Math 0413 Supplement
Logic and Proof

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1 Propositions

A *proposition* is a statement that can be true or false. Here are some examples of propositions:

- $1 = 1$
- $1 = 0$
- Every dog is an animal.
- Every animal is a dog.
- There is a real number r such that $r^2 = 2$.
- There is a rational number r such that $r^2 = 2$.

Every proposition has a *value* of either True of False, abbreviated T or F.

The term *statement* is often used as a synonym for "proposition." We will use these words interchangeably.

1.1 New Propositions from Old

If you start with one or more propositions, you can build new propositions from them by combining them with *logical operations*.

Logical Negation. The simplest logical operator is the **not** operator. If P is a proposition, then the proposition not P, abbreviated $\sim P$, has the opposite truth value to P. This is spelled out in a *truth table* which gives the value of $\sim P$ for every possible value of P. The truth table for $\sim P$ is shown in Figure 1. For example, if P is the proposition $1 = 0$, then $\sim P$ is the proposition $1 \neq 0$.

Logical And. If P and Q are propositions, the proposition "P and Q" is denoted symbolically by $P \wedge Q$. It's true when P and Q are *both* true, and false otherwise. This is summarized in Figure 2.

P	$\sim P$
T	F
F	T

Figure 1: Truth table for $\sim P$

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

Figure 2: Truth table for $P \wedge Q$

Logical Or. The proposition "P or Q" is denoted symbolically by $P \vee Q$. It is true if one or both of P, Q is true. It is false if both P and Q are false. This is summarized in Figure 3.

Conditional Statements. The proposition "If P then Q" is a *conditional statement*. It asserts that Q is true whenever P is true. It is denoted symbolically by $P \Rightarrow Q$. Its truth value is false when P is true and Q is false. Otherwise, the $P \Rightarrow Q$ is true. This is summarized in Figure 4. The proposition P is called the *hypothesis* of the conditional statement $P \Rightarrow Q$. The proposition Q is the *conclusion*.

Converse. The *converse* of the if-then proposition $P \Rightarrow Q$ is the if-then proposition $Q \Rightarrow P$. It's important to note that an if-then proposition is *not* logically equivalent to its converse. Let's illustrate with a simple example. Consider the conditional statement

If n is prime, then n is an integer.

Its converse is

If n is an integer, then n is prime.

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

Figure 3: Truth table for $P \vee Q$

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Figure 4: Truth table for $P \Rightarrow Q$

P	Q	$\sim Q$	$\sim P$	$\sim Q \Rightarrow \sim P$
T	T	F	F	T
T	F	T	F	F
F	T	F	T	T
F	F	T	T	T

Figure 5: Truth table for $\sim Q \Rightarrow \sim P$

The first proposition is true, since every prime number is an integer. The second is false, since there are integers that are not prime.

Contrapositive. The contrapositive of the proposition $P \Rightarrow Q$ is the proposition $\sim Q \Rightarrow \sim P$. For example, the contrapositive of

> If x is prime then x is an integer

is

> If x is not an integer then x is not prime.

If you read these two statements carefully, you will see that they say exactly the same thing, but in different ways. You will also note that both statements are true.

In general, the truth value of $\sim Q \Rightarrow \sim P$ depends on the truth values of P and Q. You can work out the dependence with a truth table, as shown in Figure 5. You can see from Figure 5 that for every combination of truth values for P and Q, the truth value of $\sim Q \Rightarrow \sim P$ is the same as that of $P \Rightarrow Q$. In this case, we say that the two statements are *logically equivalent*. Thus, any if-then statement is logically equivalent to its contrapositive.

Biconditionals. The proposition "P if and only if Q" is denoted symbolically $P \Leftrightarrow Q$. It is true when P and Q have the same truth values and false otherwise, as shown in Figure 6.

P	Q	$P \Leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

Figure 6: Truth table for $P \Leftrightarrow Q$

2 Quantifiers

2.1 Existential Quantifiers

Consider the statement

> There is a rational number r such that $r^2 = 2$.

