

CS 441: Functions

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[Motivation] Functions

The URL Router

When you type a URL into your browser, like `https://www.example.com/users/profile/123`, a web server needs to figure out which specific piece of code should handle that request. It needs to:

- Identify the part of the URL that's important (`/users/profile/123`).
- Extract any variables from the URL (in this case, the user ID `123`).
- Map this specific URL to a function that will generate the user's profile page.

This is a classic function problem: a well-defined input (the URL) needs to be mapped to a single, specific output (the correct function to execute).



Source: Gemini

Today's topics

- Set Functions
 - Important definitions
 - Relationships to sets, relations
 - Specific functions of particular importance

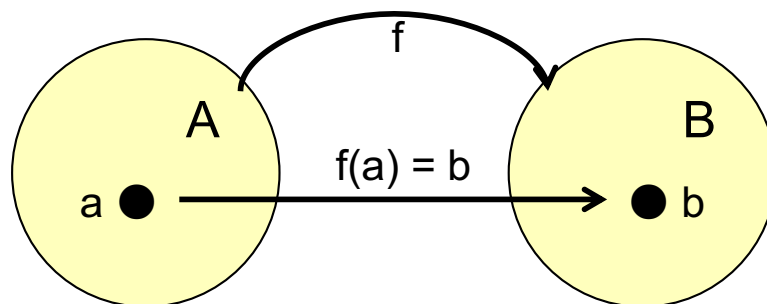


Sets give us a way to formalize the concept of a function

Definition: Let A and B be nonempty sets. A **function**, f , is an assignment of exactly one element of set B to each element of set A .

Note: We write $f : A \rightarrow B$ to denote that f is a function from A to B

Note: We say that $f(a) = b$ if the element $a \in A$ is mapped to the unique element $b \in B$ by the function f



Functions can be defined in a number of ways

1. Explicitly

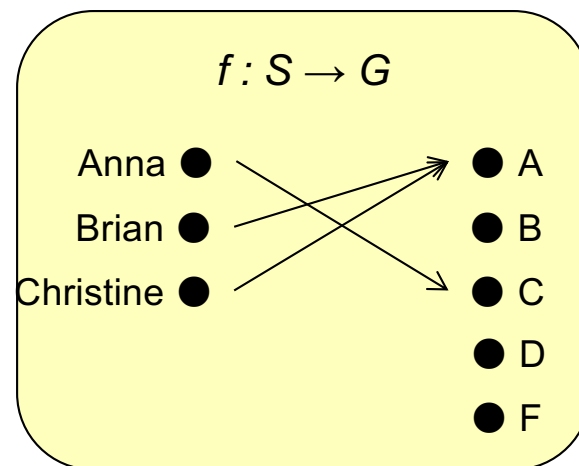
- $f: \mathbf{Z} \rightarrow \mathbf{Z}$
- $f(x) = x^2 + 2x + 1$

2. Using a programming language

- `int min(int x, int y) = { x < y ? return x : return y; }`

3. Using a relation

- Let $S = \{\text{Anna, Brian, Christine}\}$
- Let $G = \{A, B, C, D, F\}$



More terminology

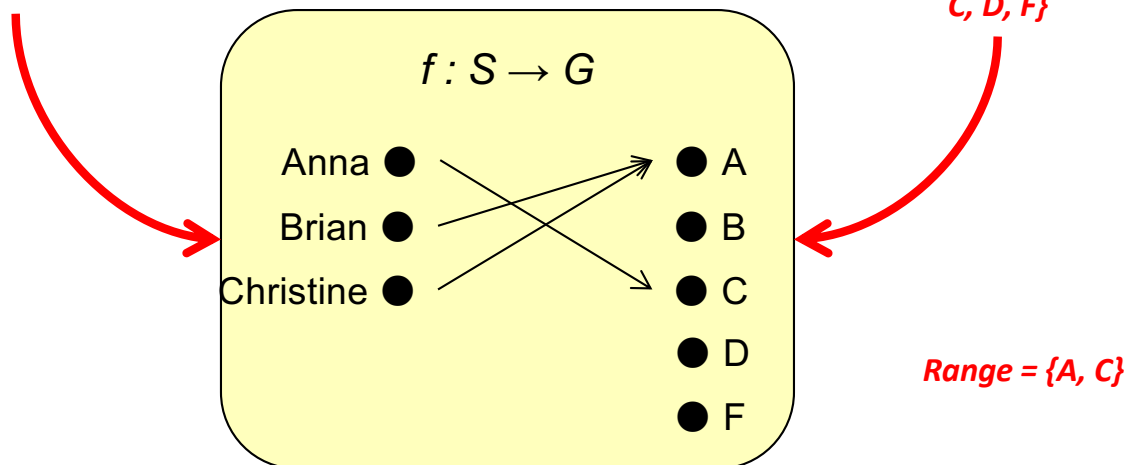
The **domain** of a function is the set that the function maps from, while the **codomain** is the set that is mapped to

If $f(a) = b$, b is called the **image** of a , and a is called the **preimage** of b

The **range** of a function $f : A \rightarrow B$ is the set of all images of elements of A

Domain = $S = \{Anna, Brian, Christine\}$

Codomain = $G = \{A, B, C, D, F\}$



What are the domain, codomain, and range of the following functions?

