CS 441: Divisibility and Modular Arithmetic

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Today's topics

- Integers and division
  - The division algorithm
  - Modular arithmetic
  - Applications of modular arithmetic

What is number theory?

Number theory is the branch of mathematics that explores the integers and their properties.

Number theory has many applications within computer science, including:

- Organizing data
- Encrypting sensitive data
- Developing error correcting codes
- Generating “random” numbers
- …

We will only scratch the surface…

The notion of divisibility is one of the most basic properties of the integers

Definition: If a and b are integers and $a \neq 0$, we say that a divides b iff there is an integer c such that $b = ac$. We write $a \mid b$ to say that a divides b, and $a \nmid b$ to say that a does not divide b.

Mathematically: $a \mid b \Leftrightarrow \exists\, c \in \mathbf{Z}\ (b = ac)$

Note: If $a \mid b$, then

- a is called a factor of b
- b is called a multiple of a

We've been using the notion of divisibility all along!

- $E = \{x \mid x = 2k \wedge k \in \mathbf{Z}\}$

Division examples

Examples:

- Does $4 \mid 16$?
- Does $3 \mid 11$?
- Does $7 \mid 42$?

Question: Let n and d be two positive integers. How many positive integers not exceeding n are divisible by d?

Answer: We want to count the number of integers of the form dk that are no more than n. That is, we want to know the number of integers k with $0 < dk \leq n$, or $0 < k \leq n/d$. Therefore, there are $\lfloor n/d \rfloor$ positive integers not exceeding n that are divisible by d.

Important properties of divisibility

Property 1: If $a \mid b$ and $a \mid c$, then $a \mid (b + c)$

Property 2: If $a \mid b$, then $a \mid bc$ for all integers c.

Property 3: If $a \mid b$ and $b \mid c$, then $a \mid c$.

Division algorithm

Theorem: Let a be an integer and let d be a positive integer. There are unique integers q and r, with

$$0 \leq r < d, \text{ such that } a = dq + r.$$

For historical reasons, the above theorem is called the division algorithm, even though it isn't an algorithm!

Terminology: Given $a = dq + r$

- a is called the dividend
- d is called the divisor
- q is called the quotient
- r is called the remainder
- $q = a$ **div** d
- $r = a$ **mod** d

div and mod are operators

Examples

Question: What are the quotient and remainder when 123 is divided by 23?

Answer: We have that 123 = 23 × 5 + 8. So the quotient is 123 **div** 23 = 5, and the remainder is 123 **mod** 23 = 8.

Question: What are the quotient and remainder when -11 is divided by 3?

Answer: Since -11 = 3 × -4 + 1, we have that the quotient is -4 and the remainder is 1.

Recall that since the remainder must be non-negative, 3 × -3 − 2 is not a valid use of the division theorem!

Many programming languages use the **div** and **mod** operations

For example, in Java, C, and C++

- / corresponds to **div** when used on integer arguments
- % corresponds to **mod**

```
public static void main(String[] args)
{
    int x = 2;
    int y = 5;
    float z = 2.0;

    System.out.println(y/x);
    System.out.println(y%x);
    System.out.println(y/z);
}
```

Prints out 2, not 2.5! (y/x)

Prints out 1 (y%x)

Prints out 2.5 (y/z)

This can be a source of many errors, so be careful in your future classes!

In-class exercises

Problems 1–4: On Top Hat

Sometimes, we care only about the remainder of an integer after it is divided by some other integer

Example: What time will it be 22 hours from now (11:00 am)?

Answer: If it is 11 am now, it will be (11 + 22) **mod** 24 = 33 **mod** 24 = 9 am in 22 hours.

Since remainders can be so important, they have their own special notation!

Definition: If *a* and *b* are integers and *m* is a positive integer, we say that *a* is congruent to *b* modulo *m* iff *m* | (a – b). We write this as $a \equiv b \pmod{m}$.

Note: $a \equiv b \pmod{m}$ iff *a* **mod** *m* = *b* **mod** *m*.

Examples:

- Is 17 congruent to 5 modulo 6?
- Is 24 congruent to 14 modulo 6?

Properties of congruencies

Theorem: Let m be a positive integer. The integers a and b are congruent modulo m ($a \equiv b \pmod{m}$) iff there is an integer k such that $a = b + km$.

