CS 441: Proof Methods

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Today's topics

- Proof techniques
  - Proof by exhaustion
  - Proof by cases
  - Existence proofs
  - Uniqueness proofs

- Proof strategies
  - Backward reasoning

Not all theorems are of the form $p \rightarrow q$

Sometimes, we need to prove a theorem of the form:

$$p_1 \vee p_2 \vee \ldots \vee p_n \rightarrow q$$

So, we might need to examine multiple cases!

Prove that $n^2 + 1 \geq 2n$ where n is a positive integer with $1 \leq n \leq 4$

***Proof*:**

Since we have verified each case, we have shown that $n^2 + 1 \geq 2n$ where n is a positive integer with $1 \leq n \leq 4$.
□

With only 4 cases to consider, exhaustive proof was a good choice!

Sometimes, exhaustive proof isn't an option, but we still need to examine multiple possibilities

Example: Prove the triangle inequality. That is, if x and y are real numbers, then $|x| + |y| \geq |x + y|$.

Clearly, we can't use exhaustive proof here since there are infinitely many real numbers to consider.

We also can't use a simple direct proof either, since our proof depends on the signs of x and y.

What should we do?

Example: Prove that if x and y are real numbers, then |x| + |y| ≥ |x + y|.

- Note: If $x \geq 0$, $|x| = x$, otherwise $|x| = -x$
- Cases:

1) $x \geq 0$ and $y \geq 0$
   - $|x| + |y| = x + y$ and $|x + y| = x + y$
   - $x + y \geq x + y$ ✓
2) $x < 0$ and $y < 0$
   - $|x| + |y| = -x - y$ and $|x + y| = -x - y$
   - $-x - y \geq -x - y$ ✓
3) $x \geq 0$ and $y < 0$
   - If $\boxed{x \geq |y|}$, then $|x + y| = x - |y|$ and $|x| + |y| = x + |y|$
     $x + |y| \geq x - |y|$ ✓
   - If $\boxed{x < |y|}$, then $|x + y| = |y| - x$ and $|x| + |y| = x + |y|$
     $|y| + x \geq |y| - x$ ✓
4) Symmetrical to Case 3 □

Making mistakes when using proof by cases is all too easy!

Mistake 1: Proof by "a few cases" is not equivalent to proof by cases.

This is a "there exists" proof, not a "for all" proof!

Example: Prove that all odd numbers are prime.

"Proof:"

- Case (i): The number 1 is both odd and prime
- Case (ii): The number 3 is both odd and prime
- Case (iii): The number 5 is both odd and prime
- Case (iv): The number 7 is both odd and prime

Thus, we have shown that odd numbers are prime. ☐

Making mistakes when using proof by cases is all too easy!

Mistake 2: Leaving out critical cases.

Example: Prove that $x^2 > 0$ for all integers x

"Proof:"

- Case (i): Assume that $x < 0$. Since the product of two negative numbers is always positive, $x^2 > 0$.
- Case (ii): Assume that $x > 0$. Since the product of two positive numbers is always positive, $x^2 > 0$.

Since we have proven the claim for all cases, we can conclude that $x^2 > 0$ for all integers x. □

What about the case in which x = 0?

Sometimes we need to prove the existence of a given element

There are two ways to do this

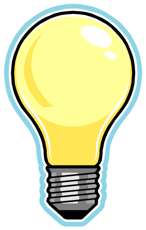
The constructive approach

The non-constructive approach

A constructive existence proof

Prove: Show that there is a positive integer that can be written as the sum of cubes of positive integers in two different ways.

Proof: $1729 = 10^3 + 9^3 = 12^3 + 1^3$ □



Constructive existence proofs are really just instances of "existential generalization."

A non-constructive existence proof

Prove: Show that there exist two irrational numbers x and y such that x^y is rational.

Proof:

Note: We don't know whether $\sqrt{2}^{\sqrt{2}}$ is rational or irrational. However, in either case, we can use it to construct a rational number.

Sometimes, existence is not enough, and we need to prove uniqueness

This process has two steps:

1.
2.

Example: Prove that if *a* and *b* are real numbers, then there exists a unique real number *r* such that $ar + b = 0$

Existence

Proof:

- Note that r = -b/a is a solution to this equality since a(-b/a) + b = -b + b = 0.
- Assume that as + b = 0
- Then as = -b, so s = -b/a = r, which means s is just r ☐

Uniqueness

In-class exercises

Problem 1: Prove that there exists a positive integer that is equal to the sum of all positive integers less than it. Is your proof constructive or non-constructive?

Problem 2: Prove that there is no positive integer n such that $n^2 + n^3 = 100$.

Top Hat

Final Thoughts

- Proving theorems is not always straightforward
- Having several proof strategies at your disposal will make a huge difference in your success rate!
- We are “done” with our intro to logic and proofs
- Next lecture:
  - Intro to set theory
  - Please read sections 2.1 and 2.2