1. $f : \mathbf{Z} \rightarrow \mathbf{Z}, f(x) = x^3$

- Domain:
- Codomain:
- Range:

2. $g : \mathbf{R} \rightarrow \mathbf{R}, g(x) = x - 2$

- Domain:
- Codomain:
- Range:

3. `int foo(int x, int y) = { return (x*y)%2; }`

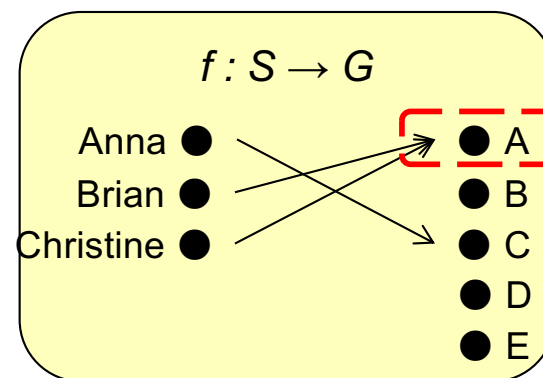
- Domain:
- Codomain:
- Range:

A one-to-one function never assigns the same image to two different elements

Definition: A function $f : A \rightarrow B$ is **one-to-one**, or **injective**, iff $\forall x, y \in A [(f(x) = f(y)) \rightarrow (x = y)]$

Are the following functions **injections**?

- $f : \mathbf{R} \rightarrow \mathbf{R}, f(x) = x + 1$
- $f : \mathbf{Z} \rightarrow \mathbf{Z}, f(x) = x^2$
- $f : \mathbf{R}^+ \rightarrow \mathbf{R}^+, f(x) = \sqrt{x}$
- $f : S \rightarrow G$

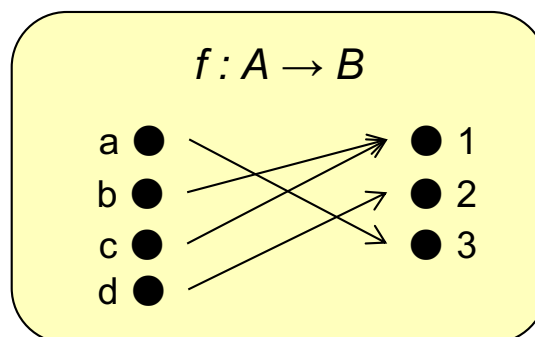


An onto function “uses” every element of its codomain

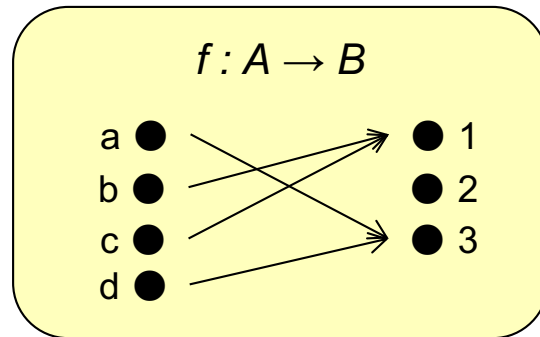
Definition: We call a function $f : A \rightarrow B$ **onto**, or **surjective**, iff for every element $b \in B$, there is some element $a \in A$ such that $f(a) = b$.

Think about an onto function as “covering” the entirety of its codomain.

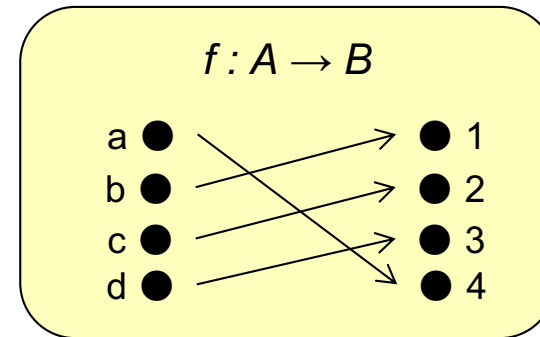
The following function is a **surjection**:



Are the following functions one-to-one, onto, both, or neither?