Theorem: Let m be a positive integer. If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then

- $(a + c) \equiv (b + d) \pmod{m}$
- $ac \equiv bd \pmod{m}$

These properties mean we can use addition and multiplication as usual, even if we are only considering remainders (mod m)

Proving one of these congruence rules

WTP: If m is a positive integer, $a \equiv b \pmod{m}$, and $c \equiv d \pmod{m}$, then $(a + c) \equiv (b + d) \pmod{m}$

Assume $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$

- So $m \mid (a - b)$ and $m \mid (c - d)$
- This means $a - b = km$ and $c - d = jm$, for some integers k and j

Consider $(a + c) - (b + d) = (a - b) + (c - d) = km + jm = m(j + k)$

- Thus, $m \mid \big((a + c) - (b + d)\big)$
- This means $(a + c) \equiv (b + d) \pmod{m}$. $\square$

WTP: Want to Prove

Defining arithmetic restricted to remainders when dividing by m

$\mathbf{Z}_m$ denotes the set of nonnegative integers less than m

- i.e., the remainders when dividing by m

From our previous theorem we can show that **mod** "preserves" addition and multiplication

- (a+b) **mod** m = ((a **mod** m) + (b **mod** m)) **mod** m
- (ab) **mod** m = ((a **mod** m)(b **mod** m)) **mod** m

Thus, we can define versions of addition and multiplication that are restricted to this set

- $a +_m b$ = (a+b) **mod** m
- $a \cdot_m b$ = $(a \cdot b)$ **mod** m

These operations form arithmetic modulo m

Examples

Use the definition of addition and multiplication in $\mathbf{Z}_m$ to evaluate each of the following.

$6 +_{12} 10$

- $6 +_{12} 10 = (6 + 10)$ **mod** 12
- $= 16$ **mod** $12 = 4$

$6 \cdot_{12} 10$

- $6 \cdot_{12} 10 = (6 \cdot 10)$ **mod** 12
- $= 60$ **mod** $12 = 0$

$15 \cdot_{12} 22$

- $15 \cdot_{12} 22 = (15 \cdot 22)$ **mod** 12
- $= (3 \cdot 10)$ **mod** 12
- $= 30$ **mod** $12 = 6$

Congruencies have many applications within computer science

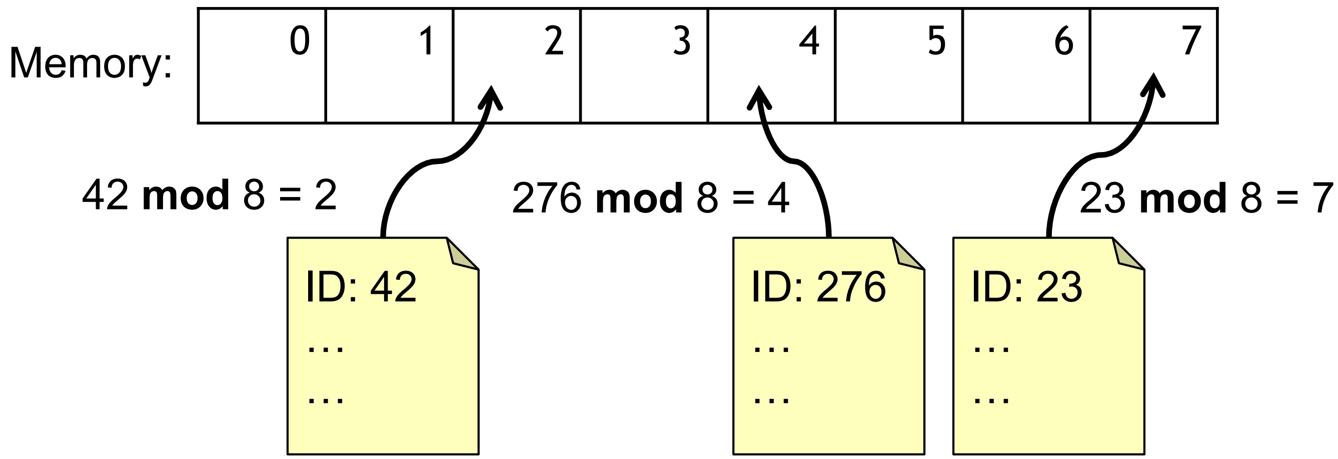
Today we'll look at three:

1. Hash functions
2. The generation of pseudorandom numbers
3. Cryptography

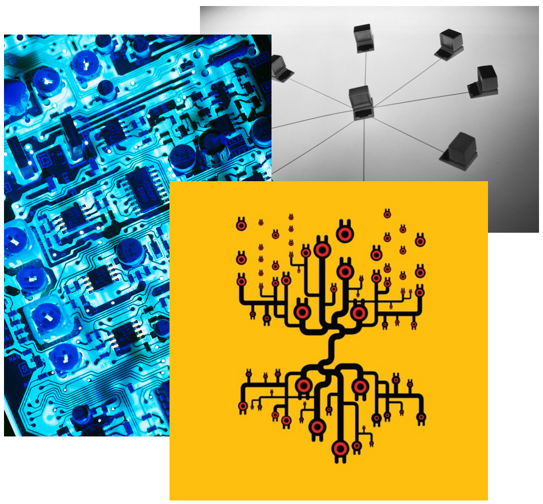
Hash functions allow us to quickly and efficiently locate data

Problem: Given a large collection of records, how can we find the one we want quickly?

