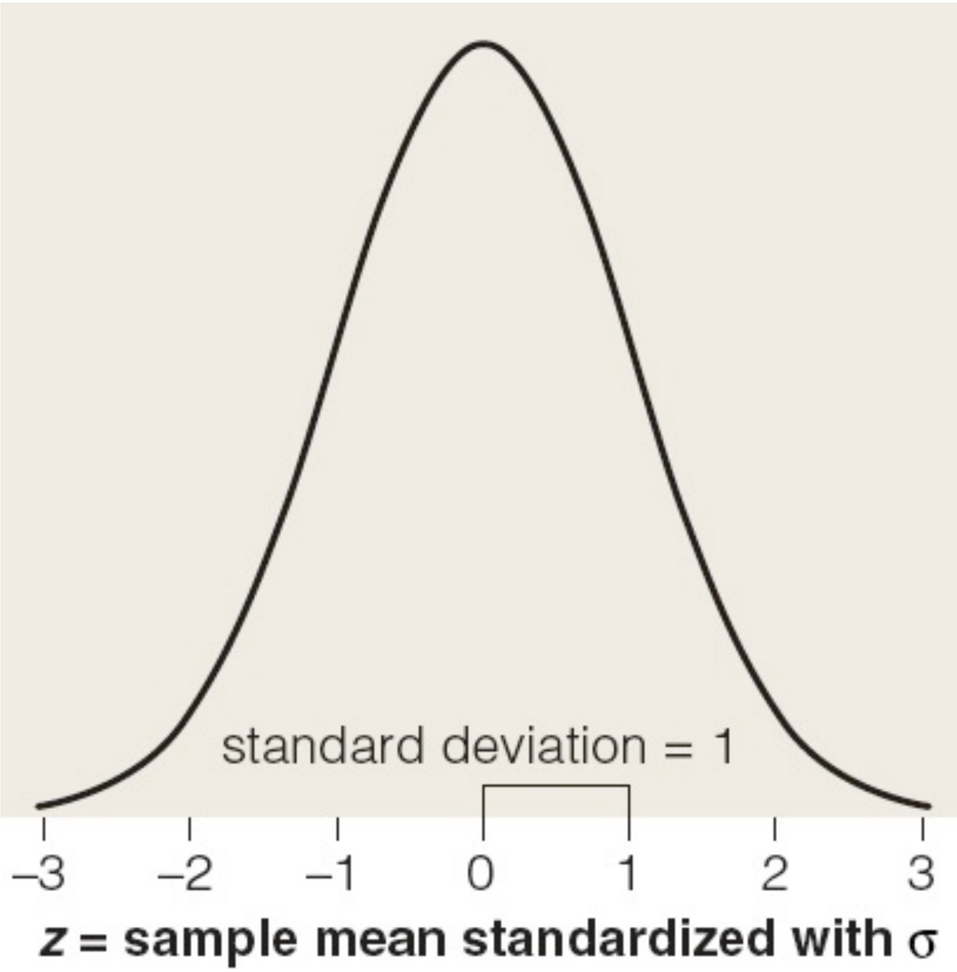
Inference *by Hand* Based on z or t

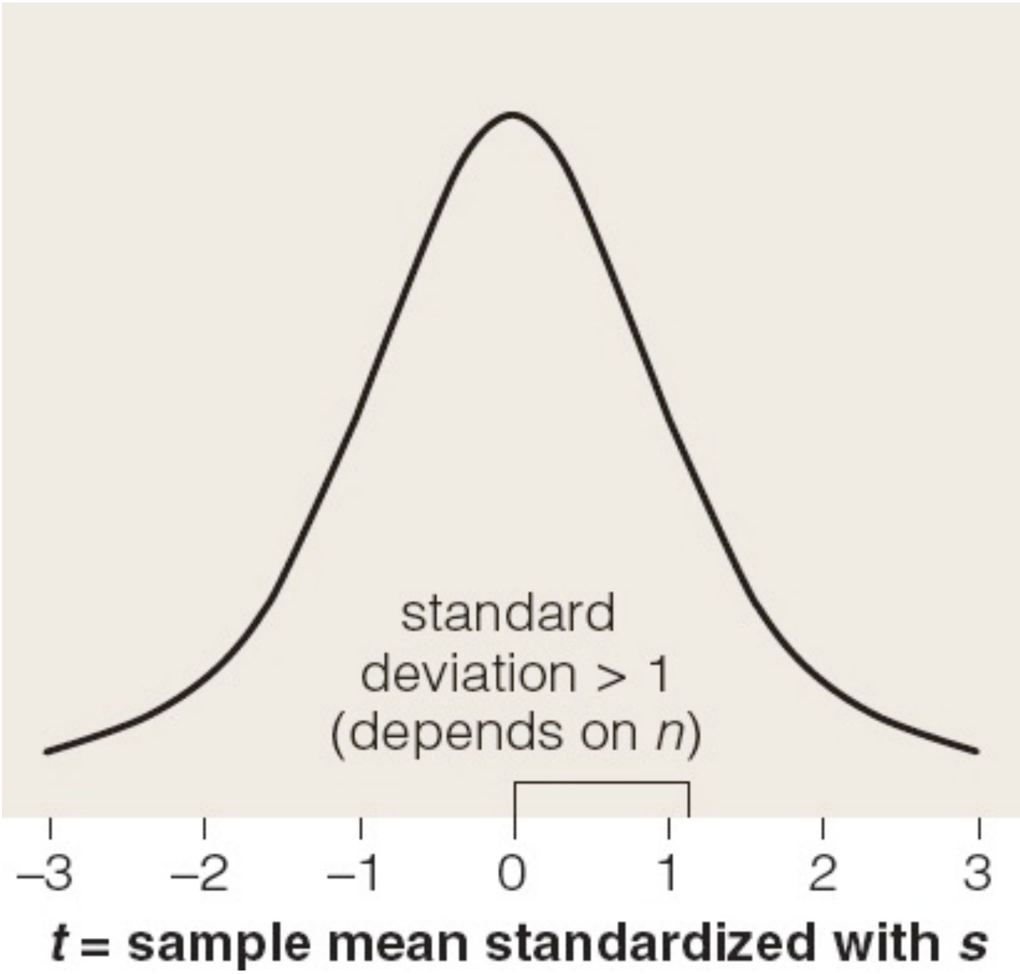
	σ known	σ unknown
small sample $(n < 30)$	$\frac{\bar{x}-\mu}{\sigma/\sqrt{n}} = z$	$\frac{\bar{x}-\mu}{s/\sqrt{n}} = t$
large sample $(n \geq 30)$	$\frac{\bar{x}-\mu}{\sigma/\sqrt{n}} = z$	$\frac{\bar{x}-\mu}{s/\sqrt{n}} \approx z$