This statement, which, by the way, is false, asserts the existence of a member of some set, in this case the set of rational numbers, having a certain property. The first part of this statement can be expressed symbolically as $\exists r \in \mathbb{Q}$. The symbol $\exists$ is an *existential quantifier*." It may be read as "there exists." The whole statement becomes

$$\big(\exists r \in \mathbb{Q}\big)\ r^2 = 2.$$

The general form of an existential proposition is

$$\big(\exists x \in A\big)\ P(x)$$

which asserts that there is a member x of the set A such that the property P holds for x.

2.2 Universal Quantifiers

Consider the statement

> For every real number x we have $\cos^2 x + \sin^2 x = 1$.

The first part of this statement can be expressed symbolically as $\forall x \in \mathbb{R}$. Here the symbol $\forall$ is a *universal quantifier*. It is read "for all" or "for every." The entire statement can be expressed symbolically as

$$\big(\forall x \in \mathbb{R}\big)\ \cos^2 x + \sin^2 x = 1.$$

2.3 Nested Quantifiers

You will often have to deal with statements that involve two or more quantifiers. Here is a typical example with two quantifiers.

> For every $\varepsilon > 0$ there is a natural number n such that $|\sin n| < \varepsilon$.

Proposition	**Negation**
P	$\sim P$
$\sim P$	P
$P \wedge Q$	$(\sim P) \vee (\sim Q)$
$P \vee Q$	$(\sim P) \wedge (\sim Q)$
$P \Rightarrow Q$	$P \wedge (\sim Q)$
$(\exists x)\ P(x)$	$(\forall x)\ \sim P(x)$
$(\forall x)\ P(x)$	$(\exists x)\ \sim P(x)$

Figure 7: Negation rules

In symbols,

$$\big(\forall \varepsilon \in (0,\infty)\big)\ \big(\exists n \in \mathbb{N}\big)\ |\sin n| < \varepsilon.$$

Here's one with three quantifiers:

> For every $\varepsilon > 0$ there is a natural number K such that for every $n \geq K$ we have $|n \sin(1/n) - 1| < \varepsilon$.

In symbols,

$$\big(\forall \varepsilon \in (0,\infty)\big)\ \big(\exists K \in \mathbb{N}\big)\ \big(\forall n \in \mathbb{N} \cap [K,\infty)\big)\ |n \sin(1/n) - 1| < \varepsilon. \qquad (1)$$

3 Negating a Proposition

You will often find yourself having to negate a complex proposition. For example, the negation of the proposition (1) is

$$\big(\exists \varepsilon \in (0,\infty)\big)\ \big(\forall K \in \mathbb{N}\big)\ \big(\exists n \in \mathbb{N} \cap [K,\infty)\big)\ |n \sin(1/n) - 1| \geq \varepsilon.$$

Some rules to help you with this task are given in Figure 7.

4 Proofs

A proof of a proposition establishes that the proposition is true by a sequence of logical steps. The purpose of this section is to familiarize you with several proof strategies that you will use frequently.

4.1 Proving conditional statements

Recall that a conditional statement has the form "If P then Q." The statement P is the hypothesis, and Q is the conclusion.

A *direct proof* proceeds by assuming that the hypothesis is true, and arguing that the conclusion must be true. Here is an example.

Theorem 4.1. *If a and b are even integers then $a + b$ is even.*

Proof. Assume that a and b are even integers. Then there are integers k and ℓ such that $a = 2k$ and $b = 2\ell$. Therefore

$$a + b = 2k + 2\ell = 2(k + \ell).$$

Since $a + b$ is a multiple of 2, it is even. □

Another way to prove a conditional statement is by *contraposition*. This means that instead of proving the statement directly, you prove its contrapositive. Since any conditional statement is logically equivalent to its contrapositive, this will establish the original statement. A proof by contraposition begins by assuming that the conclusion fails. From there, you argue that the hypothesis must fail as well. Here's an example.

