

# CS 441: Sequences and Summations

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PhD. Nils Murrugarra-Llerena  
[nem177@pitt.edu](mailto:nem177@pitt.edu)



# Today's topics

- Sequences and Summations
  - Specifying and recognizing sequences
  - Summation notation
  - Closed forms of summations



# Sequences are ordered lists of elements

**Definition:** A **sequence** is a function from a subset of the set of integers to a set  $S$ . We use the notation  $a_n$  to denote the image of the integer  $n$ .  $a_n$  is called a **term** of the sequence.

**Examples:**

- 1, 3, 5, 7, 9, 11                      A sequence with 6 terms
- 1, 1/2, 1/3, 1/4, 1/5, ...              An infinite sequence

**Note:** The second example can be described as the sequence  $\{a_n\}$  where  $a_n = 1/n$

# What makes sequences so special?

**Question:** Aren't sequences just sets?

**Answer:** The elements of a sequence are members of a set, but a sequence is **ordered**, a set is not.

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**Question:** How are sequences different from ordered n-tuples?

**Answer:** An ordered n-tuple is ordered, but always contains n elements. Sequences can be infinite!

## Some special sequences

**Geometric progressions** are sequences of the form  $\{ar^m\}$  where  $a$  and  $r$  are real numbers

**Examples:**

- 1, 1/2, 1/4, 1/8, 1/16, ...
- 1, -1, 1, -1, 1, -1, ...

**Arithmetic progressions** are sequences of the form  $\{a + nd\}$  where  $a$  and  $d$  are real numbers.

**Examples:**

- 2, 4, 6, 8, 10, ...
- -10, -15, -20, -25, ...

## Sometimes we need to figure out the formula for a sequence given only a few terms

Questions to ask yourself:

1. Are there runs of the same value?
2. Are terms obtained by multiplying the previous value by a particular amount? (Possible geometric sequence)
3. Are terms obtained by adding a particular amount to the previous value? (Possible arithmetic sequence)
4. Are terms obtained by combining previous terms in a certain way?
5. Are there cycles amongst terms?

# What are the formulas for these sequences?

**Problem 1:** 1, 5, 9, 13, 17, ...

**Problem 2:** 1, 3, 9, 27, 81, ...

**Problem 3:** 2, 3, 3, 5, 5, 5, 7, 7, 7, 7, 11, 11, 11, 11, 11, ...

**Problem 4:** 1, 1, 2, 3, 5, 8, 13, 21, 34, ...

***This is called the Fibonacci sequence.*** 

## Sequences are often specified using recurrence relations

This is a **recursive** approach to specifying the terms

- Later terms are specified from earlier terms

For instance, consider this definition of the Fibonacci sequence:

- $f_0 = 0$
- $f_1 = 1$
- For any  $n > 1$ ,  $f_n = f_{n-1} + f_{n-2}$

Note that we need at least one **initial condition**

- Like a base case when writing recursive code
- We'll return to recursion later in the term



Sometimes we want to find the sum of the terms in a sequence

**Summation notation** lets us compactly represent the sum of terms  $a_m + a_{m+1} + \dots + a_n$

The diagram shows two equivalent summation notations:  $\sum_{j=m}^n a_j$  and  $\sum_{m \leq j \leq n} a_j$ . Red circles highlight the components:  $n$  is the upper limit,  $m$  is the lower limit, and  $j$  is the index of summation. Red arrows point from the labels 'Upper limit', 'Lower limit', and 'Index of summation' to their respective parts in both notations.