z used if σ known **or** n large

t used if σ unknown **and** n small

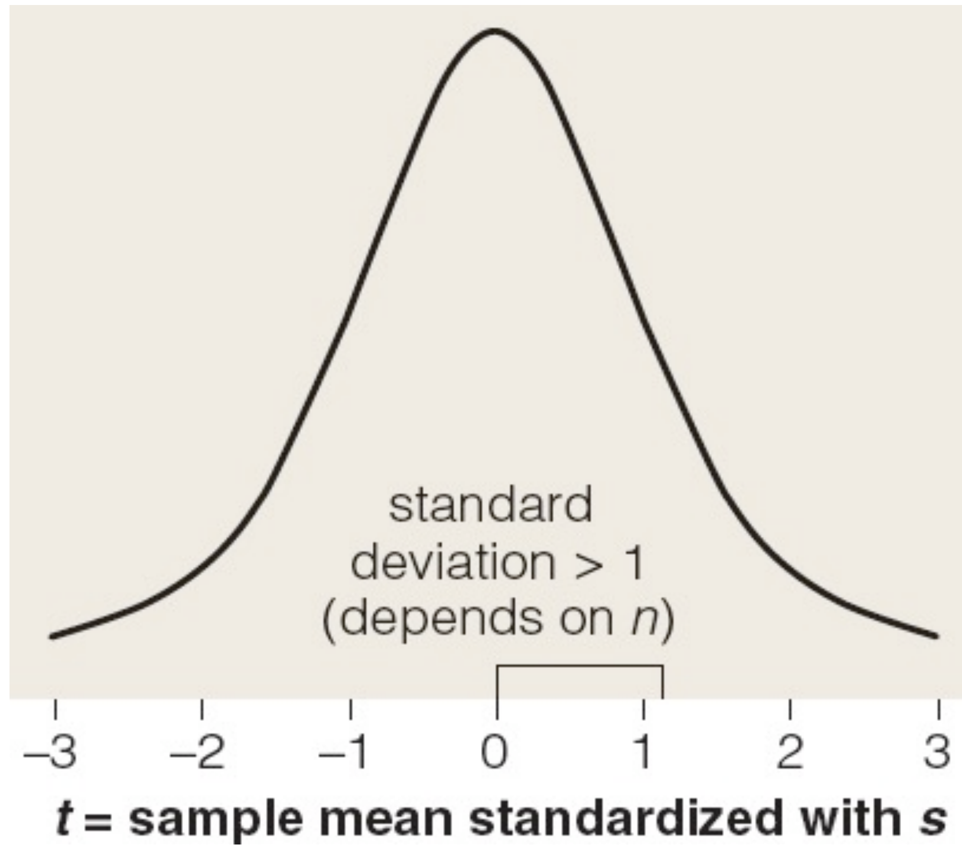
z vs. *t*: How the Sample Mean is Standardized





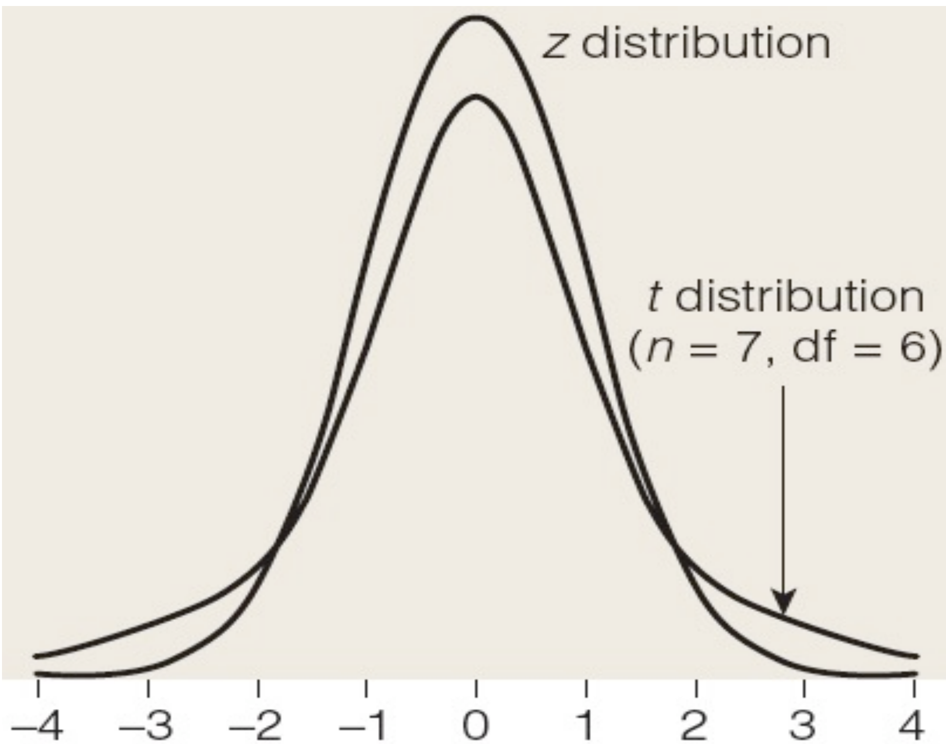
z vs. *t*: How the Sample Mean is Standardized

