CS 441: Nested Quantifiers

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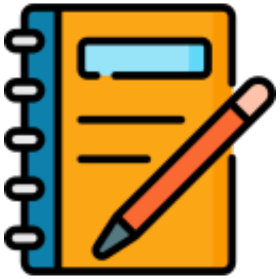
Today's topics

- Predicates
- Quantifiers
- Logical equivalences in predicate logic
- Translations using quantifiers



Nested quantifiers?!?

Many times, we need the ability to nest one quantifier within the scope of another quantifier

Example: All integers have an additive inverse. That is, for any integer x, we can choose an integer y such that the sum of x and y is zero.

There is no way to express this statement using only a single quantifier!

Deciphering nested quantifiers isn't as scary as it looks…

… if you remember to read from left to right!

$$\forall x \; \exists y \; \forall z \; [(x + y) \times z = 0]$$

For all x…

… there exists a y such that…

… for all z…

… $(x + y) \times z = 0$

And think about scope of variables like with programming!

A few more examples…

This is the commutative law for addition!

$\forall x\ \forall y\ (x + y = y + x)$

- For all integers x and for all integers y, $x + y = y + x$

This is the associative law for addition!

$\forall x\ \forall y\ \forall z\ [(x+y)+z = x+(y+z)]$

- For all integers x, for all integers y, and for all integers z, $(x+y)+z = x+(y+z)$

$\exists x\ \forall y\ (x \times y = 0)$

- There exists an x such that for all y, $x \times y = 0$

Since we always read from left to right, the order of quantifiers matters!

Consider: $\forall x \, \exists y \, (x + y = 0)$ ← Clearly true! Just set y = -x

Transpose: $\exists y \, \forall x \, (x + y = 0)$ ← Not true…

Remember: When reading from left to right, later quantifiers are within the scope of earlier ones

Many mathematical statements can be translated into logical statements with nested quantifiers

Translating mathematical expressions is often easier than translating English statements!

Steps:

1. Rewrite statement to make quantification and logical operators more explicit
2. Determine the order in which quantifiers should appear
3. Generate logical expression

Let’s try a translation…

Statement: Every real number except zero has a multiplicative inverse

Universal quantifier

$x \times y = 1$

Singular—suggestive of an existential quantifier

$\forall x$

Rewrite: For every real number x, if $x \neq 0$, then there exists a real number y such that $x \times y = 1$.

$\ldots\ \exists y\ (x \times y = 1)$

$(x \neq 0) \rightarrow \ldots$

More examples…

Statement: The product of any two negative integers is always positive

Statement: For any real number a, it is possible to choose real numbers b and c such that $a^2 + b^2 = c^2$

Translating quantified statements to English is as easy as reading a sentence!

Let:

- C(x) ≡ x is enrolled in CS441
- P(x) ≡ x has an iPhone
- F(x, y) ≡ x and y are friends
- Domain of x and y is “all students”

Statement: $\forall x\ [C(x) \rightarrow P(x) \vee (\exists y\ (F(x,y) \wedge P(y))]$

For every student x…

… if x is enrolled in CS441, then…

… x has an iPhone…

… or there exists another student y such that…

… x and y are friends…

… and y has an iPhone.

Every CS 441 student has an iPhone or a friend with an iPhone.

Translate the following expressions into English

Let:

- $O(x,y) \equiv$ x is older than y
- $F(x,y) \equiv$ x and y are friends
- The domain for variables x and y is "all students"

Statement: $\exists x\ \forall y\ O(x,y)$

Statement: $\exists x\ \exists y\ [F(x,y) \land \forall z\ [(y \neq z) \rightarrow \neg F(x,z)]]$

In-class exercises

Problem 1: Translate the following mathematical statement into predicate logic: Every even number is a multiple of 2. Assume that the predicate E(x) means "x is even." (Domains: All integers)

- **Hint:** What does "x is a multiple of 2" mean algebraically? Try not to use "mod."

