# CS 441: Sequences and Summations

**PhD. Nils Murrugarra-Llerena**
nem177@pitt.edu

University of **Pittsburgh**

# [Key CS Link] Summations

## 💻 Analyzing a Nested Loop

Consider a simple, but common, programming task: multiplying every element of an array with every other element. A naive approach would use two nested for loops. The number of operations, and thus the algorithm's runtime, can be expressed as a summation.

Here's a pseudocode example:

```
let n be the size of the array
count = 0
for i from 1 to n:
  for j from 1 to n:
    count = count + 1
```

# Today's topics

- Sequences and Summations
  - Specifying and recognizing sequences
  - Summation notation
  - Closed forms of summations

# Sequences are ordered lists of elements

*Definition:* A sequence is a function from a subset of the set of integers to a set S. We use the notation $a_n$ to denote the image of the integer $n$. $a_n$ is called a term of the sequence.

*Examples:*

- 1, 3, 5, 7, 9, 11                        A sequence with 6 terms
- 1, 1/2, 1/3, 1/4, 1/5, …                An infinite sequence

Note: The second example can be described as the sequence $\{a_n\}$ where $a_n$ = 1/n

# In-class Activities

Submit on

**TOP HAT**

# What makes sequences so special?

Question: Aren't sequences just sets?

Answer: The elements of a sequence are members of a set, but a sequence is ordered, a set is not.

---

Question:  How are sequences different from ordered n-tuples?

Answer: An ordered n-tuple is ordered, but always contains n elements. Sequences can be infinite!

# Some special sequences

Geometric progressions are sequences of the form $\{ar^n\}$ where $a$ and $r$ are real numbers

*Examples:*

- 1, 1/2, 1/4, 1/8, 1/16, …
- 1, -1, 1, -1, 1, -1, …

Arithmetic progressions are sequences of the form $\{a + nd\}$ where $a$ and $d$ are real numbers.

*Examples:*

- 2, 4, 6, 8, 10, …
- -10, -15, -20, -25, …

# In-class Activities

Submit on

TOP HAT

# Sometimes we need to figure out the formula for a sequence given only a few terms

Questions to ask yourself:

1. Are there runs of the same value?

2. Are terms obtained by multiplying the previous value by a particular amount? (Possible geometric sequence)

3. Are terms obtained by adding a particular amount to the previous value? (Possible arithmetic sequence)

4. Are terms obtained by combining previous terms in a certain way?

5. Are there cycles amongst terms?

# What are the formulas for these sequences?

*Problem 1:* 1, 5, 9, 13, 17, …

*Problem 2:* 1, 3, 9, 27, 81, …

*Problem 3:* 2, 3, 3, 5, 5, 5, 7, 7, 7, 7, 11, 11, 11, 11, 11, …

*Problem 4:* 1, 1, 2, 3, 5, 8, 13, 21, 34, …

*This is called the Fibonacci sequence.*

# Sequences are often specified using recurrence relations

This is a recursive approach to specifying the terms
  - Later terms are specified from earlier terms

For instance, consider this definition of the Fibonacci sequence:
  - $f_0 = 0$
  - $f_1 = 1$
  - For any $n > 1$, $f_n = f_{n-1} + f_{n-2}$

Note that we need at least one initial condition
  - Like a base case when writing recursive code
  - We'll return to recursion later in the term

# Sometimes we want to find the sum of the terms in a sequence

Summation notation lets us compactly represent the sum of terms $a_m + a_{m+1} + \ldots + a_n$

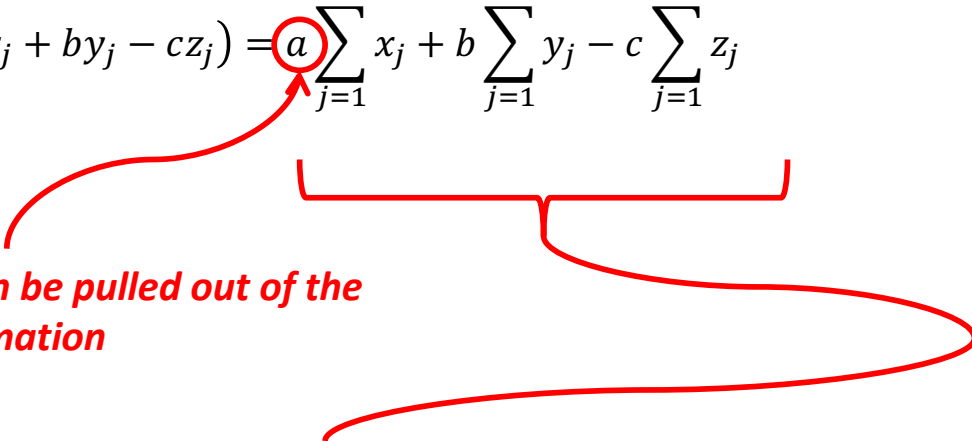$$\sum_{j=m}^{n} a_j = \sum_{m \leq j \leq n} a_j$$

