

CS 441: Sequences and Summations

PhD. Nils Murrugarra-Llerena
nem177@pitt.edu



[Key CS Link] Summations

Analyzing a Nested Loop

Consider a simple, but common, programming task: multiplying every element of an array with every other element. A naive approach would use two nested for loops. The number of operations, and thus the algorithm's runtime, can be expressed as a summation.

Here's a pseudocode example:



```
let n be the size of the array
count = 0
for i from 1 to n:
  for j from 1 to n:
    count = count + 1
```

Today's topics

- Sequences and Summations
 - Specifying and recognizing sequences
 - Summation notation
 - Closed forms of summations



Sequences are ordered lists of elements

Definition: A **sequence** is a function from a subset of the set of integers to a set S . We use the notation a_n to denote the image of the integer n . a_n is called a **term** of the sequence.

Examples:

- 1, 3, 5, 7, 9, 11 A sequence with 6 terms
- 1, 1/2, 1/3, 1/4, 1/5, ... An infinite sequence

Note: The second example can be described as the sequence $\{a_n\}$ where $a_n = 1/n$

To join, go to: ahaslides.com/ZG5SP 



Quiz question 1 of 6

0 players ready

Waiting for players to join...

 Edit slides

 Add slides

Start 



 0  0 / 100 

What makes sequences so special?

Question: Aren't sequences just sets?

Answer: The elements of a sequence are members of a set, but a sequence is **ordered**, a set is not.

Question: How are sequences different from ordered n-tuples?

Answer: An ordered n-tuple is ordered, but always contains n elements. Sequences can be infinite!

Some special sequences

Geometric progressions are sequences of the form $\{ar^n\}$ where a and r are real numbers

Examples:

- 1, 1/2, 1/4, 1/8, 1/16, ...
- 1, -1, 1, -1, 1, -1, ...

Arithmetic progressions are sequences of the form $\{a + nd\}$ where a and d are real numbers.

Examples:

- 2, 4, 6, 8, 10, ...
- -10, -15, -20, -25, ...

To join, go to: ahaslides.com/ZG5SP



Quiz question 2 of 6

0 players ready

Waiting for players to join...

 Edit slides

 Add slides

Start 



 0  0 / 100 

To join, go to: ahaslides.com/ZG5SP



Quiz question 3 of 6

0 players ready

Waiting for players to join...

 Edit slides

 Add slides

Start 



 0  0 / 100 

Sometimes we need to figure out the formula for a sequence given only a few terms

Questions to ask yourself:

1. Are there runs of the same value?
2. Are terms obtained by multiplying the previous value by a particular amount? (Possible geometric sequence)
3. Are terms obtained by adding a particular amount to the previous value? (Possible arithmetic sequence)
4. Are terms obtained by combining previous terms in a certain way?
5. Are there cycles amongst terms?

What are the formulas for these sequences?

Problem 1: 1, 5, 9, 13, 17, ...

Problem 2: 1, 3, 9, 27, 81, ...

Problem 3: 2, 3, 3, 5, 5, 5, 7, 7, 7, 7, 11, 11, 11, 11, 11, ...

Problem 4: 1, 1, 2, 3, 5, 8, 13, 21, 34, ...

This is called the Fibonacci sequence.



Sequences are often specified using recurrence relations

This is a **recursive** approach to specifying the terms

- Later terms are specified from earlier terms

For instance, consider this definition of the Fibonacci sequence:

- $f_0 = 0$
- $f_1 = 1$
- For any $n > 1$, $f_n = f_{n-1} + f_{n-2}$

Note that we need at least one **initial condition**

- Like a base case when writing recursive code
- We'll return to recursion later in the term

Sometimes we want to find the sum of the terms in a sequence

Summation notation lets us compactly represent the sum of terms $a_m + a_{m+1} + \dots + a_n$

The diagram shows two equivalent summation notations: $\sum_{j=m}^n a_j$ and $\sum_{m \leq j \leq n} a_j$. Red circles highlight the components: n is the upper limit, $j=m$ is the lower limit and index of summation, and $m \leq j \leq n$ is the range. Red arrows and labels identify these parts: 'Upper limit' points to n , 'Index of summation' points to j , and 'Lower limit' points to both m in the first notation and the first m in the second notation.

