CS 441: Integer representation and Algorithms

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[Key CS Link] Integer Representations

How does a computer understand / interact with images?

🎨 RGB Color Codes

In a computer, every color on the screen is represented by numbers. The most common system is RGB (Red, Green, Blue), where a color is created by mixing varying intensities of these three primary colors of light.

[How it works] Each of the three colors (Red, Green, Blue) is assigned an intensity value, typically ranging from 0 to 255. A value of 0 means the color is completely off, while 255 means it is at full intensity.

The range of 0 to 255 isn't arbitrary; it's a direct result of how computers store information. We often see these RGB values represented in hexadecimal (base-16).

Today's topics

- Integer representations
  - Base b expansions
  - Common bases: Binary, octal, hexadecimal
  - Base conversions

- Integer algorithms
  - Addition
  - Multiplication
  - Connection to computing and pen-and-paper arithmetic

While we typically use decimal, other base systems work very similarly

Recall: Decimal expansion of integers, e.g.,

$$3528 = 3 * 10^3 + 5 * 10^2 + 2 * 10^1 + 8 * 10^0$$

Theorem: Let b be an integer greater than 1. Any $n \in \mathbf{Z}^+$ can be expressed uniquely in the form:

$$n = a_k b^k + a_{k-1} b^{k-1} + \cdots + a_1 b^1 + a_0 b^0$$

where $k \in \mathbf{N}$, each $a_i \in \mathbf{N}$ where $a_i < b$, and $a_k \neq 0$.

This representation is called the base b expansion of n, which we write compactly as $(a_k a_{k-1} \ldots a_1 a_0)_b$

- When $b > 10$, we write each a_i as a single symbol in an extended “alphabet” of digits
  - e.g., 0123456789ABCDEFGH…

Examples of base b expansions

Express each of these expansions in decimal:

- $(675)_8 =$

  $=$
- $(110101)_2 =$

  $=$
- $(2A4)_{12} =$

  $=$

Here, A has value 10

Common base expansions

These base systems are very common in computing:

- Base 2, binary: Expansions are bit strings
  $412 = (110011100)_2$
- Base 8, octal: Each digit a_i is $0 \leq a_i < 8$
  $412 = (634)_8$
- Base 16, hexadecimal: Each digit $a_i \in \{0, 1, \ldots, 9, \mathrm{A}, \mathrm{B}, \ldots, \mathrm{F}\}$
  $412 = (19\mathrm{C})_{16}$

Trends to note:

- Base b requires b digits in the “alphabet”
- Lesser b yields longer expansions, greater b yields shorter expansions

Why are these important?

- Data is stored in binary, octal represents 3 bits per digit, hexadecimal represents 4 bits per digit

Constructing base b expansions

```
procedure base b expansion($n$, $b$: positive integers with $b > 1$)
    $q := n$
    $k := 0$
    while $q \neq 0$
        $a_k := q$ mod $b$
        $q := q$ div $b$
        $k := k + 1$
    return $(a_{k-1}, a_{k-2} \ldots, a_1, a_0)$
    {$(a_{k-1}a_{k-2}\cdots a_1a_0)_b$ is the base $b$ expansion of $n$}
```

Digits are produced right-to-left

Repeat: Divide q by b;
remainder becomes a digit,
quotient replaces q

Return when q = 0

Constructing base *b* expansions, examples

1. Express 1501 in hex
   - 1501 divided by 16
     q = 93, r = 13 = $(\mathrm{D})_{16}$
   - 93 divided by 16
     q = 5, r = 13 = $(\mathrm{D})_{16}$
   - 5 divided by 16
     q = 0, r = 5
   - Thus, 1501 = $(\mathrm{5DD})_{16}$

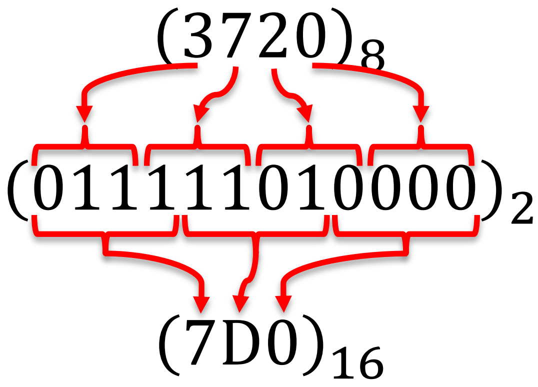
2. Express 441 in octal
   - 441 divided by 8
     q = 55, r = 1
   - 55 divided by 8
     q = 6, r = 7
   - 6 divided by 8
     q = 0, r = 6
   - Thus, 441 = $(671)_{8}$

3. Express 441 in base-30
   - 441 divided by 30
     q = 14, r = 21 = $(\mathrm{I})_{30}$
   - 14 divided by 30
     q = 0, r = 14 = $(\mathrm{E})_{30}$
   - Thus, 441 = $(\mathrm{EI})_{30}$

4. Express 441 in base-4
   - 441 divided by 4
     q = 110, r = 1
   - 110 divided by 4
     q = 27, r = 2
   - 27 divided by 4
     q = 6, r = 3
   - 6 divided by 4
     q = 1, r = 2
   - 1 divided by 4
     q = 0, r = 1
   - Thus, 441 = $(12321)_{4}$