A Closer Look: *We say t is "heavy-tailed" (compared to z).*



Example: *Distribution of t* (6 df) *vs. z*

- **Background**: For *n*=7, $\frac{\bar{x}-\mu}{s/\sqrt{n}} = t$ has 6 df.



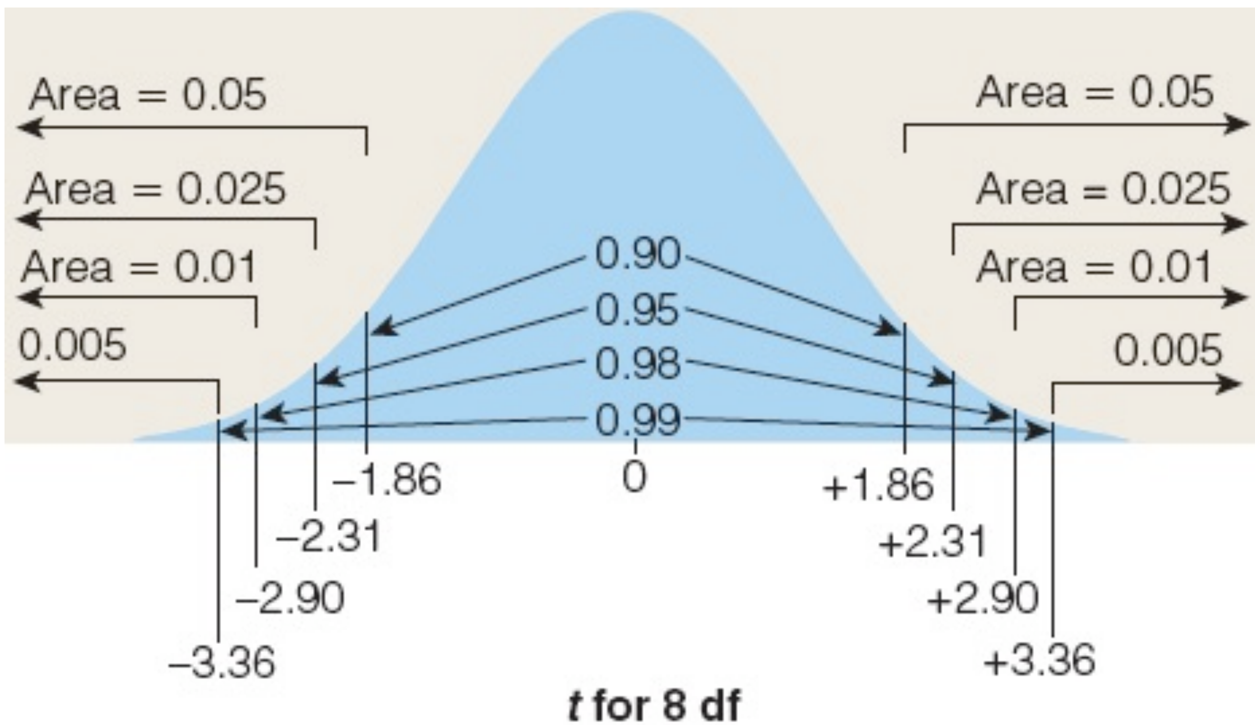
***A Closer Look:** In fact, P(t > 2) is about 0.05; P(z > 2) is about 0.025.*

- **Question:** How does P(t > 2) compare to P(z > 2)?
- **Response:** P(t > 2) ________________________ P(z > 2).

