CS 441: Informal Proofs

PhD. Nils Murrugarra-Llerena
nem177@pitt.edu
University of Pittsburgh

Today's topics

- Proof techniques
  - How can I prove an implication is true?
  - What forms can an informal proof take?
- Proof strategies
  - Which proof techniques should I try?
  - How do I find a proof without trying every proof technique?

Mathematical theorems are often stated in the form of an implication

Example: If $x > y$, where x and y are positive real numbers, then $x^2 > y^2$.

- $\forall x, y\ [(x > 0) \land (y > 0) \land (x > y) \rightarrow (x^2 > y^2)]$
- $\forall x, y\ \big(P(x, y) \rightarrow Q(x, y)\big)$

We will discuss three applicable proof methods:

- Direct proof
- Proof by contraposition
- Proof by contradiction

Direct proof

In a direct proof, we prove $p \rightarrow q$ by showing that if p is true, then q must necessarily be true

Example: Prove that if n is an odd integer, then n^3 is an odd integer.

Proof:

Direct proofs are not always the easiest way to prove a given conjecture.

In this case, we can try proof by contraposition

How does this work?

- Recall that $p \rightarrow q \equiv \neg q \rightarrow \neg p$
- Therefore, a proof of $\neg q \rightarrow \neg p$ is also a proof of $p \rightarrow q$

Proof by contraposition is an indirect proof technique since we don't prove $p \rightarrow q$ directly.

Let's take a look at an example…

Prove: If n is an integer and $\underbrace{n^2 \text{ is even}}_{p}$, then $\underbrace{n \text{ is even}}_{q}$.

First, attempt a direct proof:

- Assume that n^2 is even, thus $n^2 = 2k$ for some integer k
- Can solve to find that $n = \sqrt{2k}$
- … ?

Now, try proof by contraposition:

- Assume $\underbrace{n \text{ is odd}}_{\neg q}$, thus $n = 2k + 1$ for some integer k
- $n^2 = (2k+1)^2 = 4k^2 + 4k + 1$
- $= \underbrace{2(2k^2 + 2k) + 1}_{\neg p}$
- Thus, n^2 is odd, and we proved ¬"n is even" → ¬"n^2 is even", we can conclude that "n^2 is even" → "n is even" ☐

Proof by contradiction

Given a conditional $p \rightarrow q$, the only way to reject this claim is to prove that $p \land \neg q$ is true.

In a proof by contradiction we:

1. Assume that $p \land \neg q$ is true
2. Proceed with the proof
3. If this assumption leads us to a contradiction, we can conclude that $p \rightarrow q$ is true

Prove: If n is an integer and $\underbrace{3n + 2 \text{ is odd}}_{p}$, then $\underbrace{n \text{ is odd}}_{q}$.

Proof:

- Assume the negation, that $\underbrace{3n + 2 \text{ is odd}}_{p}$, $\underbrace{n \text{ is even (i.e., } n = 2k)}_{\neg q}$
- 3n + 2 = 3(2k) + 2 = 6k + 2 = 2(3k + 1)
- The above statement tells us that 3n + 2 is even, which is a contradiction of our assumption that 3n + 2 is odd.
- Therefore, we have shown that if 3n + 2 is odd, then n is also odd. ☐

Note: If we prove this by contraposition, we use a lot of the same mechanics/algebra!

- But the line of argumentation is different

We can also use proof by contradiction in cases where were the theorem to be proved is not of the form p → q

Prove: At least 10 of any 64 dates fall on the same day of the week

Proof:

- Let p ≡ “At least 10 of any 64 dates fall on the same day of the week”
- Assume ¬p is true, that is “64 dates can be chosen such that at most 9 fall on the same day of the week”
- When choosing dates, since there are 7 days in a week, at most $7 \times 9 = 63$ dates can be chosen in total
- This is a contradiction of the statement that we can chose 64 dates
- Therefore, the assumption is false, and we can conclude that at least 10 of any 64 dates fall on the same day of the week. ☐

This proof is an example of the <u>pigeonhole principle</u>, which we will study during our combinatorics unit.

