# CS 441: Primes, GCDs, and LCMs

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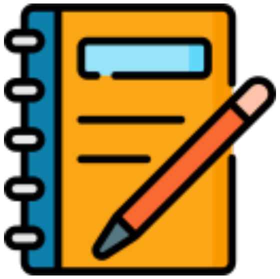
# Today's topics

- Primes & Greatest Common Divisors
  - Prime factorizations
  - Important theorems about primality
  - Greatest Common Divisors
  - Least Common Multiples
  - Euclid’s algorithm



# Let’s (finally) define the primes formally

***Definition:*** A prime number is a positive integer *p* greater than 1 that is divisible by only 1 and itself. If a number is not prime, it is called a composite number.

***Mathematically:*** $p$ is prime $\Leftrightarrow p>1 \land \forall\, x \in \mathbf{Z}^+ [(x \neq 1 \land x \neq p) \rightarrow x \nmid p]$

***Examples:*** Are the following numbers prime or composite?

- 23
- 42
- 17
- 3
- 9

# Any positive integer can be represented as a unique product of prime numbers!

***Theorem*** (The Fundamental Theorem of Arithmetic)**:** Every positive integer greater than 1 can be written uniquely as a prime or the product of two or more primes where the prime factors are written in order of non-decreasing size.

***Examples:***

- $100 = 2 \times 2 \times 5 \times 5 = 2^2 \times 5^2$
- $641 = 641$
- $999 = 3 \times 3 \times 3 \times 37 = 3^3 \times 37$
- $1024 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^{10}$

Note: Proving the fundamental theorem of arithmetic requires some mathematical tools that we have not yet learned.

# This leads to a related theorem…

***Theorem:*** If n is a composite integer, then n has a prime divisor less than or equal to $\sqrt{n}$.

***Proof:***

- If n is composite, then it has a positive integer factor a with $1 < a < n$ by definition. This means that $n = ab$, where b is an integer greater than 1.
- Assume $a > \sqrt{n}$ and $b > \sqrt{n}$. Then $ab > \sqrt{n}\sqrt{n} = n$, which is a contradiction. So either $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$.
- Thus, n has a divisor less than or equal to $\sqrt{n}$.
- By the fundamental theorem of arithmetic, this divisor is either prime, or is a product of primes. In either case, n has a prime divisor less than or equal to $\sqrt{n}$. ❑

# Applying contraposition leads to a naive primality test

***Corollary:*** If n is a positive integer that does not have a prime divisor less than or equal to $\sqrt{n}$, then n is prime.

***Example:*** Is 101 prime?

- The primes less than or equal to $\sqrt{101}$ are 2, 3, 5, and 7
- Since 101 is not divisible by 2, 3, 5, or 7, it must be prime

***Example:*** Is 1147 prime?

- The primes less than or equal to $\sqrt{1147}$ are 2, 3, 5, 7, 11, 13, 17, 23, 29, and 31
- $1147 = 31 \times 37$, so 1147 must be composite

# This approach can be generalized

The Sieve of Eratosthenes is a brute-force algorithm for finding all prime numbers less than some value *n*

*Step 1:* List the numbers less than *n*

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 2 | 3 | ✗ | 5 | ✗ | 7 | ✗ | ✗ | ✗ | 11 |
| ✗ | 13 | ✗ | ✗ | ✗ | 17 | ✗ | 19 | ✗ | ✗ |
| ✗ | 23 | ✗ | ✗ | ✗ | ✗ | ✗ | 29 | ✗ | 31 |
| ✗ | ✗ | ✗ | ✗ | ✗ | 37 | ✗ | ✗ | ✗ | 41 |
| ✗ | 43 | ✗ | ✗ | ✗ | 47 | ✗ | ✗ | ✗ | ✗ |
| ✗ | 53 | ✗ | ✗ | ✗ | ✗ | ✗ | 59 | ✗ | 61 |
| ✗ | ✗ | ✗ | ✗ | ✗ | 67 | ✗ | ✗ | ✗ | 71 |

*Step 2:* If the next available number is less than √n, cross out all of its multiples

*Step 3:* Repeat until the next available number is > √n

*Step 4:* All remaining numbers are prime

# How many primes are there?

***Theorem:*** There are infinitely many prime numbers.

***Proof:*** By contradiction

- Assume that there are only a finite number of primes $p_1, \ldots, p_n$
- Let $Q = p_1 \times p_2 \times \ldots \times p_n + 1$
- By the fundamental theorem of arithmetic, Q can be written as the product of two or more primes.
- Q is not divisible by any of the primes $p_1, p_2, p_3, \ldots, p_n$ because dividing Q by any of these primes would leave a remainder of 1.
- Since Q is not divisible by any of the previous prime number,there must be some prime number not in our list. This prime number is either Q (if Q is prime) or a prime factor of Q (if Q is composite).
- This is a contradiction since we assumed that all primes were listed. Therefore, there are infinitely many primes. ❑

***This is a non-constructive existence proof!***

# In-class exercises

**Problem 1:** What is the prime factorization of 984?

**Problem 2:** Is 157 prime? Is 97 prime?

**Problem 3:** Is the set of all prime numbers countable or uncountable? If it is countable, show a 1-to-1 correspondence between the prime numbers and the natural numbers.

# Greatest common divisors

***Definition:*** Let $a$ and $b$ be integers, not both zero. The largest integer $d$ such that $d \mid a$ and $d \mid b$ is called the greatest common divisor of $a$ and $b$, denoted by $\gcd(a, b)$.

Note: We can (naively) find GCDs by comparing the common divisors of two numbers.