Practice: 10.14 p.489

Example: *Distribution of t* (8 df) *vs. z*

- **Background**: According to 90-95-98-99 Rule for z, P(z > 2) is between 0.01 and 0.025 because 2 is between 1.96 and 2.576. Consider the t curve for 8 df.



- **Question:** What is a range for P(t > 2) when t has 8 df?
- **Response:** P(t > 2) is between ______ and ______.

Practice: 10.60d p.512

Looking Back: *Review*

- **4 Stages of Statistics**
  - Data Production (discussed in Lectures 1-3)
  - Displaying and Summarizing (Lectures 3-8)
  - Probability (discussed in Lectures 9-14)
  - Statistical Inference
    - 1 categorical (discussed in Lectures 14-16)
    - 1 quantitative: z CI (L16), z test, t CI, t test
    - categorical and quantitative
    - 2 categorical
    - 2 quantitative

Confidence Interval for Mean *(Review)*

95% confidence interval for μ (σ known) is

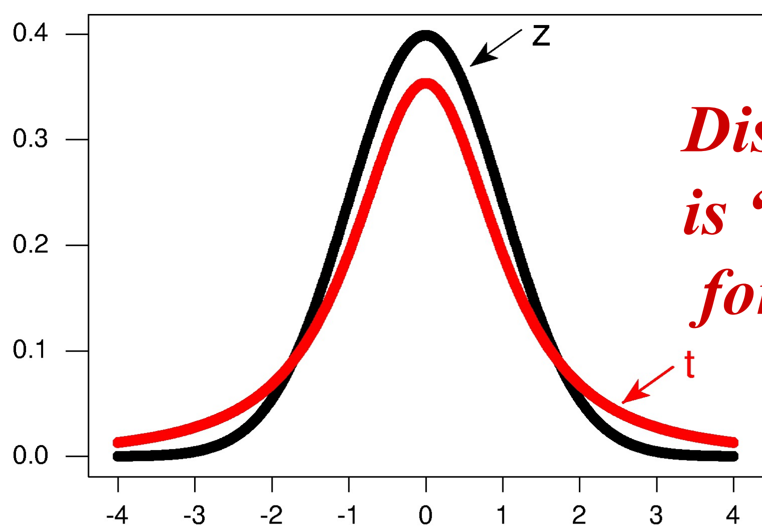
$$\bar{x} \pm \boxed{2} \frac{\sigma}{\sqrt{n}}$$

- multiplier 2 is from *z* distribution

(95% of normal values within 2 s.d.s of mean)

For *n* small, σ unknown can't say 95% C.I. is

$$\bar{x} \pm 2 \frac{s}{\sqrt{n}}$$



Distribution of t is "heavy tailed" for small n.

Confidence Interval for Mean: σ Unknown

95% confidence interval for μ is

$$\bar{x} \pm \text{multiplier}\left(\frac{s}{\sqrt{n}}\right)$$

- multiplier from *t* distribution with *n*-1 *degrees of freedom* (df)
- multiplier at least 2, closer to 3 for *very* small *n*

Degrees of Freedom

- **Mathematical** explanation of df: not needed for elementary statistics
- **Practical** explanation of df: several useful distributions like t, F, chi-square are *families* of similar curves; df tells us which one applies (depends on sample size n).

