

CS 441: Proof Methods

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Today's topics

- Proof techniques
 - Proof by exhaustion
 - Proof by cases
 - Existence proofs
 - Uniqueness proofs
- Proof strategies
 - Backward reasoning



Not all theorems are of the form $p \rightarrow q$

Sometimes, we need to prove a theorem of the form:

$$p_1 \vee p_2 \vee \dots \vee p_n \rightarrow q$$

So, we might need to examine **multiple cases!**

Prove that $n^2 + 1 \geq 2n$ where n is a positive integer with $1 \leq n \leq 4$

Proof:

Since we have verified each case, we have shown that $n^2 + 1 \geq 2n$ where n is a positive integer with $1 \leq n \leq 4$.

□

With only 4 cases to consider, exhaustive proof was a good choice!

Sometimes, **exhaustive proof** isn't an option, but we still need to examine multiple possibilities

Example: Prove the triangle inequality. That is, if x and y are real numbers, then $|x| + |y| \geq |x + y|$.

Clearly, we can't use exhaustive proof here since there are **infinitely many** real numbers to consider.

We also can't use a simple direct proof either, since our proof depends on the signs of x and y .

What should we do?

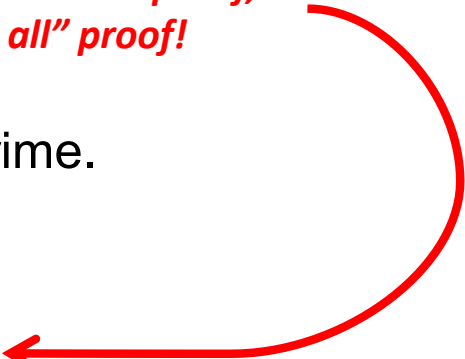
Example: Prove that if x and y are real numbers, then $|x| + |y| \geq |x + y|$.

- **Note:** If $x \geq 0$, $|x| = x$, otherwise $|x| = -x$
- **Cases:**
 - 1) $x \geq 0$ and $y \geq 0$
 - $|x| + |y| = x + y$ and $|x + y| = x + y$
 - $x + y \geq x + y$ ✓
 - 2) $x < 0$ and $y < 0$
 - $|x| + |y| = -x - y$ and $|x + y| = -x - y$
 - $-x - y \geq -x - y$ ✓
 - 3) $x \geq 0$ and $y < 0$
 - If $x \geq |y|$, then $|x + y| = x - |y|$ and $|x| + |y| = x + |y|$
 $x + |y| \geq x - |y|$ ✓
 - If $x < |y|$, then $|x + y| = |y| - x$ and $|x| + |y| = x + |y|$
 $|y| + x \geq |y| - x$ ✓
 - 4) Symmetrical to Case 3 □

Making mistakes when using **proof by cases** is all too easy!

Mistake 1: Proof by “a few cases” is **not** equivalent to proof by cases.

*This is a “there exists” proof,
not a “for all” proof!*



Example: Prove that all odd numbers are prime.

“Proof:”

- **Case (i):** The number 1 is both odd and prime
- **Case (ii):** The number 3 is both odd and prime
- **Case (iii):** The number 5 is both odd and prime
- **Case (iv):** The number 7 is both odd and prime

Thus, we have shown that odd numbers are prime. $\square$

Making mistakes when using **proof by cases** is all too easy!

Mistake 2: Leaving out critical cases.

Example: Prove that $x^2 > 0$ for all integers x

“Proof:”

- **Case (i):** Assume that $x < 0$. Since the product of two negative numbers is always positive, $x^2 > 0$.
- **Case (ii):** Assume that $x > 0$. Since the product of two positive numbers is always positive, $x^2 > 0$.

Since we have proven the claim for all cases, we can conclude that $x^2 > 0$ for all integers x . $\square$

What about the case in which $x = 0$?

Sometimes we need to prove the **existence** of a given element

There are two ways to do this



The **constructive** approach



The **non-constructive** approach

A constructive existence proof

Prove: Show that there is a positive integer that can be written as the sum of cubes of positive integers in two different ways.

Proof: $1729 = 10^3 + 9^3 = 12^3 + 1^3 \quad \square$



*Constructive existence proofs are really just instances of
“existential generalization.”*

A non-constructive existence proof

Prove: Show that there exist two irrational numbers x and y such that x^y is rational.

Proof:

Note: We don't know whether $\sqrt{2}^{\sqrt{2}}$ is rational or irrational. However, in either case, we can use it to construct a rational number.

Sometimes, existence is not enough, and we need to prove **uniqueness**

This process has two steps:

- 1.
- 2.

Example: Prove that if a and b are real numbers, then there exists a unique real number r such that $ar + b = 0$

Proof:

- Note that $r = -b/a$ is a solution to this equality since $a(-b/a) + b = -b + b = 0$.
- Assume that $as + b = 0$
- Then $as = -b$, so $s = -b/a = r$, which means s is just r $\square$

Existence



Uniqueness

In-class Activities

To join, go to: ahaslides.com/PY4SA 

 AhaSlides

Quiz question 3 of 3

1 player ready

 eric

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In-class Activities

Activity 1: Prove that there exists a positive integer that is equal to the sum of all positive integers less than it. Is your proof constructive or non-constructive?
[miro]

Activity 2: Prove that there is no positive integer n such that $n^2 + n^3 = 100$.
[miro]

Steps:

1. Introduce to a classmate
2. Work in pairs on the exercise
3. Submit answers on miro
4. Volunteers to share answers

Final Thoughts

- Proving theorems is not always straightforward
- Having several **proof strategies** at your disposal will make a huge difference in your success rate!
- We are “done” with our intro to logic and proofs
- Next lecture:
 - Intro to set theory
 - Please read sections 2.1 and 2.2