$$\sum_{j=m}^n a_j = \sum_{m \leq j \leq n} a_j$$

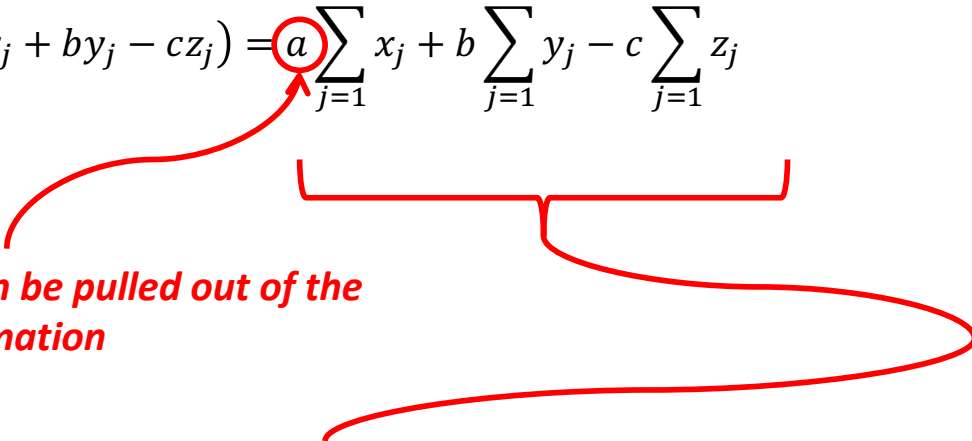
*Upper limit*

*Lower limit*

*Index of summation*

**Example:**  $\sum_{1 \leq i \leq 5} i^2 = 1 + 4 + 9 + 16 + 25 = 55$

## Summations are linear: The usual laws of algebra apply

$$\sum_{j=1}^n (ax_j + by_j - cz_j) = a \sum_{j=1}^n x_j + b \sum_{j=1}^n y_j - c \sum_{j=1}^n z_j$$


*Constant factors can be pulled out of the summation*

*A summation over a sum (or difference) can be split into a sum (or difference) of smaller summations*

### **Example:**

- $\sum_{1 \leq j \leq 3} (4j + j^2) =$
- $4 \sum_{1 \leq j \leq 3} j + \sum_{1 \leq j \leq 3} j^2 =$

## Example sums

**Example:** Express the sum of the first 50 terms of the sequence  $1/n^2$  for  $n = 1, 2, 3, \dots$

**Answer:** 
$$\sum_{j=1}^{50} \frac{1}{j^2}$$

**Example:** What is the value of  $\sum_{k=4}^8 (-1)^k$

**Answer:** 
$$\begin{aligned} \sum_{k=4}^8 (-1)^k &= \\ &= \\ &= \end{aligned}$$

We can also compute the summation of the elements of some set

**Example:** Compute  $\sum_{s \in \{0,2,4,6\}} (s + 2)$

**Answer:**  $(0 + 2) + (2 + 2) + (4 + 2) + (6 + 2) = 20$

**Example:** Let  $f(x) = x^3 + 1$ . Compute  $\sum_{s \in \{1,3,5,7\}} f(s)$

**Answer:**  $f(1) + f(3) + f(5) + f(7) = 2 + 28 + 126 + 344 = 500$

## Sometimes it is helpful to shift the index of a summation

This is particularly useful when **combining** two or more summations. For example:

$$\begin{aligned}
 S &= \sum_{j=1}^{10} j^2 + \sum_{k=2}^{11} (2k - 1) && \text{Let } j = k - 1 \\
 &= \sum_{j=1}^{10} j^2 + \sum_{j=1}^{10} (2(j + 1) - 1) && \text{Need to add 1} \\
 &&& \text{to each } j \\
 &= \sum_{j=1}^{10} (j^2 + 2(j + 1) - 1) \\
 &= \sum_{j=1}^{10} (j^2 + 2j + 1) \\
 &= \sum_{j=1}^{10} (j + 1)^2
 \end{aligned}$$

# In-class exercises

## On Top Hat

## Summations can be nested within one another

Often, you'll see this when analyzing nested loops within a program (i.e., CS 1501/1502)

**Example:** Compute  $\sum_{j=1}^4 \sum_{k=1}^3 (jk)$

**Solution:**

$$\sum_{j=1}^4 \sum_{k=1}^3 (jk) = \sum_{j=1}^4 (j + 2j + 3j)$$

*Expand inner sum*

$$= \sum_{j=1}^4 6j$$

*Simplify if possible*

$$= 6 + 12 + 18 + 24 = 60$$

*Expand outer sum*

Computing the sum of a geometric series by hand is time consuming...

Would you **really** want to calculate  $\sum_{j=0}^{20} (6 \times 2^j)$  by hand?

Fortunately, we have a **closed-form solution** for computing the sum of a geometric series:

$$\sum_{j=0}^n ar^j = \begin{cases} \frac{ar^{n+1} - a}{r-1} & \text{if } r \neq 1 \\ (n+1)a & \text{if } r = 1 \end{cases}$$

$$\text{So } \sum_{j=0}^{20} (6 \times 2^j) = \frac{6 \times 2^{21} - 6}{2 - 1} = 12,582,906$$

**Why?**



# Proof of geometric series closed form

On Whiteboard

There are other closed form summations that you should know

<i>Sum</i>	<i>Closed Form</i>

# Final thoughts

- **Sequences** allow us to represent (potentially infinite) ordered lists of elements
- **Summation notation** is a compact representation for adding together the elements of a sequence
- Next time:
  - Midterm exam review