CS 441: Infinite Cardinalities

PhD. Nils Murrugarra-Llerena
nem177@pitt.edu

University of Pittsburgh

Today's topics

- Defining cardinality for infinite sets
  - How can sequences help?
  - Countability and proving sets countable
  - Proving a set uncountable

We can use the notion of sequences to analyze the cardinality of infinite sets

Definition: Two sets A and B have the same cardinality if and only if there is a one-to-one correspondence (a bijection) from A to B.

Definition: A finite set or a set that has the same cardinality as the natural numbers (or the positive integers) is called countable. A set that is not countable is called uncountable.

Implication: Any sequence $\{a_n\}$ ranging over the natural numbers is countable.

Yes, the cardinalities of the natural numbers and positive integers are the same!

f: **N** → $\mathbf{Z}^+$, f(x) = x + 1

- This maps natural numbers to positive integers
- Every positive integer k (codomain) is mapped by natural number k-1 [surjection]
- No two natural numbers have the same mapping [injection]
  - That is, if x+1 = y+1, then x = y
- Thus, f is a bijection, and |**N**| = |$\mathbf{Z}^+$|
- Both have cardinality **countably infinite**
- Even though **N** contains 0 and $\mathbf{Z}^+$ does not, cardinality is equal

What about **Z**?

- *Seemingly* twice as many elements as $\mathbf{Z}^+$
- Exercise on the board

Yes, the cardinalities of the natural numbers and positive integers are the same!

f: **Z** → **Z⁺**,

$$f(x) = \begin{cases} 2x, & \text{if } x > 0, \\ 1, & \text{if } x = 0, \\ -2x + 1, & \text{if } x < 0. \end{cases}$$

- This maps integers to positive integers
- Every positive integer k (codomain) is mapped by interleaved positive/negative integers [surjection]
- No two integer numbers have the same mapping [injection]
  - That is, if f(x) = f(y), then x = y
- Thus, f is a bijection, and |**Z**| = |$\mathbf{Z}^+$|
- Both have cardinality **countably infinite**
- Even though **Z** contains 0 and negative numbers, and $\mathbf{Z}^+$ does not, cardinality is equal

Show that the set of even positive integers is countable

Proof #1 (Graphical): We have the following one-to-one correspondence between the positive integers and the even positive integers:

So, the even positive integers are countable. ❑

Proof #2: We can define the even positive integers as the sequence $\{2k\}$ for all $k \in \mathbf{Z}^+$, so it has the same cardinality as $\mathbf{Z}^+$, and is thus countable. ❑

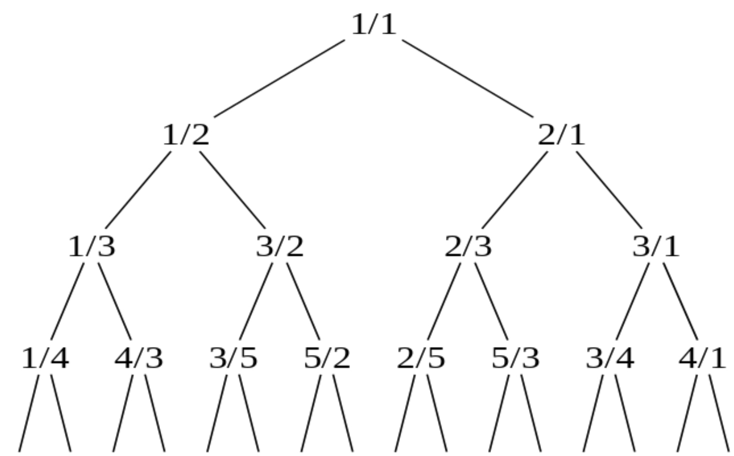
Surprisingly, the set of positive rationals is also countable

Consider a binary tree of rationals, with root node $\frac{1}{1}$

- For each node containing $\frac{a}{b}$, let its children be $\frac{a}{a+b}$ and $\frac{a+b}{b}$

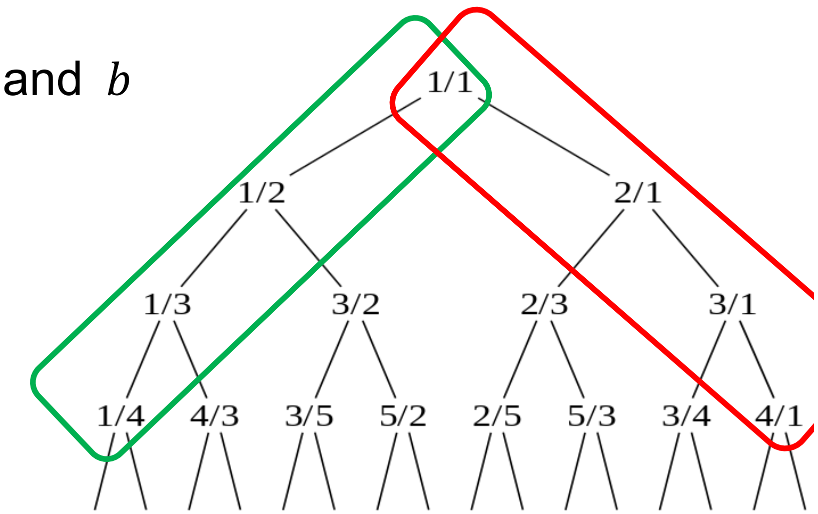
Traverse this tree in level-order fashion, assigning to the natural numbers in order