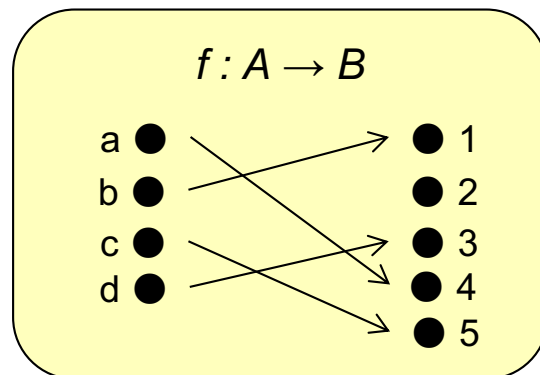


Neither!

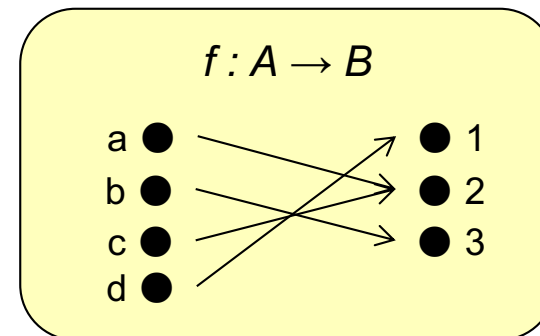


One-to-one and onto

(Aside: Functions that are both one-to-one and onto are called *bijections*)



One-to-one

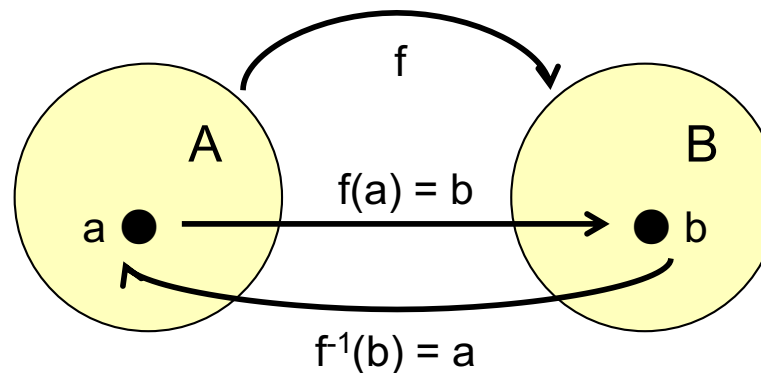


Onto

Bijections have inverses

Definition: If $f : A \rightarrow B$ is a bijection, the **inverse** of f is the function $f^{-1} : B \rightarrow A$ that assigns to each $b \in B$ the unique value $a \in A$ such that $f(a) = b$. That is, $f^{-1}(b) = a$ iff $f(a) = b$.

Graphically:



Note: Only a bijection can have an inverse. (Why?)

Reversal is only possible:

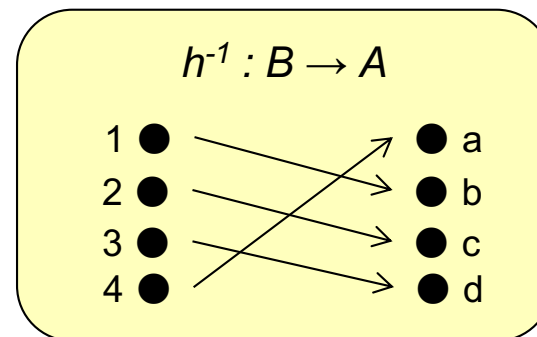
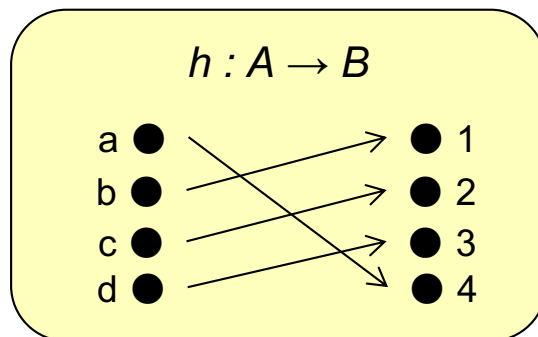
- Each element in the domain maps to a unique element in the codomain (**injective**).
- Every element in the codomain has a corresponding element in the domain (**surjective**).

Do the following functions have inverses?