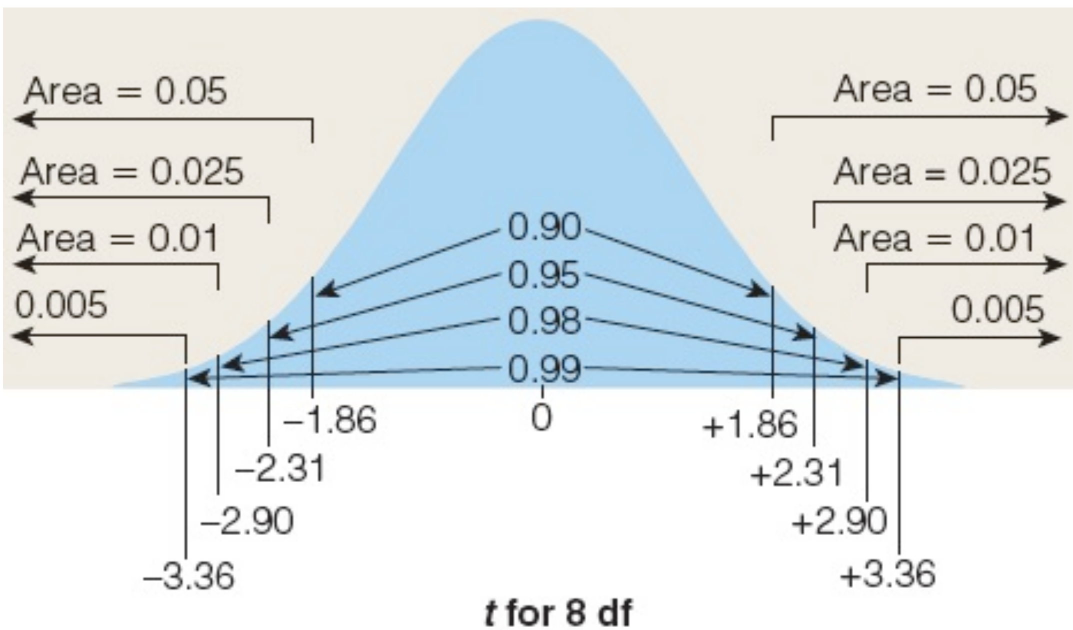
z or t: Which to Concentrate On?

- For purpose of **learning**, start with z (know what to expect from 68-95-99.7 Rule, etc.) (only one z distribution)
- For **practical** purposes, t more realistic (usually don't know population s.d. σ)

Software automatically uses appropriate t distribution with n-1 df: just enter data.

Example: *Confidence Interval with t Curve*

- **Background**: Random sample of shoe sizes for 9 college males: 11.5, 12.0, 11.0, 15.0, 11.5, 10.0, 9.0, 10.0, 11.0
- **Question:** What is 95% C.I. for population mean?
- **Response:** Mean 11.222, s= 1.698, n=9, multiplier 2.31:



Example: *t Confidence Interval* ***with Software***

- **Background**: Random sample of shoe sizes for 9 college males: 11.5, 12.0, 11.0, 15.0, 11.5, 10.0, 9.0, 10.0, 11.0
- **Question:** How do we find a 95% C.I. for the population mean, using software?
- **Response:**

```
One-Sample T: Shoe
Variable          N      Mean     StDev   SE Mean         95.0% CI
Shoe              9    11.222     1.698     0.566  (   9.917,  12.527)
```

Example: *Compare t and z Confidence Intervals*

- **Background**: Random sample of shoe sizes for 9 college males: 11.5, 12.0, 11.0, 15.0, 11.5, 10.0, 9.0, 10.0, 11.0
We produced 95% *t* confidence interval:

$$11.222 \pm 2.31\left(\frac{1.698}{\sqrt{9}}\right) = 11.222 \pm 1.307 = (9.92, 12.53)$$

If 1.698 had been population s.d., would get *z* C.I.:

$$11.222 \pm 1.96\left(\frac{1.698}{\sqrt{9}}\right) = 11.222 \pm 1.109 = (10.11, 12.33)$$

- **Question:** How do the *t* and *z* intervals differ?
- **Response:** *t* multiplier is 2.31, *z* multiplier is 1.96:

t interval width about ____________ [_______________]

z interval width about ____________ [_____________]

σ known→______ info→___________ interval

Practice: 10.16b p.489

Example: *t vs. z Confidence Intervals, Large n*

- **Background**: Earnings for sample of 446 students at a university averaged $3,776, with s.d. $6,500. The *t* multiplier for 95% confidence and 445 df is 1.9653.
- **Question:** How different are the *t* and *z* intervals?
- **Response:** The intervals will be ______________, whether we use
  - *t* multiplier 1.9653
  - precise *z* multiplier 1.96
  - approximate *z* multiplier 2

