# CS 441: Logic Puzzles and Propositional Equivalence

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# [Motivation] Logic Puzzle

**The Case of the Missing Laptop**

A brand-new, top-of-the-line laptop has gone missing from the university's computer science lab. There are three suspects: Alice, Bob, and Carol, all known to have been in the lab that day. Your job, as the Logic Detective Agency, is to use the provided clues to deduce who took the laptop.

**The Suspects:** Alice, Bob, and Carol

**The Clues:**

- If Alice took the laptop, then she was in the computer lab at 2 PM.
- Carol was in the lab at 2 PM.
- If Bob took the laptop, then he was not in the lab at 2 PM.
- Either Alice took the laptop or Carol took the laptop.
- If Carol was in the lab at 2 PM, then she did not take the laptop.

**Who took the Laptop?**

# Today's topics

- Logic puzzles
- Propositional equivalences

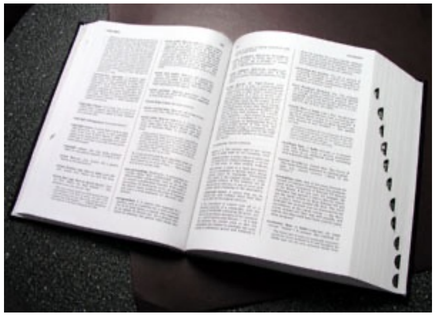
# A technical support riddle

Alice and Bob are technical support agents. If an agent is having a bad day, they will always lie to you. If an agent is having a good day, they will always tell you the truth. Alice tells you that Bob is having a bad day. Bob tells you that he and Alice are both having the same type of day. Can you trust the advice you receive from Alice during your call?

**How do we solve this type of problem?**

# Solving logic puzzles systematically

**Step 1:** Identify rules and constraints



**Step 2:** Assign propositions to key concepts

**Step 3:** Make assumptions and reason logically!

# Technical support revisited

Alice and Bob are technical support agents. If an agent is having a bad day, they will always lie to you. If an agent is having a good day, they will always tell you the truth. Alice tells you that Bob is having a bad day. Bob tells you that he and Alice are both having the same type of day. Can you trust the advice you receive from Alice during your call?

**Step 1:** Identify the rules of the puzzle

**Step 2:** Assign propositions to the key concepts in the puzzle

# Step 3: Make assumptions and reason logically

- A ≡ "Alice is having a good day"
- B ≡ "Bob is having a good day"

**[Case 1]**

Let's assume A is True,

So Alice is telling the truth.

Therefore, Alice's Statement "Bob is having a bad day" is True.

Bob lies

Bob's statement that he and Alice are both having the same type of day is a lie --> Alice and Bob have a different type of day

All these statements are consistent with our assumptions ☺

**[Case 2]**

Let's assume A is False --> "Alice is having a bad day",

So Alice lies.

Therefore, Alice's Statement "Bob is having a bad day" is False. → "Bob is having a good day"

Bob is telling the truth.

Bob's statement that he and Alice are both having the same type of day is True --> However ….

There is a contradiction.

# Another example

Consider a group of friends: Fredrik, Anuradha, and Cai. If Fredrik is not the oldest, then Anuradha is. If Anuradha is not the youngest, then Cai is the oldest. Determine the relative ages of Fredrik, Anuradha, and Cai.

Propositions:

Rules:

# Step 3: Make assumptions and reason logically

R1: $\neg f \rightarrow a$ (If Fredrik is not the oldest, Anuradha is the oldest)

R2: $\neg a' \rightarrow c$ (If Anuradha is not the youngest, then Cai is the oldest)

**[Case 1: Fredrik is the oldest]**

R1 does not apply,

It is irrelevant to consider *Anuradha is the oldest*

Consider R2,

Let's consider Anuradha is "in the middle", so Cai is the oldest [Contradiction]

Let's consider Anuradha is the youngest, R2 does not apply

Cai is in "in the middle"

All these statements are consistent with our assumptions ☺

**[Case 2: Fredrik is not the oldest]**

Consider R1,

Anuradha is the oldest

Consider R2,

Cai is the oldest [Contradiction]

# Sometimes no solution is a solution!

Alice and Bob are technical support agents. Alice says, “I am having a good day.” Bob says, “I am having a good day.” Can you trust either Alice or Bob?

**Step 1:** Identify rules

**Step 2:** Assign propositions

# Step 3: Make assumptions and reason logically

- A ≡ "Alice is having a good day"
- B ≡ "Bob is having a good day"

**[Case 1]**

Let's assume A is True,

Unfortunately, we can not say anything about B.

**[Case 2]**

Let's assume B is True,

Unfortunately, we can not say anything about A.

→ [No solution ] We need more information to make any conclusion.

