CS 441: Applications of Number Theory

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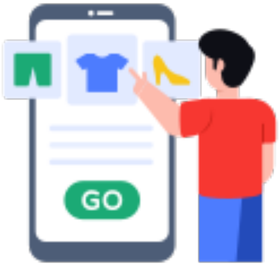
[Motivation] Cryptography

🛒 Shopping Online

When you buy something online, your credit card number must travel from your computer to the merchant's server. This information needs to be kept secret from anyone who might be eavesdropping on the network, such as a hacker.

How can we ensure that the credit card number is unreadable if intercepted?



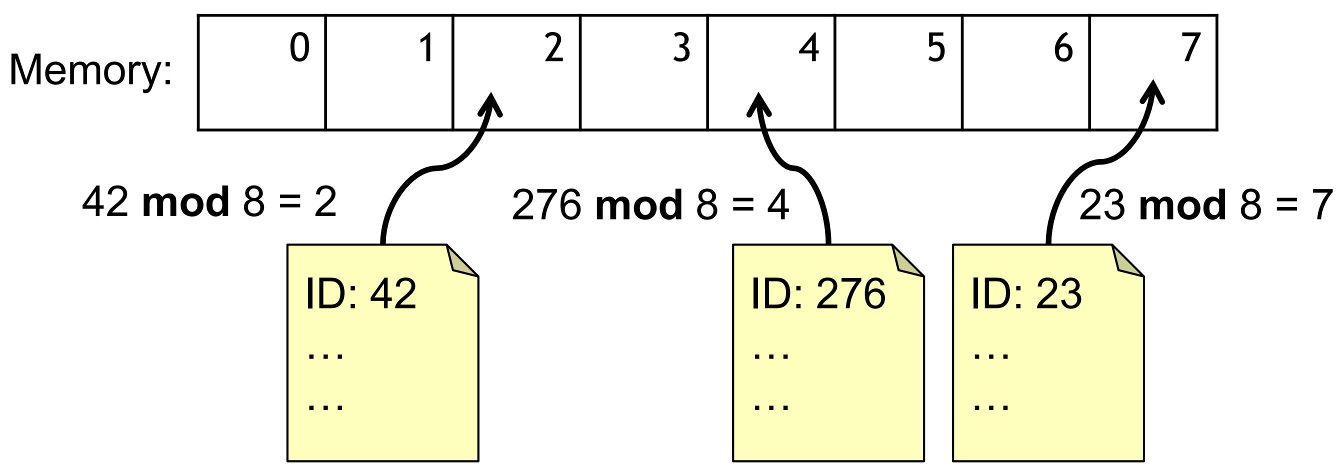
Today's topics

- Hashing non-numeric data
  - Horner’s method for efficient computation
- Check digits and error-correcting codes
  - ISBN, Luhn, Reed–Solomon
- Cryptography
  - Block ciphers
  - Public-key cryptography
  - RSA

Hash functions, recap

Problem: Given a large collection of records, how can we find the one we want quickly?

Solution: Apply a hash function that determines the storage location of the record based on the record's ID. A common hash function is $h(k) = k \textbf{ mod } n$, where n is the number of available storage locations.



What if we want to use non-numeric IDs?

For example, say your IDs are alphanumeric strings

Thanks to base-b expansion, we can interpret these as integers!

- Let $b = 36$, with digits 0–9 then A–Z
- If case sensitive, $b = 62$ to include 0–9, A–Z, a–z
- Base64 is a standard where $b = 64$ using A–Z, a–z, 0–9, +, /
    - Why might this be preferred over $b = 62$?

We can even hash arbitrary binary data

- Any binary data can be interpreted as a base-b integer! Let $b = 2$
- Or, we can read k bits at a time (say, 1 byte = 8 bits) and let $b = 2^k$, interpreting each "block" as an integer in $\mathbf{Z}_b$

To calculate these more efficiently, we can use Horner's method

An k-digit string in base b:

$$a_{k-1}b^{k-1} + a_{k-2}b^{k-2} + \cdots + a_1 b^1 + a_0 b^0$$

- If k and b are large, values like b^{k-1} are very time-consuming to calculate

Instead, we can use Horner's method:

$$((\ldots(a_{k-1} * b + a_{k-2}) * b + \cdots + a_2) * b + a_1) * b + a_0$$

procedure *Horner*(b, a_0, a_1, …, a_{k-1})

  $y := a_{k-1}$

  for i : k–2 **to** 0

    $y := y * b + a_i$

  return y

Each a_i is multiplied by b a total of i times

We can add "mod n" here if using this for hashing!

Congruences are also used for check bits and check digits

The textbook describes parity bits and UPC/ISBN check digits

The Luhn algorithm is used for credit cards

- Calculates the 16th digit based on the first 15
- Double the rightmost digit and every second digit, moving right to left
- Let s be the sum of the resulting digits
- Set the check digit to $10 - (s \bmod 10)$

10-(57 mod 10)
10-7 = 3

1234 5678 4321 555

2 6 10 14 8 4 10 10

2 4 6 8 3 1 5

- $s = 2 + 2 + 6 + 4 + (1 + 0) + 6 + (1 + 4) + 8 + 8 + 3 + 4 + 1 + (1 + 0) + 5 + (1 + 0) = 57$

Recall the Caesar cipher

To encode a message using the Caesar cipher:

- Choose a shift index *s*
- Convert each letter A-Z into a number 0-25
- Compute $f(p) = (p + s) \textbf{ mod } 26$

Example: Let *s* = 9. Encode "ATTACK".