Theorem 4.2. *If n is an integer such that $3n + 5$ is odd, then n is even.*

Proof. Assume that n is not even. Then n is odd. This means that there is an integer k such that $n = 2k + 1$. Then

$$3n + 5 = 3(2k + 1) + 5 = 6k + 8 = 2(3k + 4)$$

and so $3n + 5$ is even, contrary to hypothesis. □

4.2 Proving Biconditionals

Recall that a biconditional statement has the form "P if and only if Q." The biconditional is logically equivalent to

$$(P \Rightarrow Q) \wedge (Q \Rightarrow P).$$

Proof of a biconditional usually proceeds by proving the two conditional statements separately. Here is an example

Theorem 4.3. *An integer n is even if and only if its square is even.*

Proof. We first prove the "only if" part: If n is even, then n^2 is even. Since n is even, we can write $n = 2k$ for some integer k. Therefore

$$n^2 = (2k)^2 = 4k^2 = 2(2k^2).$$

Since n^2 is a multiple of 2, it is even.

We next prove the "if" part: If n^2 is even, then n is even. We prove this part by contraposition. Suppose that n is not even. Then n is odd, so $n = 2k + 1$ for some integer k. Therefore

$$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1.$$

Thus, n^2 has the form $2\ell + 1$ with $\ell = 2k^2 + 2k$, so n^2 is odd, contrary to hypothesis. □

4.3 Proof by Contradiction

To prove a theorem by contradiction, you assume statement is false, and derive from that another statement which is demonstrably false. Here is an example

Theorem 4.4. *There is no rational square root of* 2.

Proof. By way of contradiction, suppose there is a rational number r such that $r^2 = 2$. Since r is rational, it can be expressed as a quotient

$$r = \frac{m}{n}$$

where m and n are integers with no common factors. In particular, m and n are not both even. Since $r^2 = 2$, we have

$$m^2 = 2n^2. \tag{2}$$

Therefore m^2 is even. But, since the square of any odd number is odd, it follows that m is even. Therefore

$$m = 2k$$

for some integer k. Inserting this into (2) gives

$$4k^2 = 2n^2.$$

Dividing through by 2 gives

$$2k^2 = n^2,$$

and so n^2 is even. Since the square of any odd number is odd, it follows that n is even. We have now shown that m and n are both even, contradicting the fact that m and n have no common factors. Thus, the assumption that 2 has a rational square root leads to a contradiction. It follows that 2 has no rational square root. □

5 Exercises

1. Use truth tables to show that the conditional statement $P \Rightarrow Q$ is logically equivalent to the statement $(\sim P) \vee Q$.

2. Use truth tables to show that the statement $\sim(P \wedge Q)$ is logically equivalent to $(\sim P) \vee (\sim Q)$.

3. Use truth tables to show that the statement $\sim(P \vee Q)$ is logically equivalent to $(\sim P) \wedge (\sim Q)$.

4. Show that $\sim(P \Rightarrow Q)$ is logically equivalent to $P \wedge (\sim Q)$.

5. Negate the statement "For every $x \in \mathbb{R}$, $x^2 > 0$."

6. Negate the statement "There is a prime number p satisfying $p > 10^{1000}$."

7. Negate the statement "For every $a, b \in \mathbb{R}$ with $a < b$ there is an $r \in \mathbb{Q}$ with $a < r < b$.
8. Negate the statement "For every $\varepsilon > 0$ there is an $n \in \mathbb{N}$ such that $|\sin n| < \varepsilon$.
9. Negate the statement "For every $x \in \mathbb{R}$ there is an $n \in \mathbb{N}$ such that $x < n$."
10. Negate the statement "There is an $a \in \mathbb{R}$ such that for every $\varepsilon > 0$ there is a $K \in \mathbb{N}$ such that for every $n \in \mathbb{N}$ with $n \geq K$ we have $|(1 + 1/n)^n - a| < \varepsilon$."