PDE AND LINEAR PROGRAMMING APPROACH TO THE OPTIMAL TRANSPORTATION PROBLEM

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Abstract. The Optimal Transportation problem has been the subject of a great deal of attention by theoreticians in last couple of decades. The Wasserstein (or Earth Mover) distance allows for the metrization of the space of probability measures. As a consequence, it is now possible to more effectively measure the distance between new classes of objects. This important in many fields, particularly in image registration, and parameter identification. However the effective computation of these distances (and the associated maps) has been intractable, except for very small problems.

Recent advances have allowed for more efficient computation of solutions of the Monge-Kantorovich problem of optimal transportation. In the special, but important case of quadratic costs, the map can be obtained from the solution of the elliptic Monge-Ampere partial differential equation with nonstandard boundary conditions. For more general costs, the Kantorovich plan can be approximated by a finite dimensional linear program. In this talk we will compare the cost and quality of the solutions obtained by two different methods.

In the PDE approach, the measures are approximated by densities, and map is obtained as the gradient of a convex function. The target domain is a convex set. The gradient of the solution is required to lie in the convex target domain, which leads to non-standard boundary conditions. The boundary conditions are defined implicitly by a Hamilton-Jacobi (first order nonlinear partial differential) equation. We use a wide stencil finite difference scheme, which is augmented by a second order accurate finite difference scheme, resulting in a second order method. We prove convergence to the unique viscosity solution, using a modification of the Barles-Souganidis result to filtered schemes. One step of an interative method provides an approximate solution which is the seed for a Newton iteration. Solutions are computed in a small number of iterations using directs solvers and two dimension and for moderate problem sizes in three dimensions. The open problem of scalable solvers in three dimensions is partially addressed by a multigrid method for the finite difference scheme.

In the Kantorovich Linear Programming approach, the measures are approximated by weighted sums of Dirac deltas. A direct implementation of the Kantorovich approach allows for low resolution approximations (thousands of Dirac deltas). However when the solution is known to be a map, we present a refinement to the method allows solutions to be computed at high resolutions (hundreds of thousands or more Dirac deltas). The linear programming approach has the advantage that it easily extends to more general problems: partial transportation, barycentre problems, etc. However the solutions computed are very coarse: convergence is weak (in the sense of measures) so oscillations can appear in the solutions. Barycentric projection is a theoretical tool which allows us to improve the accuracy of the weak solutions.

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