CS 441: Propositional Logic

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Motivation: Propositional Logic

- Let's play Tic-Tac-Toe
  https://playtictactoe.org/

- What is our strategy?
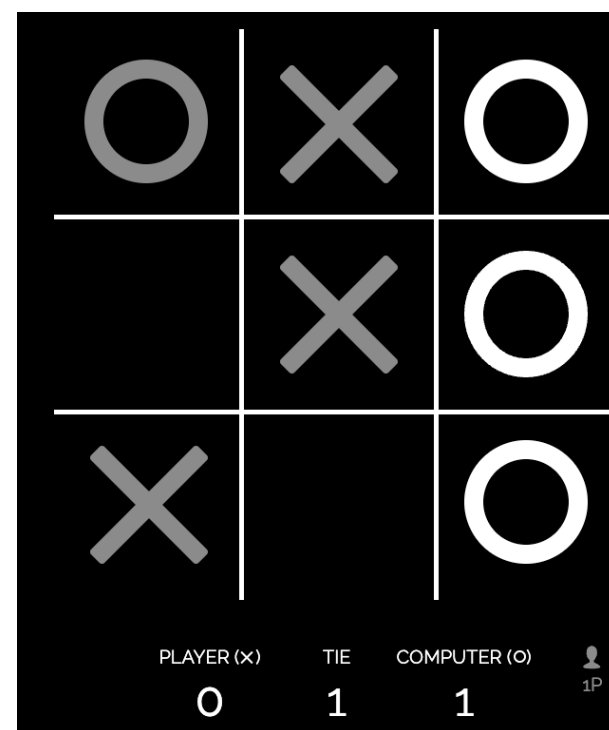
Today's Topic: Propositional Logic

- What is a proposition?

- Logical connectives and truth tables

- Translating between English and propositional logic

Logic is the basis of all mathematical and analytical reasoning

Given a collection of known truths, logic allows us to deduce new truths

Example

Base facts:
If it is raining, I will not go outside
If I am inside, Lisa will stay home
Lisa and I always play video games if we are together during the weekend
Today is a rainy Saturday

Conclusion: Lisa and I will play video games today

Logic allows us to advance mathematics through an iterative process
of conjecture and proof

Propositional logic is a very simple logic

Definition: A proposition is a precise statement that is either true or false, but not both.

Examples:

- 2 + 2 = 4 (true)
- All dogs have 3 legs (false)
- $x^2 < 0$ (false)
- Washington, D.C. is the capital of the USA (true)

Not all statements are propositions

- Eliana is cool
  - "Cool" is a subjective term.

- $x^3 < 0$
  - True if x < 0, false otherwise.

- Springfield is the capital
  - True in Illinois, false in Massachusetts.

We can use logical connectives to build complex propositions

We will discuss the following logical connectives:

- ¬ (not)
- ∧ (conjunction / and)
- ∨ (disjunction / or)
- ⊕ (exclusive disjunction / xor)
- → (implication)
- ↔ (biconditional)

Negation

The negation of a proposition is true iff the proposition is false

What we know

What we want
to know

One row for
each possible
value of "what
we know"

p	¬p

The truth table for negation

(I'll sometimes use ⊤ and ⊥)

Negation Examples

Negate the following propositions

- Today is Monday

- 21 * 2 = 42

What is the truth value of the following propositions

- ¬(9 is a prime number)

- ¬(Pittsburgh is in Pennsylvania)

Conjunction

The conjunction of two propositions is true iff both propositions are true

p	q	p ∧ q
T	T	
T	F	
F	T	
F	F	

The truth table for conjunction

2^2 = 4 rows since we know both p and q!

Disjunction

The disjunction of two propositions is true iff *at least one* proposition is true

p	q	p ∨ q
T	T	
T	F	
F	T	
F	F	

The truth table for disjunction

Conjunction and disjunction examples

*This symbol means "is defined as"
or "is equivalent to"
(sometimes seen as ≜)*

Let:

- $p \equiv x^2 \geq 0$ True
- $q \equiv$ A lion weighs less than a mouse False
- $r \equiv 10 < 7$ False
- $s \equiv$ Pittsburgh is located in Pennsylvania True

What are the truth values of these expressions:

- $p \wedge q$
- $p \wedge s$
- $p \vee q$
- $q \vee r$

In-class Activities
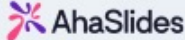
Activity 1: Let p ≡ 2+2=5, q ≡ eagles can fly, r ≡ 1=1. Determine the value for each of the following:

- p ∧ q
- ¬p ∨ q
- p ∨ (q ∧ r)
- (p ∨ q) ∧ (¬r ∨ ¬p)

Steps:
1. Introduce to a classmate
2. Work in pairs on the exercise
3. Submit answers

Exclusive or (XOR)

The exclusive or of two propositions is true iff *exactly one* proposition is true

p	q	p ⊕ q
T	T	
T	F	
F	T	
F	F	

The truth table for exclusive or

Note: Exclusive or is typically used to natural language to identify *choices*. For example, "You may have a soup or salad with your entree."

Implication

The implication p → q is false if p is true, and q is false; p → q is true otherwise

Terminology

- p is called the hypothesis
- q is called the conclusion

p	q	p → q
T	T	
T	F	
F	T	
F	F	

The truth table for implication

Implication (cont.)