Solution: Apply a hash function that determines the storage location of the record based on the record's ID. A common hash function is $h(k) = k \textbf{ mod } n$, where n is the number of available storage locations.

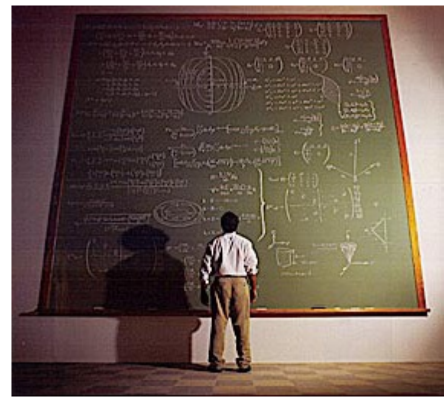


Many areas of computer science rely on the ability to generate pseudorandom numbers

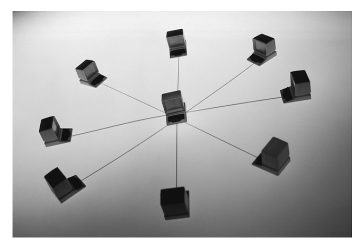


Hardware, software, and network simulation

Security



Coding algorithms



Network protocols

Congruencies can be used to generate pseudorandom sequences

Step 1: Choose

- A modulus m
- A multiplier a
- An increment c
- A seed x_0

Step 2: Apply the following

- $x_{n+1} = (ax_n + c)$ **mod** m

Example: $m = 9$, $a = 7$, $c = 4$, $x_0 = 3$

- $x_1 = (7x_0 + 4) \textbf{ mod } 9 = (7 \times 3 + 4) \textbf{ mod } 9 = 25 \textbf{ mod } 9 = 7$
- $x_2 = (7x_1 + 4) \textbf{ mod } 9 = (7 \times 7 + 4) \textbf{ mod } 9 = 53 \textbf{ mod } 9 = 8$
- $x_3 = (7x_2 + 4) \textbf{ mod } 9 = (7 \times 8 + 4) \textbf{ mod } 9 = 60 \textbf{ mod } 9 = 6$
- $x_4 = (7x_3 + 4) \textbf{ mod } 9 = (7 \times 6 + 4) \textbf{ mod } 9 = 46 \textbf{ mod } 9 = 1$
- $x_5 = (7x_4 + 4) \textbf{ mod } 9 = (7 \times 1 + 4) \textbf{ mod } 9 = 11 \textbf{ mod } 9 = 2$
- …

The field of cryptography makes heavy use of number theory and congruencies

Cryptography is the study of secret messages

Uses of cryptography:

- Protecting medical records
- Storing and transmitting military secrets
- Secure web browsing
- …

Congruencies are used in cryptosystems from antiquity, as well as in modern-day algorithms

Since modern algorithms require quite a bit of background to discuss, we'll examine an ancient cryptosystem

The Caesar cipher is based on congruencies

To encode a message using the Caesar cipher:

- Choose a shift index s
- Convert each letter A-Z into a number 0-25
- Compute $f(p) = (p + s)$ **mod** 26

Example: Let $s = 9$. Encode "ATTACK".

- ATTACK = 0 19 19 0 2 10
- $f(0) = 9$, $f(19) = 2$, $f(2) = 11$, $f(10) = 19$
- Encrypted message: 9 2 2 9 11 19 = JCCJLT

Decryption involves using the inverse function

That is, $f^{-1}(p) = (p - s)$ **mod** 26

Example: Assume that $s = 3$. Decrypt the message “UHWUHDW”.

- UHWUHDW = 20 7 22 20 7 3 22
- $f^{-1}(20) = 17$, $f^{-1}(7) = 4$, $f^{-1}(22) = 19$, $f^{-1}(3) = 0$
- Decrypted result: 17 4 19 17 4 0 19 = RETREAT

In-class exercises

Problem 5–7: On Top Hat

Final thoughts

- Number theory is the study of integers and their properties
- Divisibility, modular arithmetic, and congruency are used throughout computer science
  - Later in Chapter 4 we will see how these are used throughout computer science
- Next time:
  - Integer representations (Section 4.2)