- ATTACK = 0 19 19 0 2 10
- *f*(0) = 9, *f*(19) = 2, *f*(2) = 11, *f*(10) = 19
- Encrypted message: 9 2 2 9 11 19 = JCCJLT

Affine ciphers use bijections of the form $f(p) = (ap + b) \textbf{ mod } 26$

- To be a bijection, requires $\gcd(a, 26) = 1$

These simple ciphers are not secure!

Patterns become obvious even though the letters are replaced

- Frequencies of letters, digraphs, trigraphs, etc.

However, modern secure block ciphers might look similar at first!

- Replace each k-bit block with another k-bit block
  - Say, $k = 128$, as with Advanced Encryption Standard (AES)
- Use a key (e.g., also 128 bits) to determine the substitution
- There are 2^k possible blocks, similar number of keys, so there are way more combinations
- Even so, to be used securely, a block cipher needs to be combined with a secure mode of operation (block mode)
- A secure block mode ensures that the same block encrypts differently depending on its location

These ciphers, including modern block ciphers, are examples of secret-key cryptography

Symmetric-key (or private-key, or secret-key) cryptography uses the same key to encrypt and decrypt

- The ability to encrypt is the same as the ability to decrypt
- This means we need to share a key with each other party we want to communicate with!

Public-key cryptography: The encryption and decryption key are distinct

- The two keys are paired, but the decryption key is hard to get from the encryption key
- I can generate a key pair, **keep the decryption key**, and **distribute the encryption key** to multiple parties!

The RSA cryptosystem

To generate an RSA key pair:

- Choose two primes, p and q, and let $n = pq$
- Compute $\varphi(n) = (p-1)(q-1)$ 1
  - The count of integers less than n that are coprime with n
- Choose an integer e such that is coprime with $\varphi(n)$
  - How can we ensure this?
- Calculate d such that $ed \equiv 1 \pmod{\varphi(n)}$ 2
  - How can we do this?
- Let (n, e) and be the public key
- Let d be the private key

To use an RSA key pair:

- Encrypt a message $m \in \mathbf{Z}_n$ to public key (n, e): $c = m^e \textbf{ mod } n$
- Decrypt a ciphertext to (n, e): $r = c^d \textbf{ mod } n = m^{ed} \textbf{ mod } n = m$

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Why does RSA decryption work?

Why does $m^{ed} \mathbf{\ mod\ } n = m$? 2 1

- Recall: $ed \equiv 1 \pmod{\varphi(n)}$, and $\varphi(n) = (p-1)(q-1)$
- 3 $r = m^{ed} = m^{k\varphi(n)+1} = m^{k(p-1)(q-1)+1}$
- 4 By Fermat's Little Theorem, this means $r \equiv m \pmod{p}$ and $r \equiv m \pmod{q}$
- Thus, $p \mid (r-m)$ and $q \mid (r-m)$
- Since p and q are distinct and both prime, they are coprime
- Lemma: If $a \mid c$ and $b \mid c$ for coprime a and b, then $ab \mid c$
  - Good practice proof!
  - If $p \mid (r-m)$ and $q \mid (r-m)$ then $\mathrm{p} * q \mid (r-m)$
    - $n \mid (r-m)$
- This means that $n \mid (r-m)$, so $r \equiv m \pmod{n}$

Let's assume for now. We will dive more later

Why does RSA decryption work?: Fermat's Little Theorem

- $r = m^{ed} = m^{k\varphi(n)+1} = m^{k(p-1)(q-1)+1}$
- 4 By Fermat's Little Theorem, this means $r \equiv m \pmod{p}$ and $r \equiv m \pmod{q}$
  - Let's assume* gcd(m, p) = 1 and gcd(m,q) = 1. m and p are coprimes; m and q are coprimes [See slide 42 from congruences]
  - Since p is prime, and m is not divisible by p [gcd(m, p) = 1] → $m^{(p-1)}$ = 1 (mod p)
  - Since q is prime, and m is not divisible by q [gcd(m, q) = 1] → $m^{(q-1)}$ = 1 (mod q)
  - $$\begin{aligned} r = m^{k(p-1)(q-1)+1} &= m * m^{k(p-1)(q-1)} \\ &= m * 1\ (mod\ p) \\ &= m\ (mod\ p) \end{aligned}$$
  - $$\begin{aligned} r = m^{k(p-1)(q-1)+1} &= m * m^{k(p-1)(q-1)} \\ &= m * 1\ (mod\ q) \\ &= m\ (mod\ q) \end{aligned}$$
- Thus, $p \mid (r - m)$ and $q \mid (r - m)$

Note: *m* is the message to encrypt

* Which mostly holds, except in rare cases

RSA Encryption: Example

To encrypt a message using RSA using a key (*n*,*e*) :

i. Translate the plaintext message *M* into sequences of two-digit integers representing the letters. Use 00 for A, 01 for B, etc.

ii. Concatenate the two-digit integers into strings of digits.

iii. Divide this string into equally sized blocks of 2*N* digits where 2*N* is the largest even number 2525…25 with 2N digits that does not exceed n.

iv. The plaintext message M is now a sequence of integers $m_1, m_2, \ldots, m_K$.

v. Each block (an integer) is encrypted using the function $C = M^e \textbf{ mod } n$.