The implication $p \rightarrow q$ can be read in a number of (equivalent) ways:

- If p then q
- p only if q
- p is sufficient for q
- q whenever p

Implication examples

Let:

- p ≡ Jane gets a 100% on her final exam
- q ≡ Jane gets an A on her final exam

What are the truth values of these implications:

- p → q   true
- q → p

Other conditional statements

Given an implication p → q:

- q → p is its converse
- ¬q → ¬p is its contrapositive
- ¬p → ¬q is its inverse

Why might this be useful?

Note: An implication and its contrapositive *always* have the same truth value

Biconditional

The biconditional p ↔ q is true if and only if p and q assume the same truth value

p	q	p ↔ q
T	T	
T	F	
F	T	−
F	F	

The truth table for the biconditional

Note: The biconditional statement p ↔ q is often read as "p if and only if q" or "p is a necessary and sufficient condition for q."

Truth tables can also be made for more complex expressions

Example: What is the truth table for $(p \land q) \to \lnot r$?

Subexpressions of "what we want to know"

What we want to know

p	q	r			

$2^3 = 8$ rows

Like mathematical operators, logical operators are assigned precedence levels

1. Negation
   - $\neg q \vee r$ means $(\neg q) \vee r$, not $\neg(q \vee r)$

2. Conjunction

3. Disjunction
   - $q \wedge r \vee s$ means $(q \wedge r) \vee s$, not $q \wedge (r \vee s)$

4. Implication
   - $q \wedge r \rightarrow s$ means $(q \wedge r) \rightarrow s$, not $q \wedge (r \rightarrow s)$

5. Biconditional

In general, we will use parentheses to disambiguate and to override precedence rules.

In-class Activities

Activity 2: Show that an implication p → q and its contrapositive ¬q → ¬p always have the same value [app.mural]

- **Hint:** Construct two truth tables

Activity 3: Construct the truth table for the compound proposition p ∧ (¬q ∨ r) → r [app.mural]

Steps:
1. Introduce to a classmate
2. Work in pairs on the exercise
3. Volunteers to share answers

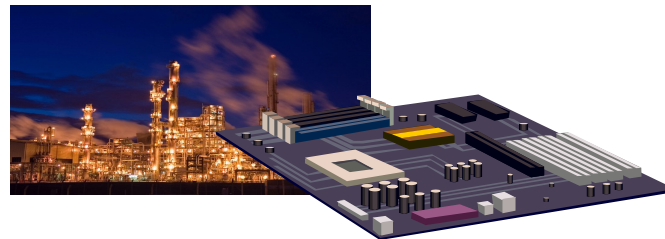# English sentences can often be translated into propositional sentences

But *why* would we do that?

Philosophy and epistemology

Reasoning about law

Verifying complex system specifications

Example #1

Example: You can see an R-rated movie only if you are over 17 or you are accompanied by your legal guardian.

Find logical connectives

Let:

Translate fragments

Create logical expression

Example #2

Example: You can have free coffee if you are a senior citizen and it is a Tuesday

Let:

Example #3

Example: If you are under 17 and are not accompanied by your legal
guardian, then you cannot see the R-rated movie.

Let:

Note: The above translation is the contrapositive of the
translation from example 1!

Logic also helps us understand bitwise operations

- Computers represent data as sequences of bits
  - e.g., 0101 1101 1010 1111

- Bitwise logical operations are often used to manipulate these data

- If we treat 1 as true and 0 as false, our logic truth tables tell us how to carry out bitwise logical operations

Bitwise logic examples

$$\begin{array}{r} 1010\ 1110 \\ \wedge\ 1110\ 1010 \\ \hline \end{array}$$

$$\begin{array}{r} 1010\ 1110 \\ \vee\ 1110\ 1010 \\ \hline \end{array}$$

$$\begin{array}{r} 1010\ 1110 \\ \oplus\ 1110\ 1010 \\ \hline \end{array}$$

In-class Activities

Activity 4: Translate the following sentences

Translate the following English sentence into propositional logic.

- "You get a free salad only if you order off of the extended menu and it is a Wednesday". Use the following propositions:
  s≡"You get a free salad"
  x≡"You order off of the extended menu"
  w≡"It is a Wednesday"

- "If it is raining and today is Saturday, then I will either play video games or watch a movie". Use the following propositions:
  r≡"It is raining"
  s≡"Today is Saturday"
  g≡"I will play video games"
  m≡"I will watch a movie"

Steps:
1. Introduce to a classmate
2. Work in pairs on the exercise
3. Volunteers to share answers

In-class Activities

Activity 5: Solve the following bitwise problems

$$\begin{array}{r} 1011\ 1000 \\ \oplus\ 1010\ 0110 \\ \hline \end{array}$$

$$\begin{array}{r} 1011\ 1000 \\ \wedge\ 1010\ 0110 \\ \hline \end{array}$$

Steps:
1. Introduce to a classmate
2. Work in pairs on the exercise
3. Volunteers to share answers

Final Thoughts

- Propositional logic is a simple logic that allows us to reason about a variety of concepts

- In recitation:
    - More examples and practice problems
    - Be sure to attend!

- Next:
    - Logic puzzles and propositional equivalence
    - Please read sections 1.2 and 1.3
        - In general: do the assigned reading!