In-class exercises

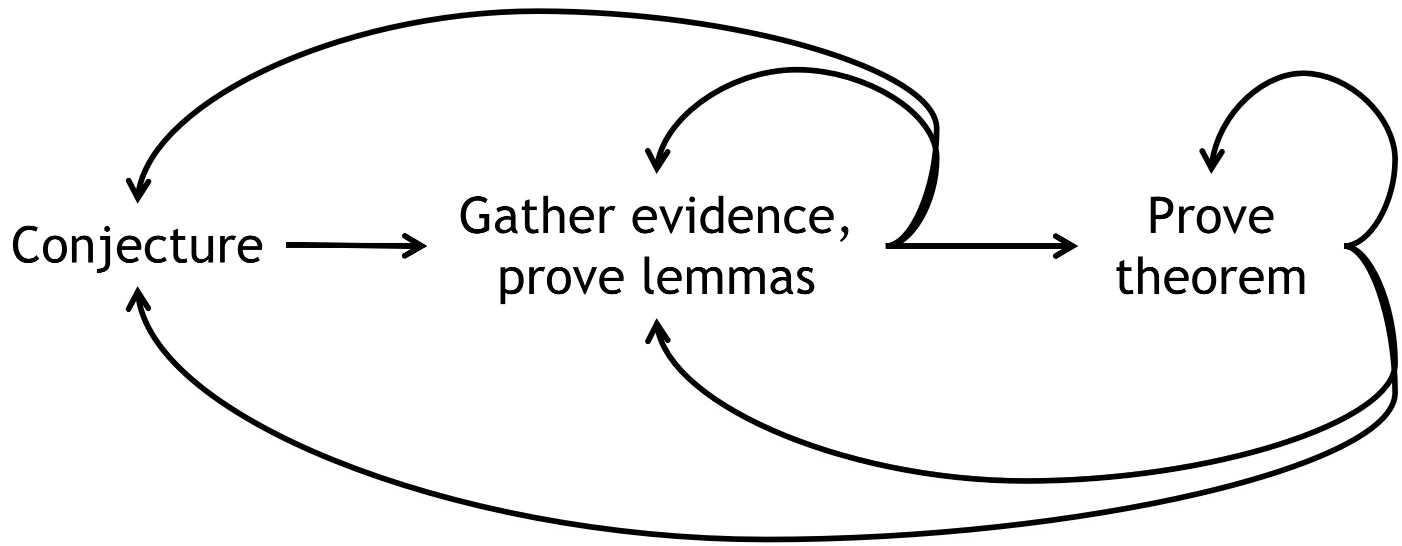
Problem 1: Prove the following claims

a) Use a direct proof to show that the square of an even number is an even number.
   - Note that we proved the converse earlier, so this will prove it is an “if and only if” relationship

b) Use proof by contraposition to show that if n is an integer and $n^3 + 5$ is odd, then n is even.

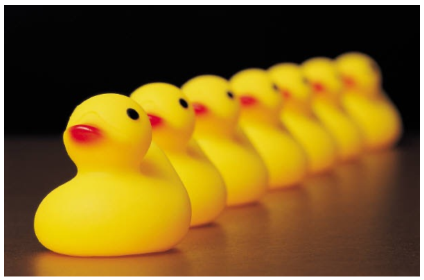
Problem 2: On Top Hat

The scientific process is not always straightforward…



The scientific process is not always straightforward…

Proof strategies help us…



Organize our problem solving approach

Effectively use all of the tools at our disposal

Develop a coherent plan of attack

Types of proof strategy

Today we’ll discuss four types of strategy:

1. Forward reasoning
2. Backward reasoning
3. Searching for counterexamples
4. Adapting existing proofs

Sometimes forward reasoning doesn't work

In these cases, it is often helpful to reason backwards, starting with the goal that we want to prove.

Example: Prove that given two distinct positive real numbers x and y, the arithmetic mean of x and y is always greater than the geometric mean of x and y.

$(x + y)/2$

$\sqrt{(xy)}$

Sanity check: Let x=8 and y=4. $(8+4)/2 = 6$. $\sqrt{(8 \times 4)} = \sqrt{(32)} \cong 5.66$. $6 > 5.66$ ✓

Prove that $(x+y)/2 > \sqrt{(xy)}$ for all distinct pairs of positive real numbers x and y.

Proof:

$(x + y)/2 > \sqrt{(xy)}$

$(x + y)^2/4 > xy$

$(x + y)^2 > 4xy$

$x^2 + 2xy + y^2 > 4xy$

$x^2 - 2xy + y^2 > 0$

$(x - y)^2 > 0$

$(x - y) > 0$

$x > y$

Since $(x - y)^2 > 0$ whenever $x \neq y$, the final inequality is true. Since all of these inequalities are equivalent, it follows that $(x + y)/2 > \sqrt{(xy)}$. □

Other times, searching for a counterexample is helpful

Proof by counterexample is helpful if:

- Proof attempts repeatedly fail
- The conjecture to be proven looks “funny”

Example: Prove that every positive integer is the sum of two squares.

This seems suspicious to me, since other factorizations (e.g., prime factorizations) can be complex.

Counterexample:

3 is not the sum of two squares, so the claim is false. ☐

These four proof strategies are just a start!

A great tool for programmers AND logicians!

When trying to prove a new conjecture, a good "meta strategy" is to:

1. If possible, try to reuse an existing proof (analogy!)
2. If the conjecture looks fishy, check for a counterexample
3. Attempt a "real" proof
   a) Apply the forward reasoning strategy
   b) Or, apply the backward reasoning strategy
   c) Possibly alternate between forward and backward reasoning

Unfortunately, not every proof can be solved using this nice little meta strategy…

In fact, there are many, many proof strategies out there, and NONE of them can be guaranteed to find a proof!

Final Thoughts

- Proving theorems is not always straightforward
- Having several proof techniques at your disposal will make a huge difference in your success rate!
    - And use proof strategies to organize yourself
- Next lecture:
    - More proof techniques
    - Please read section 1.8