- i.e., go across the first level, then second level, etc.
- $\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{3}{2}, \frac{2}{3}, \frac{3}{1}, \dots$
- We just need to show that all positive rational numbers appear exactly once

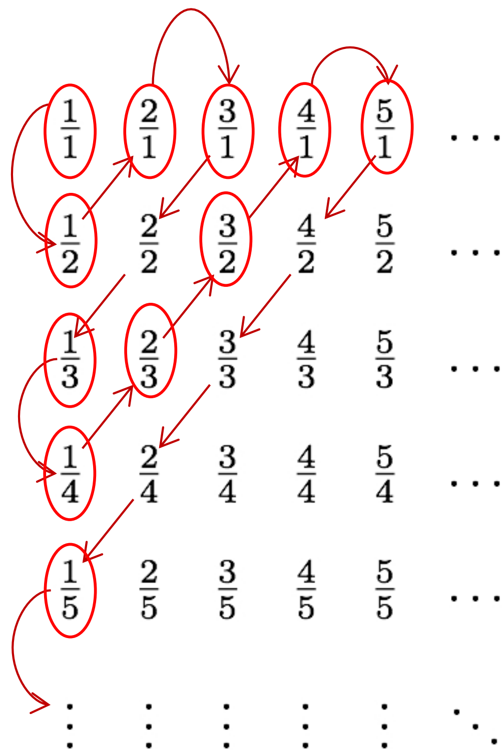


Proof sketch that Calkin-Wilf tree contains every positive rational

- First, note that every child has a larger sum of numerator + denominator than its parent
- Consider an arbitrary positive rational, $\frac{a}{b}$, where a and b are positive integers
  - If $\frac{a}{b} = 1$ and thus $a = b$:
    - This is the root, so it is in the tree
  - If $\frac{a}{b} < 1$ and thus $a < b$:
    - This would be the left child of $\frac{a}{b-a}$, also a positive rational
  - If $\frac{a}{b} > 1$ and thus $a > b$:
    - This would be the right child of $\frac{a-b}{b}$, also a positive rational
  - Since all non-root cases have a parent that is closer to $\frac{1}{1}$, repeatedly applying this logic will eventually reach the root
    - Analyze most to the left, and most to the right branches
    - Apply this logic for all intermediate node in the tree



Another way to show the rationals are countable



This yields the sequence 1/1, 1/2, 2/1, 3/1, 1/3, …, so the set of rational numbers is countable. ❑

Is the set of real numbers countable?

No, it is not. We can prove this using a proof method called diagonalization, invented by Georg Cantor.

Proof: Assume that the set of real numbers is countable. Then the subset of real numbers between 0 and 1 is also countable, by definition. This implies that the real numbers can be listed in some order, say, *r1, r2, r3 ….*

Let the decimal representation these numbers be:

$r1 = 0.d_{11}d_{12}d_{13}d_{14}\ldots$

$r2 = 0.d_{21}d_{22}d_{23}d_{24}\ldots$

$r3 = 0.d_{31}d_{32}d_{33}d_{34}\ldots$

…

Where $d_{ij} \in \{0,1,2,3,4,5,6,7,8,9\}\ \forall i,j$

Proof (continued)

Now, form a new decimal number $r=0.d_1d_2d_3\ldots$ where $d_i = 0$ if $d_{ii} = 1$, and $d_i=1$ otherwise.

Example:

$r_1 = 0.123456\ldots$

$r_2 = 0.234524\ldots$

$r_3 = 0.631234\ldots$

…

$r = 0.010\ldots$

Note that the i^{th} decimal place of r differs from the i^{th} decimal place of each r_i, by construction. Thus r is not included in the list of all real numbers between 0 and 1. This is a contradiction of the assumption that all real numbers between 0 and 1 could be listed. Thus, not all real numbers can be listed, and **R** is uncountable. ❑

Note: r can not be the same number as r_1, r_2, r_3, …. because it already has a different digit.

Final thoughts

- We can use sequences to help us compare the cardinality of infinite sets
  - Prove a set is countable by demonstrating a bijection to another countable set
  - Prove a set uncountable using diagonalization

- Next time:
  - Algorithms (Section 3.1)