$$\sum_{j=m}^n a_j = \sum_{m \leq j \leq n} a_j$$

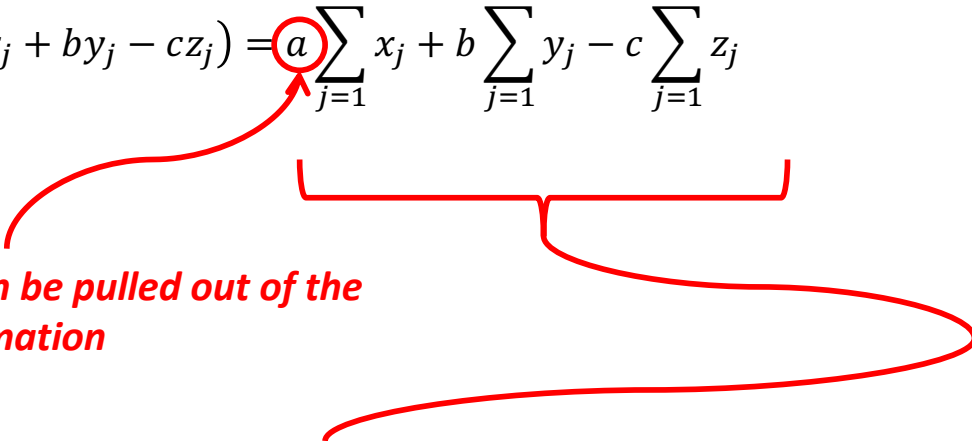
Upper limit

Index of summation

Lower limit

Example: $\sum_{1 \leq i \leq 5} i^2 = 1 + 4 + 9 + 16 + 25 = 55$

Summations are linear: The usual laws of algebra apply

$$\sum_{j=1}^n (ax_j + by_j - cz_j) = a \sum_{j=1}^n x_j + b \sum_{j=1}^n y_j - c \sum_{j=1}^n z_j$$


Constant factors can be pulled out of the summation

A summation over a sum (or difference) can be split into a sum (or difference) of smaller summations

Example:

- $\sum_{1 \leq j \leq 3} (4j + j^2) =$
- $4 \sum_{1 \leq j \leq 3} j + \sum_{1 \leq j \leq 3} j^2 =$

Example sums

Example: Express the sum of the first 50 terms of the sequence $1/n^2$ for $n = 1, 2, 3, \dots$

Answer: $\sum_{j=1}^{50} \frac{1}{j^2}$

Example: What is the value of $\sum_{k=4}^8 (-1)^k$

Answer: $\sum_{k=4}^8 (-1)^k =$
 $=$
 $=$

We can also compute the summation of the elements of some set

Example: Compute $\sum_{s \in \{0,2,4,6\}} (s + 2)$

Answer: $(0 + 2) + (2 + 2) + (4 + 2) + (6 + 2) = 20$

Example: Let $f(x) = x^3 + 1$. Compute $\sum_{s \in \{1,3,5,7\}} f(s)$

Answer: $f(1) + f(3) + f(5) + f(7) = 2 + 28 + 126 + 344 = 500$

To join, go to: ahaslides.com/ZG5SP 



Quiz question 5 of 6

0 players ready

Waiting for players to join...

 Edit slides

 Add slides

Start 



 0  0 / 100 

Sometimes it is helpful to shift the index of a summation

This is particularly useful when **combining** two or more summations. For example:

$$\begin{aligned}
 S &= \sum_{j=1}^{10} j^2 + \sum_{k=2}^{11} (2k - 1) && \text{Let } j = k - 1 \\
 &= \sum_{j=1}^{10} j^2 + \sum_{j=1}^{10} (2(j + 1) - 1) && \text{Need to add 1 to each } j \\
 &= \sum_{j=1}^{10} (j^2 + 2(j + 1) - 1) \\
 &= \sum_{j=1}^{10} (j^2 + 2j + 1) \\
 &= \sum_{j=1}^{10} (j + 1)^2
 \end{aligned}$$

Summations can be nested within one another

Often, you'll see this when analyzing nested loops within a program (i.e., CS 1501/1502)

Example: Compute $\sum_{j=1}^4 \sum_{k=1}^3 (jk)$

Solution:

$$\begin{aligned}
 \sum_{j=1}^4 \sum_{k=1}^3 (jk) &= \sum_{j=1}^4 (j + 2j + 3j) && \text{Expand inner sum} \\
 &= \sum_{j=1}^4 6j && \text{Simplify if possible} \\
 &= 6 + 12 + 18 + 24 = 60 && \text{Expand outer sum}
 \end{aligned}$$

Computing the sum of a geometric series by hand is time consuming...

Would you **really** want to calculate $\sum_{j=0}^{20} (6 \times 2^j)$ by hand?

Fortunately, we have a **closed-form solution** for computing the sum of a geometric series:

$$\sum_{j=0}^n ar^j = \begin{cases} \frac{ar^{n+1} - a}{r - 1} & \text{if } r \neq 1 \\ (n + 1)a & \text{if } r = 1 \end{cases}$$

So $\sum_{j=0}^{20} (6 \times 2^j) = \frac{6 \times 2^{21} - 6}{2 - 1} = 12,582,906$

Why?

Proof of geometric series closed form

On Whiteboard

There are other closed form summations that you should know

<i>Sum</i>	<i>Closed Form</i>

Final thoughts

- **Sequences** allow us to represent (potentially infinite) ordered lists of elements
- **Summation notation** is a compact representation for adding together the elements of a sequence
- Next time:
 - Midterm exam review