***Example:*** What is the GCD of 24 and 36?

- Factors of 24: 1, 2, 3, 4, 6, 8, 12, 24
- Factors of 36: 1, 2, 3, 4, 6, 9, 12, 18, 36
- ∴ gcd(24, 36) = 12

# Sometimes, the GCD of two numbers is 1

***Example:*** What is gcd(17, 22)?

- Factors of 17: 1, 17
- Factors of 22: 1, 2, 11, 22
- ∴ gcd(17, 22) = 1

***Definition:*** If gcd($a$, $b$) = 1, we say that $a$ and $b$ are relatively prime, or coprime. We say that $a_1$, $a_2$, …, $a_n$ are pairwise relatively prime if gcd($a_i$, $a_j$) = 1 $\forall i, j$.

***Example:*** Are 10, 17, and 21 pairwise coprime?

- Factors of 10: 1, 2, 5, 10
- Factors of 17: 1, 17
- Factors of 21: 1, 3, 7, 21

# We can leverage the fundamental theorem of arithmetic to develop a better algorithm

***Let:*** $a = p_1^{a_1} p_2^{a_2} \cdots p_n^{a_n}$ and $b = p_1^{b_1} p_2^{b_2} \cdots p_n^{b_n}$

***Then:***

$$gcd(a,b) = p_1^{min(a_1,b_1)} p_2^{min(a_2,b_2)} \cdots p_n^{min(a_n,b_n)}$$

***Greatest multiple of $p_1$ in both a and b***

***Greatest multiple of $p_2$ in both a and b***

***Example:*** Compute gcd(120, 500)

- $120 = 2^3 \times 3 \times 5$
- $500 = 2^2 \times 5^3$
- So $gcd(120, 500) = 2^2 \times 3^0 \times 5 = 20$

# Better still is Euclid's algorithm

***Observation:*** If $a = bq + r$, then $\gcd(a, b) = \gcd(b, r)$

*Proved in section 4.3 of the book*

So, let $r_0 = a$ and $r_1 = b$. Then:

- $r_0 = r_1 q_1 + r_2$ $\quad 0 \le r_2 < r_1$
- $r_1 = r_2 q_2 + r_3$ $\quad 0 \le r_3 < r_2$
- …
- $r_{n-2} = r_{n-1} q_{n-1} + r_n$ $\quad 0 \le r_n < r_{n-1}$
- $r_{n-1} = r_n q_n$

$\gcd(a, b) = r_n$

# Examples of Euclid's algorithm

***Example:*** Compute gcd(414, 662)

- 662 = 414 × 1 + 248
- 414 = 248 × 1 + 166
- 248 = 166 × 1 + 82
- 166 = 82 × 2 + 2
- 82 = 2 × 41

gcd(414, 662) = 2

***Example:*** Compute gcd(9888, 6060)

- 9888 = 6060 × 1 + 3828
- 6060 = 3828 × 1 + 2232
- 3828 = 2232 × 1 + 1596
- 2232 = 1596 × 1 + 636
- 1596 = 636 × 2 + 324
- 636 = 324 × 1 + 312
- 324 = 312 × 1 + 12
- 312 = 12 × 26

gcd(9888, 6060) = 12

# Least common multiples

***Definition:*** The least common multiple of the integers *a* and *b*, where neither is 0, is the smallest positive integer that is divisible by both *a* and *b*. The least common multiple of *a* and *b* is denoted lcm(*a*, *b*).

***Example:*** What is lcm(3,12)?

- Multiples of 3: 3, 6, 9, 12, 15, …
- Multiples of 12: 12, 24, 36, …
- So lcm(3,12) = 12

Note: lcm(*a*, *b*) is guaranteed to exist, since a common multiple exists (i.e., *ab*).

# We can leverage the fundamental theorem of arithmetic to develop a better algorithm

***Let:*** $a = p_1^{a_1} p_2^{a_2} \cdots p_n^{a_n}$ and $b = p_1^{b_1} p_2^{b_2} \cdots p_n^{b_n}$

***Then:***

$$lcm(a, b) = p_1^{max(a_1,b_1)} p_2^{max(a_2,b_2)} \cdots p_n^{max(a_n,b_n)}$$

***Greatest multiple of $p_1$ in either a or b***

***Greatest multiple of $p_2$ in either a or b***

***Example:*** Compute lcm(120, 500)

- $120 = 2^3 \times 3 \times 5$
- $500 = 2^2 \times 5^3$
- So lcm(120, 500) = $2^3 \times 3 \times 5^3$ = 3000 << 120 × 500 = 60,000

# LCMs are closely tied to GCDs

Note: $ab = \text{lcm}(a, b) \times \text{gcd}(a, b)$

***Example:*** $a = 120 = 2^3 \times 3 \times 5$, $b = 500 = 2^2 \times 5^3$

- $120 = 2^3 \times 3 \times 5$
- $500 = 2^2 \times 5^3$
- $\text{lcm}(120, 500) = 2^3 \times 3 \times 5^3 = 3000$
- $\text{gcd}(120, 500) = 2^2 \times 3^0 \times 5 = 20$
- $\text{lcm}(120, 500) \times \text{gcd}(120, 500)$

✔

# In-class exercises

**Problem 4:** Use Euclid's algorithm to compute gcd(92928, 123552).

**Problem 5:** Compute gcd(24, 36) and lcm(24, 36). Verify that gcd(24, 36) × lcm(24, 36) = 24 × 36.

(Submit both on Top Hat)

# Final Thoughts

- Prime numbers play an important role in number theory
- There are an infinite number of prime numbers
- Any number can be represented as a product of prime numbers; this has implications when computing GCDs and LCMs
- Next time:  Solving congruences, modular inverses