Interval approximately

Practice: 10.27a p.500

Behavior of Sample Mean *(Review)*

For random sample of size n from population with mean μ, standard deviation σ, sample mean $\bar{X}$ has

- mean μ
- standard deviation $\frac{\sigma}{\sqrt{n}}$
- shape approx. normal for large enough n

→If σ is unknown and n small, $$\frac{\bar{x}-\mu}{s/\sqrt{n}} = t$$

Guidelines for $\bar{X}$ Approx. Normal *(Review)*

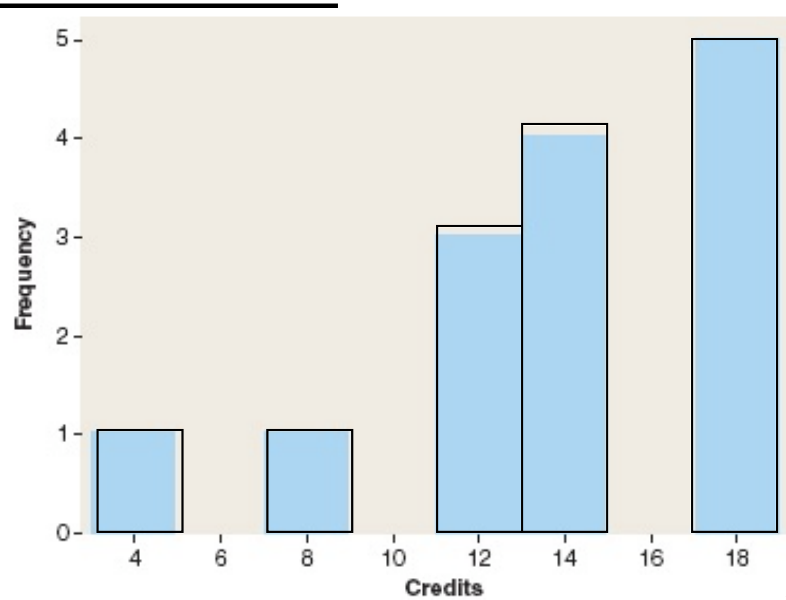
Can assume shape of $\bar{X}$ for random samples of size *n* is approximately normal if

- Graph of sample data appears normal; or
- Sample data fairly symmetric, *n* at least 15; or
- Sample data moderately skewed, *n* at least 30; or
- Sample data very skewed, *n* much larger than 30

If $\bar{X}$ **is not normal,** $\frac{\bar{x}-\mu}{s/\sqrt{n}}$ **is not *t*.**

Example: *Small, Skewed Data Set*

- **Background**: Credits taken by 14 non-traditional students: 4, 7, 11, 11, 12, 13, 13, 14, 14, 17, 17, 17, 17, 18
- **Question:** What is a 95% confidence interval for population mean?
- **Response:** *n* small, shape of credits left-skewed → ________________



Looking Ahead: *Non-parametric methods can be used for small n, skewed data.*

Lecture Summary
(Inference for Means: z Hypothesis Tests; t Dist.)

- z test about population mean: 4 steps
- Examples: 1-sided and 2-sided alternatives
- Relating test and confidence interval
- Factors in rejecting null hypothesis
  - Sample mean far from proposed population mean
  - Sample size large
  - Standard deviation small
- Inference based on z or t
  - Population sd known; standardize to z
  - Population sd unknown; standardize to t
- Comparing z and t distributions

Lecture Summary
(Inference for Means: t Confidence Intervals)

- *t* confidence interval for population mean
  - Multiplier from *t* distribution with *n*-1 df
  - When to perform inference with *z* or *t*
  - Constructing *t* CI by hand or with software
- Comparing *z* and *t* confidence intervals
- When neither *z* nor *t* applies