Upper limit

Index of summation    Lower limit

Example: $\sum_{1 \leq i \leq 5} i^2 = 1 + 4 + 9 + 16 + 25 = 55$

# Summations are linear: The usual laws of algebra apply

$$\sum_{j=1}^{n} (ax_j + by_j - cz_j) = a\sum_{j=1}^{n} x_j + b\sum_{j=1}^{n} y_j - c\sum_{j=1}^{n} z_j$$

*Constant factors can be pulled out of the summation*

*A summation over a sum (or difference) can be split into a sum (or difference) of smaller summations*

*Example:*

- $\sum_{1 \le j \le 3} (4j + j^2) =$
- $4\sum_{1 \le j \le 3} j + \sum_{1 \le j \le 3} j^2 =$

# Example sums

*Example:* Express the sum of the first 50 terms of the sequence $1/n^2$ for n = 1, 2, 3, …

Answer: $\displaystyle\sum_{j=1}^{50} \frac{1}{j^2}$

*Example:* What is the value of $\displaystyle\sum_{k=4}^{8}(-1)^k$

Answer:

$$\sum_{k=4}^{8}(-1)^k =$$

$$=$$

$$=$$

# We can also compute the summation of the elements of some set

*Example:* Compute $\displaystyle\sum_{s\in\{0,2,4,6\}} (s+2)$

Answer: (0 + 2) + (2 + 2) + (4 + 2) + (6 + 2) = 20

*Example:* Let f(x) = x$^3$ + 1. Compute $\displaystyle\sum_{s\in\{1,3,5,7\}} f(s)$

Answer: *f*(1) + *f*(3) + *f*(5) + *f*(7) = 2 + 28 + 126 + 344 = 500

# In-class Activities

Submit on

TOP HAT

# Sometimes it is helpful to shift the index of a summation

This is particularly useful when combining two or more summations. For example:

$$S = \sum_{j=1}^{10} j^2 + \sum_{k=2}^{11} (2k-1)$$

*Let j = k - 1*

$$= \sum_{j=1}^{10} j^2 + \sum_{j=1}^{10} (2(j+1)-1)$$

*Need to add 1 to each j*

$$= \sum_{j=1}^{10} (j^2 + 2(j+1) - 1)$$

$$= \sum_{j=1}^{10} (j^2 + 2j + 1)$$

$$= \sum_{j=1}^{10} (j+1)^2$$

# Summations can be nested within one another

Often, you'll see this when analyzing nested loops within a program (i.e., CS 1501/1502)

*Example:* Compute $\displaystyle\sum_{j=1}^{4}\sum_{k=1}^{3}(jk)$

***Expand inner sum***

Solution: $\displaystyle\sum_{j=1}^{4}\sum_{k=1}^{3}(jk) \;=\; \sum_{j=1}^{4}(j + 2j + 3j)$

***Simplify if possible***

$$=\; \sum_{j=1}^{4} 6j$$

$$=\; 6 + 12 + 18 + 24 = 60$$

***Expand outer sum***

# Computing the sum of a geometric series by hand is time consuming…

Would you really want to calculate $\displaystyle\sum_{j=0}^{20}(6 \times 2^j)$ by hand?

Fortunately, we have a closed-form solution for computing the sum of a geometric series:

$$\sum_{j=0}^{n} ar^j = \begin{cases} \dfrac{ar^{n+1}-a}{r-1} & \text{if } r \neq 1 \\ (n+1)a & \text{if } r = 1 \end{cases}$$

So $\displaystyle\sum_{j=0}^{20}(6 \times 2^j) = \dfrac{6 \times 2^{21} - 6}{2 - 1} = 12,582,906$   **Why?**

# Proof of geometric series closed form

## On Whiteboard

# There are other closed form summations that you should know

| Sum | Closed Form |
|-----|-------------|
|     |             |
|     |             |
|     |             |

# Final thoughts

- Sequences allow us to represent (potentially infinite) ordered lists of elements

- Summation notation is a compact representation for adding together the elements of a sequence

- Next time:
  - Midterm exam review