1. $f : \mathbf{R} \rightarrow \mathbf{R}, f(x) = x^2$

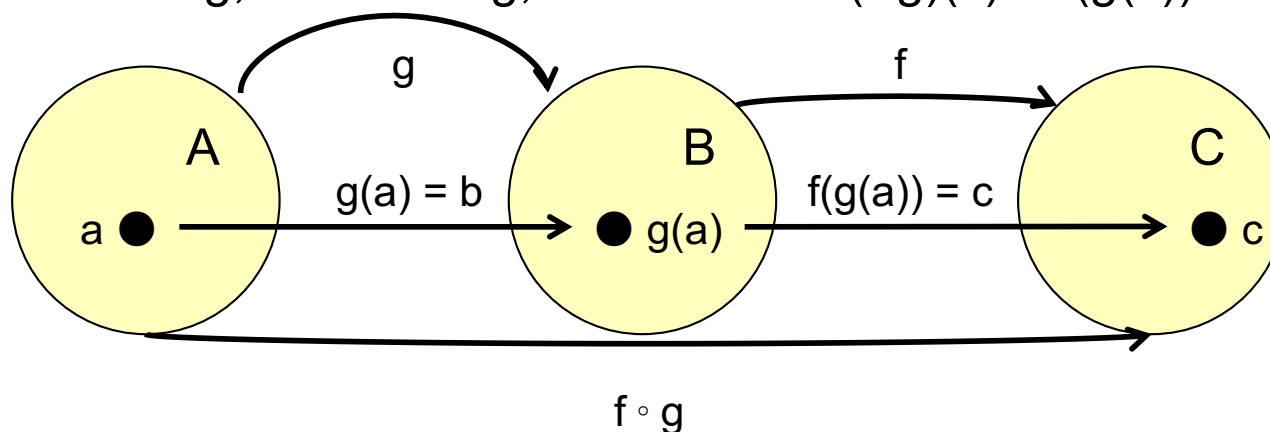
2. $g : \mathbf{Z} \rightarrow \mathbf{Z}, g(x) = x + 1$

3. $h : A \rightarrow B$



Functions can be composed with one another

Given functions $g : A \rightarrow B$ and $f : B \rightarrow C$, the **composition** of f and g , denoted $f \circ g$, is defined as $(f \circ g)(x) = f(g(x))$.



Note: For $f \circ g$ to exist, the codomain of g must be a subset of the domain of f .

Definition: If $g : A \rightarrow B$ and $f : D \rightarrow C$ and $B \subseteq D$, $f \circ g$ is a function $A \rightarrow C$ where $(f \circ g)(x) = f(g(x))$

Can the following functions be composed? If so, what is their composition?

Let $f : A \rightarrow A$ such that $f(a) = b$, $f(b) = c$, $f(c) = a$
 $g : B \rightarrow A$ such that $g(1) = b$, $g(4) = a$

1. $(f \circ g)(x)$?
2. $(g \circ f)(x)$?

Let $f : \mathbf{Z} \rightarrow \mathbf{Z}$, $f(x) = 2x + 1$
 $g : \mathbf{Z} \rightarrow \mathbf{Z}$, $g(x) = x^2$

1. $(f \circ g)(x)$?
2. $(g \circ f)(x)$?

Note: There is not a guarantee that $(f \circ g)(x) = (g \circ f)(x)$.

Important functions

Definition: The **floor** function maps a **real number** x to the **largest integer** y that is not greater than x . The floor of x is denoted $\lfloor x \rfloor$.

Definition: The **ceiling** function maps a **real number** x to the **smallest integer** y that is not less than x . The ceiling of x is denoted $\lceil x \rceil$.

Examples:

- $\lfloor 1.2 \rfloor = 1$
- $\lfloor 7.0 \rfloor = 7$
- $\lfloor -42.24 \rfloor = -43$
- $\lceil 1.2 \rceil = 2$
- $\lceil 7.0 \rceil = 7$
- $\lceil -42.24 \rceil = -42$

We actually use floor and ceiling quite a bit in computer science...

Example: A byte, which holds 8 bits, is typically the smallest amount of memory that can be allocated on most systems. How many bytes are needed to store 123 bits of data?

Answer: We need $\lceil 123/8 \rceil = \lceil 15.375 \rceil = 16$ bytes

Example: How many 1400 byte packets can be transmitted over a 14.4 kbps modem in one minute?

Answer: A 14.4 kbps modem can transmit $14,400 \times 60 = 864,000$ bits per minute. Therefore, we can transmit $\lceil 864,000 / (1400 \times 8) \rceil = \lceil 77.1428571 \rceil = 77$ packets.

In-class Activities

Activity 1: Find the **domain** and **range** of each of the following functions. [[miro](#)]

- a. The function that determines the number of zeros in some bit string
- b. The function that maps an English word to its two rightmost letters
- c. The function that assigns to an integer the sum of its individual digits

Activity 2: Suppose g is a function from A to B and f is a function from B to C . Prove that if f and g are one-to-one, then $f \circ g$ is one-to-one. [[miro](#)]

Activity 3: Solve our *URL router* case study. [[miro](#)]

Steps:

1. Introduce to a classmate
2. Work in pairs on the exercise
3. Submit answers on miro
4. Volunteers to share answers

Final thoughts

- Set identities are useful tools!
- We can prove set identities in a number of (equivalent) ways
- Sets are the basis of **functions**, which are used throughout computer science and mathematics
- Next time:
 - Summations (Section 2.4)