RSA Encryption: Example

Example: Encrypt the message HELP using the RSA cryptosystem with key (n = 2537,e = 13).

- $2537 = 43 \cdot 59$,
- $p = 43$ and $q = 59$ are primes and $\gcd(e, (p-1)(q-1)) = \gcd(13, 42 \cdot 58) = 1$

Solution: Translate the letters in HELP to their numerical equivalents 07 04 11 15.

- Divide into blocks of four digits (because 2525 < 2537 < 252525) to obtain 0704 1115.
- Encrypt each block using the mapping $C = M^{13}$ **mod** 2537.
- Since 0704^{13} mod 2537 = 0981 and 1115^{13} mod 2537 = 0461, the encrypted message is 0981 0461.

RSA Decryption: Example

To decrypt a RSA ciphertext message, the decryption key *d*, an inverse of *e* modulo $(p-1)(q-1)$ is needed. The inverse exists since gcd(e, (p-1)(q-1)) = gcd(13, $42 \cdot 58$) = 1

With the decryption key *d*, we can decrypt each block with the computation $M = c^d \textbf{ mod } p \cdot q$.

RSA works as a public key system since the only known method of finding *d* is based on a factorization of *n* into primes. There is currently no known feasible method for factoring large numbers into primes.

RSA Decryption: Example

Example: The message 0981 0461 is received. What is the decrypted message if it was encrypted using the RSA cipher from the previous example.

Solution: The message was encrypted with $n = 43 \cdot 59$ and exponent 13. An inverse of 13 module $42 \cdot 58$ (= 2436) (*exercise* 2 *in Section* 4.4) is d = 937

- To decrypt a block *C*, M= C^{937} **mod** 2537.
- Since 0981^{937} **mod** 2537 = 0704 and 0461^{937} **mod** 2537 = 1115, the decrypted message is 0704 1115. Translating back to English letters, the message is HELP.

A brief practical security note…

The textbook includes examples and exercises where they encrypt a message using RSA, one "chunk" at a time

THIS IS INSECURE, DO NOT DO THIS

Remember what we learned from block ciphers!

- Done this way, if the plaintext chunk is the same, the ciphertext chunk is the same
- Even if the encryption approach is very sophisticated in isolation, encrypting piece-by-piece reveals patterns
- Best practice is to use RSA to encrypt a single-use symmetric key, then encrypt the message using a block cipher with a secure mode of operation
  - In part, because block ciphers are way faster than RSA

Why is RSA secure?

That is, if you know $c = m^e \ \mathbf{mod}\ n$, why can't you get m?

This is called the RSA problem, and the fastest known approach is to factor n

- In turn, the fastest factoring algorithm is slower than polynomial complexity ("hard")
- Factoring n reveals p and q, and thus $\varphi(n)$, and then d can be computed from e just like in key generation
- If you can get d, then you can get p and q
  - By contrapositive, if factoring is hard, then getting d is hard
- Similarly, if you can get $\varphi(n)$, you can get p and q

To be secure for the near future, n should be 2048 bits in size

- e.g.,
  28980031691694357068918562487659336178577290872139729240999721884150682654823846774504439389267921793843771740233811602035640310196929500591908624781
  66152016032673099683618999980615311782821864256646973478297214481647222660269569400841134169754396451340590101145507012183878091040551030992366712077
  51888612680781200445138803757546069773284441936327610981983867727670435168737551110881172718728253861892500326058954623805626985122349587194747221280
  36031389620442812631321984742581817025098263901240154322179135628982031399236433383170589170534724928725807887253791412053381878561858347628938989347
  52357861795082984626 4

Final thoughts

- Number theory has many applications in computing
  - Hashing for storage
  - Check digits and error-correcting codes
  - Cryptography

- Symmetric-key cryptography relies on complex substitutions, while public-key cryptography uses number theory
  - … and mathematical problems with no known efficient algorithms

- Next: Proof by induction! (Start reading Chapter 5)

Extra

Error-correcting codes are used to store data when reading is error-prone

Examples: Optical media, QR codes

Goal: Store k digits* with extra check digits so that erasures/errors can be detected/corrected

- Store $n > k$ digits, where $t = n - k$ are check digits
- If any t digits are lost, they can be recovered
- If any t digits are changed, it can be detected
- If any $\lfloor t/2 \rfloor$ digits are changed, it can be corrected

Reed–Solomon: Let each digit represent a point on a curve

- Any k points identify a $(k-1)$-degree polynomial
- Extend the curve to generate check digits, **interpolate to recover lost digits**