# In-class Activities

**Activity 1:** Alice and Bob are technical support agents working to fix your computer. Alice tells you that Bob is having a bad day today and that you should expect a long wait before your computer is fixed. Bob tells you not to worry, Alice is just having a bad day—your computer will be ready in no time. [miro]

**Question:** Can you draw any conclusions about when your computer will be fixed? If so, what can you learn?

Steps:

1. Introduce to a classmate
2. Work in pairs on the exercise
3. Submit answers on miro

# [Motivation] Propositional Equivalences

### Debugging a Conditional Statement

Imagine you're a programmer working on an e-commerce website. There's a bug: a discount code isn't being applied correctly. You examine the code and find a complex conditional statement that determines if a user qualifies for the discount.

```
if not(isGoldMember or (hasPurchasedBefore and hasLoggedInRecently)))
    # apply discount
else:
    # don't apply discount
```

This code snippet is confusing. The logic says "if it's **NOT** true that the user is a Gold Member **OR** (they have purchased before **AND** have logged in recently), **then** apply the discount." The nested NOT and OR make it hard to read and easy to introduce bugs. How can we simplify this expression?

# Propositional equivalences: preliminaries

**Definition:** A tautology is a compound proposition that is always **true**, regardless of the truth values of the propositions occurring within it.

**Definition:** A contradiction is a compound proposition that is always **false**, regardless of the truth values of the propositions occurring within it.

**Definition:** A contingency is a compound proposition whose truth value is dependent on the propositions occurring within it.

# Examples

Are the following compound propositions tautologies, contradictions, or contingencies?

- $p \vee \neg p$ tautology
- $\neg p \wedge p$ contradiction
- $p \vee q$ contingency

# What are logical equivalences and why are they useful?

**Definition:** Compound propositions p and q are logically equivalent exactly when $p \leftrightarrow q$ is a tautology. The notation $p \equiv q$ means that p and q are logically equivalent.

Logical equivalences are extremely useful!

- Aid in the construction of proofs
- Allow us to simplify compound propositions

**Example:** $p \rightarrow q \equiv \neg p \vee q$

***How do we prove this type of statement?***

# Proving logical equivalences the "easy" (but tedious) way

We can prove simple logical equivalences using our good friend the truth table!

**Prove:** $p \rightarrow q \equiv \neg p \vee q$

| p | q | $\neg p$ | $\neg p \vee q$ | $p \rightarrow q$ |
|---|---|---|---|---|
| T | T | | | |
| T | F | | | |
| F | T | | | |
| F | F | | | |

# DeMorgan's laws allow us to distribute negation over compound propositions

Two laws:

- $\neg(p \vee q) \equiv \neg p \wedge \neg q$
- $\neg(p \wedge q) \equiv \neg p \vee \neg q$

***If "p or q" isn't true, then neither p nor q is true***

***If "p and q" isn't true, then at least one of p or q is false***

**Prove:** $\neg(p \vee q) \equiv \neg p \wedge \neg q$

| $p$ | $q$ | $p \vee q$ | $\neg(p \vee q)$ | $\neg p$ | $\neg q$ | $\neg p \wedge \neg q$ |
|---|---|---|---|---|---|---|
| T | T | | | | | |
| T | F | | | | | |
| F | T | | | | | |
| F | F | | | | | |

# Using DeMorgan's laws

Use DeMorgan's laws to negate the following expressions:

- "Tito is wearing blue pants and a sweatshirt"
  - $b \wedge s$
  - $\neg(b \wedge s) \equiv \neg b \vee \neg s$
  - Tito is not wearing blue pants or is not wearing a sweatshirt
- "I will drive or I will walk"
  - $d \vee w$
  - $\neg(d \vee w) \equiv \neg d \wedge \neg w$
  - I will not drive and I will not walk

# In-class Activities

**Activity 2:** Prove that $\neg(p \wedge q)$ and $\neg p \vee \neg q$ are logically equivalent, i.e., $\neg(p \wedge q) \equiv \neg p \vee \neg q$. This is the second DeMorgan's law. [miro]

**Activity 3:** Use DeMorgan's laws to negate the following propositions: [miro]

- Today I will go running or ride my bike
- Tom likes both pizza and beer

**Activity 4:** Rewrite conditional statement from *Debugging a Conditional Statement* problem. [miro]

Steps:

1. Introduce to a classmate
2. Work in pairs on the exercise
3. Submit answers on miro

# Sometimes using truth tables to prove logical equivalencies can become cumbersome

Recall that for an equivalence with *n* propositions, we need to build a truth table with $2^n$ rows

- Fine for tables with $n = 2, 3,$ or 4
- Consider $n = 30$—we would need 1,073,741,824 rows in the truth table!

Another option: Direct manipulation of compound propositions using known logical equivalencies

# There are many useful logical equivalences

| Equivalence | Name |
| --- | --- |
| | |
| | |
| | |
| | |
| | |

# More useful logical equivalences

| Equivalence | Name |
| --- | --- |
| | |
| | |
| | |
| | |
| | |

***These, and more equivalencies, are in the book!***

# Prove that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology

**Prove:** $(p \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$

# Final Thoughts

- Logic can help us solve real world problems and play challenging games
- Logical equivalences help us simplify complex propositions and construct proofs
  - More on proofs later in the course
- Next:
  - Predicate logic and quantification
  - Please read section 1.4