When $b = 2^i$, conversion can be done on i bits at a time

Since an octal digit encodes 3 bits and a hex digit encodes 4 bits, we can use binary to help convert



$(3720)_8$

$(011111010000)_2$

$(7D0)_{16}$

In-class Activities

Activity 1: Find the octal expansion of 100 [miro]

Activity 2: Find the octal expansion of $(100)_2$ [miro]

Activity 3: Find the octal expansion of $(100)_{16}$ [miro]

Activity 4: Find the octal expansion of $(100)_{36}$ [miro]

Submit on miro

Steps:

1. Introduce to a classmate
2. Work in pairs on the exercise
3. Submit answers on miro
4. Volunteers to share answers

Adding base b expansions

```
procedure add(x, y: positive integers, b: integer > 1)
    {The base b expansions of x and y are $(x_{n-1}x_{n-2}\cdots x_1x_0)_b$
    and $(y_{n-1}y_{n-2}\cdots y_1y_0)_b$, respectively}
    $c := 0$
    for $j := 0$ to $n-1$                {Move right-to-left}
        $t := x_j + y_j + c$             {Add the $j$th digits together}
        $c := \lfloor t/b \rfloor$       {Carry a digit if needed}
        $s_j := t - bc$                  {Remove carry and save as $s_j$}
    $s_n := c$                           {Final carry becomes $s_n$}
    return $(s_n, s_{n-1}, \ldots, s_1, s_0)$
    {The base b expansion of the sum is $(s_ns_{n-1}\cdots s_1s_0)_b$}
```

c: Quotient
S_j: Remainder

Does this sound familiar?
What is its complexity?

Addition examples in hexadecimal/octal

Hex | Octal

```
  B8C0          5630
+ 827F        + 3766


  13AC4F        723405
+ 3B9E00      +  27305
```

Multiplying base b expansions

```
procedure multiply(x, y: positive integers, b: integer > 1)
    {The base b expansions of x and y are $(x_{n-1}x_{n-2}\cdots x_1x_0)_b$ and $(y_{n-1}y_{n-2}\cdots y_1y_0)_b$, respectively}
    $p := 0$                          {Resulting product}
    for $j := 0$ to $n–1$              {Move right-to-left in y}
        $c := 0$                      {Reset carry}
        for $i := 0$ to $n-1$          {Move right-to-left in x}
            $t := x_i * y_j + c$      {Multiply digits and add carry}
            $c := \lfloor t/b \rfloor$  {Carry a digit if needed}
            $r_i := t - bc$           {Partial product, digit i}
        $r_n := c$                    {Final carry becomes $r_n$}
        r := r shifted j places       {Shift position to align with j}
        p := add(p, r)                {Add r to the result}
    return $(p_{2n}, p_{2n-1}, \ldots, p_1, p_0)$
    {The base b expansion of the sum is $(p_{2n}p_{2n-1}\cdots p_1p_0)_b$}
```

c: Quotient
r_i: Remainder

What is its complexity?

Multiplication examples in hexadecimal/octal

Hex | Octal

```
     C38              365
   * 6A4            * 457
```

How are these algorithms used in practice?

In previous exercises, didn't we assume basic arithmetic operations were $\Theta(1)$?

- This is often true! Modern CPUs can compute (at least) 32-bit integer multiplication in circuitry in a few cycles
- What about numbers bigger than your CPU's MUL?
  - e.g., for cryptography
- Let $b = 2^{32}$, consider b-bit expansions where each "digit" is a 32-bit word

We can compare a CPU's MUL (etc.) circuits to the multiplication tables we memorized in grade school

- For small enough values, we know the answer very quickly
- For larger values, we learn an algorithm that utilizes many smaller multiplications

Other multiplication algorithms for even larger values

Algorithm	Complexity	Threshold example*
Grade school	$O(n^2)$	(native MUL)
Karatsuba	$O\left(n^{\log_2 3}\right) \approx O(n^{1.585})$	832 bits
Toom–Cook (Toom-3)	$O\left(n^{\log_3 5}\right) \approx O(n^{1.46})$	6208 bits
Schönhage–Strassen	$O(n \log n \log \log n)$	159744 bits
Fürer	$O\left(n \log n \, 2^{\Theta(\log^* n)}\right)$	—
Harvey–van der Hoeven	$O(n \log n)$	—

In-class Activities

Activity 5: Use the integer addition algorithm to compute $(734)_8 + (225)_8$ [miro]

Activity 6: Use the integer multiplication algorithm to compute $(110110)_2 \times (100101)_2$ [miro]

Activity 7: Calculate $(FF)_{16} \times (FF)_{16}$, $(77)_8 \times (77)_8$, and 99×99 and compare [miro]

Submit on

miro

Steps:

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Final thoughts

- Integers can be represented uniquely in any specified base
- Integer arithmetic can be computed in other bases, and even pen-and-paper algorithms can be useful in computing
  - Arithmetic isn’t always constant
- Next time:
  - Primes and composites (Section 4.3)