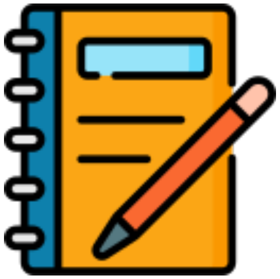
CS 441: Set Identities

PhD. Nils Murrugarra-Llerena
nem177@pitt.edu
University of Pittsburgh

Today's topics

- Set Identities
  - Methods of proof
  - Relationships to logical equivalences



Set identities help us manipulate complex expressions

Recall from last lecture that set operations bear a striking resemblance to logical operations

- Disjunction (∨) and set union (∪)
- Conjunction (∧) and set intersection (∩)
- Negation (¬) and complement ($\overline{}$)

Just as logical equivalences helped us manipulate logical expressions, set identities help us simplify and understand complex set definitions.

Some important set identities

Identity	*Name*
	Identity laws
	Domination laws
	Idempotent laws
	Complementation law
	Commutative laws
	Associative laws

Note that set difference is <u>not</u> commutative nor associative!

Some important set identities

Identity	*Name*
	Distributive laws
	DeMorgan's laws
	Absorption laws
	Complement laws

There are many ways to prove set identities

Today, we'll discuss four common methods:

1. Membership tables
   - Similar to using truth tables to prove logical equivalence.
2. Logical argument ("mutual subset" method)
   - Similar to the biconditional method for proving logical equivalence.
3. Using set builder notation
   - (No direct comparison to equivalences.)
4. Applying other known set identities
   - Similar to using existing logical equivalences to prove new ones.

Membership tables allow us to write proofs like we did using truth tables!

The membership table for an expression has columns for sub-expressions and rows to indicate the ways in which an arbitrary element may or may not be included.

Example: A membership table for set intersection

A	B	A ∩ B
1	1	1
1	0	0
0	1	0
0	0	0

An element is in A ∩ B iff it is in both A and B

Prove that A ∩ (B ∪ C) = (A ∩ B) ∪ (A ∩ C)

A	B	C	B∪C	A ∩ (B ∪ C)	A ∩ B	A ∩ C	(A ∩ B) ∪ (A ∩ C)
1	1	1					
1	1	0					
1	0	1					
1	0	0					
0	1	1					
0	1	0					
0	0	1					
0	0	0					

Since the appropriate columns of the membership table are the same, we can conclude that A ∩ (B ∪ C) = (A ∩ B) ∪ (A ∩ C). ❑

Sometimes, it's easier to make a logical argument about a set identity

Recall: A = B iff $A \subseteq B$ and $B \subseteq A$

As a result, we can prove a set identity by arguing that each side of the equality is a subset of the other.

Example: Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$

1. First prove that $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$
2. Then prove that $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$

Let's see how this is done…

- Compare this mutual subset method to the biconditional (mutual implication) method!

Prove that $\overline{A \cap B} = \bar{A} \cup \bar{B}$

First show $\overline{A \cap B} \subseteq \bar{A} \cup \bar{B}$

- Let x be an arbitrary element of $\overline{A \cap B}$
- By def'n of complement, $x \notin (A \cap B)$
- By def'n of $\notin$, $\neg(x \in A \cap B)$
- By def'n of intersection, $\neg(x \in A \wedge x \in B)$
- By DeMorgan's, $\neg(x \in A) \vee \neg(x \in B)$
  - In the first case, $x \notin A$, so $x \in \bar{A}$
  - In the second case, $x \notin B$, so $x \in \bar{B}$
  - Combining both cases, $x \in \bar{A} \cup x \in \bar{B}$
- Thus, if $x \in \overline{A \cap B}$, then $x \in \bar{A} \cup \bar{B}$
- Therefore, $\overline{A \cap B} \subseteq \bar{A} \cup \bar{B}$

Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Next show $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$

- Let x be an arbitrary element of $\overline{A} \cup \overline{B}$
- By def'n of union, $x \in \overline{A} \vee x \in \overline{B}$
- By def'n of complement, $x \notin A \vee x \notin B$
  - In the first case, $x \notin A$, so x cannot be in both A and B
  - In the second case, $x \notin B$, so x cannot be in both A and B
- Thus, if $x \in \overline{A} \cup \overline{B}$, then $x \in \overline{A \cap B}$
- Therefore, $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$

Since we have shown $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$ and $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$, we have shown that $\overline{A \cap B} = \overline{A} \cup \overline{B}$

We can use set builder notation and logical definition to make very precise proofs

Example: Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Proof:

1.	$\overline{A \cap B} = \{x \mid x \notin (A \cap B)\}$	Def'n of complement
2.	$= \{x \mid \neg(x \in (A \cap B))\}$	Def'n of $\notin$
3.	$= \{x \mid \neg(x \in A \wedge x \in B)\}$	Def'n of $\cap$
4.	$= \{x \mid \neg(x \in A) \vee \neg(x \in B)\}$	DeMorgan's law
5.	$= \{x \mid x \notin A \vee x \notin B\}$	Def'n of $\notin$
6.	$= \{x \mid x \in \overline{A} \vee x \in \overline{B}\}$	Def'n of complement
7.	$= \{x \mid x \in \overline{A} \cup \overline{B}\}$	Def'n of $\cup$
8.	$= \overline{A} \cup \overline{B}$	Set builder notation

❏

Note that the argument here uses <u>equivalence</u> rather than subset (deduction), so we do not need to argue in both directions

We can also construct proofs by repeatedly applying known set identities

Example: Prove that $\overline{A \cup (B \cap C)} = (\overline{C} \cup \overline{B}) \cap \overline{A}$

Proof:

1. $\overline{A \cup (B \cap C)} = \overline{A} \cap \overline{(B \cap C)}$ — DeMorgan's law
2. $= \overline{A} \cap (\overline{B} \cup \overline{C})$ — DeMorgan's law
3. $= (\overline{B} \cup \overline{C}) \cap \overline{A}$ — Commutative law
4. $= (\overline{C} \cup \overline{B}) \cap \overline{A}$ — Commutative law

❏

Note how similar this process is to that of proving logical equivalences using known logical equivalences.

As with set builder, only one direction is needed since we're using equivalence at every step.

In-class exercises

Problem 1: Prove DeMorgan's law for complement over intersection using a membership table.

Problem 2: Prove the complementation law using set builder notation.

Problem 3: Prove that if A and B are sets with $A \subseteq B$, then $\bar{A} \cap \bar{B} = \bar{B}$. Note that since we have "side information," we must use a deduction-based method (i.e., mutual subset).

Final thoughts

- Set identities are useful tools!
- We can prove set identities in a number of (equivalent) ways
- Next time:
  - Functions (Section 2.3)