Problem 2: Translate the following expressions into English. Assume that C(x) means "x has a car", F(x,y) means "x and y are friends", and S(x) means "x is a student." (Domains: All people)

- $\forall x\ (S(x) \rightarrow C(x) \lor \exists y\ [F(x,y) \land C(y)])$
- $\forall x\ \exists y\ \exists z\ [C(x) \lor (F(x,y) \land C(y)) \lor (F(x,y) \land F(y,z) \land C(z))]$

Translating from English to a logical expression with nested quantifiers is a little bit more work…

Steps:

1. If necessary, rewrite the sentence to make quantifiers and logical operations more explicit
2. Create propositional functions to express the concepts in the sentence
3. State the domains of the variables in each propositional function
4. Determine the order of quantifiers
5. Generate logical expression

Let's try an example…

*Statement***:** Every student has asked at least one professor a question.

Universal quantifier

Existential quantifier

*Rewrite***:** For every person x, if x is a student, then there exists a professor whom x has asked a question.

*Let***:**

- S(x) ≡ x is a student
- P(x) ≡ x is a professor
- Q(x,y) ≡ x has asked y a question

Domains for x and y are "all people"

*Translation***:** $\forall x\, (S(x) \rightarrow \exists y\, [P(y) \land Q(x,y)])$

Translate the following from English

Statement: There is a man who has tasted every type of beer.

Domain: all people

Let:

Domain: all drinks

Domains: x = all people, y = all drinks

Translation:

Negating expression with nested quantifiers is actually pretty straightforward…

… you just repeatedly apply DeMorgan's laws!

$\neg[\exists x\ (M(x) \wedge \forall y\ [B(y) \rightarrow T(x,y)])]$

$a \rightarrow b \equiv \neg a \vee b$

In English: For all people x, if x is a man, then there exists some type beer that x has not tasted.

Alternatively: No man has tasted every type of beer.

A few stumbling blocks…

Whether the negation sign is on the inside or the outside of a quantified statement makes a big difference!

Example: Let T(x) ≡ "x is tall". Consider the following:

- $\neg\forall x\, T(x)$
  - "It is not the case that all people are tall."
- $\forall x\, \neg T(x)$
  - "For all people x, it is not the case that x is tall."

Note: $\neg\forall x\, T(x) = \exists x\, \neg T(x) \neq \forall x\, \neg T(x)$

Recall: When we push negation into a quantifier,
DeMorgan's law says that we need to switch the quantifier!

A few stumbling blocks…

Let: $C(x) \equiv$ "x is enrolled in CS441"

$S(x) \equiv$ "x is smart."

Question: The following two statements look the same, what's the difference?

- $\exists x\, [C(x) \wedge S(x)]$
- $\exists x\, [C(x) \rightarrow S(x)]$

There is a smart student in CS441.

There exists a student x such that if x is in CS441, then x is smart.

Subtle note: The second statement is true if there exists one person not in CS441, because $F \rightarrow F$ or $F \rightarrow T$.

Negate $\forall x\,(S(x) \rightarrow \exists y\,[P(y) \wedge Q(x,y)])$

$\neg\forall x\,(S(x) \rightarrow \exists y\,[P(y) \wedge Q(x,y)])$

In English: There exists a student x such that for all people y, if y is a professor then x has not asked y a question.

Alternatively: There exists a student that has never asked any professor a question.

In-class exercises

Problem 3: Translate the following English sentences into predicate logic.

a) Every student has at least one friend that is dating a Steelers fan.
b) If a person is a parent and a man, then they are the father of some child.

Problem 4: Negate the results from Problem 3 and translate the negated expressions back into English.

Final Thoughts

- Quantifiers can be nested
  - Nested quantifiers are read left to right
  - Order is important!
  - Translation and negation work the same as they did before!

- Next lecture:
  - Rules of inference
  - Please read sections 1.6–1.7